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Lagrangian KAM Theorem for stat mech models

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It is an a-posteriori theorem

$$\|F(u_0)\|_0 \leq \varepsilon \quad u_0 \in X_0$$

$$f_1(u_0) \leq M_1, \quad f_m(u_m) \leq M_m$$

$$\varepsilon \leq \varepsilon^*(M_1, \dots, M_m)$$

$$\Rightarrow \exists u^* \in X_1$$

$$F(u^*) = 0$$

$$\|u^* - u\|_1 \leq \varepsilon$$

To get a real theorem
We need to specify

- Spaces
- Non-degeneracy condition
- $\mathbb{F}$

The proof is a modified
Newton method which
takes advantage of
the structure of $\mathbb{F}$

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It leads to efficient algorithms

It is a "Newton" method (quadratically convergent)

If we discretize with N terms, it requires

$O(N)$ storage

$O(N \ln N)$ operations

Straight Newton

$O(N^2)$ storage

$O(N^3)$ operations

$N \approx 10^6$ becomes easy

Numerical remarks

Newton method

+ They produce very accurate solutions

$\approx$ error in evaluation of -

- They need to start near an approximate solution

- Continuation from a ~~global~~ solution known

- Use variational methods (globally convergent but inaccurate $\approx \sqrt{\epsilon}$)

"When you
measure 6 figures
you start seeing
a new landscape"

References

R. L. Aron. Henri Poincare (2008)

R. Calleja, R L

Nonlinearity (2009)

Nonlinearity (2010)

X. Su, R. L

Preprint in arxiv (2011)

Precedents

D. Rana's thesis 1987

J. Moser & ~~Mos~~ Levi 2001
"Tent maps"

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S. Kozlov 1983

J. Moser 1987

"Reducibility in PDE"

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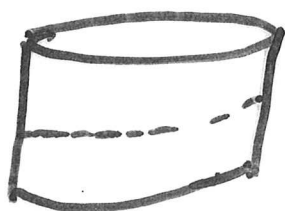
D. Salomon E. Zehnder (1989)

"Variational
formulation
with Legendre
condition"

"Works in higher
dimension"

Spaces

$$\Pi_g = \{z \in \mathbb{C}/\mathbb{Z} \mid |\operatorname{Im}(z)| \leq g\}$$



$$A_g = \{f: \Pi_g \rightarrow \mathbb{C}^n \mid$$

- Continuous
- analytic in interior
- $\|f\|_g = \sup_{z \in \Pi_g} |f(z)|$

(we are interested in the functions which are real for real values)

Some elementary facts

Cauchy bounds

$$\bullet \quad \|f'\|_{\rho-\delta} \leq C \|f\|_{\rho} \delta^{-1}$$

$$f(z) = \sum_k e^{2\pi i k z} \hat{f}_k$$

$$\bullet \quad |\hat{f}_k| \leq e^{-2\pi |k| \rho} \|f\|_{\rho}$$

Diophantine numbers

$$|W \cdot k - l|^{-1} \leq \epsilon |k|^{\tau}$$

$\mapsto$

Important tool

$$\Delta_1 \varphi \equiv \varphi - \varphi \circ T_\omega$$

Lemma (Ruzsmann, Kolmogorov)

Assume $\eta \in \mathcal{A}_g$ $\int_0^1 \eta(\alpha) d\alpha = 0$

ω Diophantine, $\delta > 0$

$\exists! \varphi \in \mathcal{A}_{g-\delta}$

$$\Delta_1 \varphi = \eta \quad \int \varphi = 0$$

$$\|\varphi\|_{g-\delta} \leq C \delta^{-\frac{\tau}{2}} \|\eta\|$$

Notice that

Δ_1 has range of
codimension 1

kernel of dimension 1

(In different spaces, it
is an isomorphism Δ_1^{-1} is
unbounded from a
space to itself, but
is bounded from one
space to another

A small remarks

Δ, Δ^{-1} identity in a
 space of codimension
 Δ^{-1}, Δ identity, projection
 to a space of
 dimension 1

$$\Delta_N \varphi = \varphi - \varphi \circ T_{N\omega}$$

~~$$\Delta_N \Delta^{-1} \varphi = \Delta_N \varphi$$~~

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$$\Delta_N \varphi = \varphi - \varphi \circ T_N \omega$$

$$= \varphi - \varphi \circ T \omega$$

$$+ \varphi \circ T \omega - \varphi \circ T_2 \omega$$

$$+ \varphi \circ T_2 \omega - \varphi \circ T_3 \omega$$

$$\vdots$$

$$+ \varphi \circ T_{N-1} \omega - \varphi \circ T_N \omega$$

$$= \Delta_1 \varphi + (\Delta_1 \varphi) \circ T \omega + \dots + (\Delta_1 \varphi) \circ T_{N-1} \omega$$

$$\|\Delta_N (\Delta_1^{-1}) \varphi\|_g \leq N \|\varphi\|_g$$

For φ in domain Δ_1^{-1}

Operators

Remember we were interested in stat-mech models given by interactions of the form

$$\mathcal{Z}(\underline{x}) =$$

$$\sum_L \sum_{\mathbf{h}} H_L(x_{\mathbf{h}}, \dots x_{\mathbf{h}+L})$$

and equilibrium equations

$$\frac{\partial \mathcal{Z}}{\partial x_i} = 0$$

but we wanted

$$x_i = h(i\omega)$$

$$h(\theta) = \theta + u(\theta)$$

$u(\theta)$ periodic

(H1) We will
assume w
is Diophantine
with constants
 (γ, δ)

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$$\begin{aligned}
 F(u) &= \sum_L \sum_{j=0}^L \partial_j H_L(h(0-jw), \\
 &\quad \vdots \\
 &\quad h(0-(L-j)w)
 \end{aligned}$$

(H2)•

We will assume that H_L are analytic in a neighborhood at uniform distance from the range of U^0

$$\alpha \equiv \sum \|H_L\|_{L^4} < \infty$$

Now we come to
specify the non-degeneracy
assumptions.

(H3.1)

$$\|(\partial_0 \partial_1 H)'(h(0), h(0+w))\| \leq T$$

Note that sign of
 $\partial_0 \partial_1 H_1$ does not matter!

(H3.2)

$$\mathcal{E}_{0,11}(s) = \partial_0 \partial_1 H_1(h(s), h(s+w)) \\ \cdot h'(s) h'(s+w)$$

$$\left\| \left(\int \mathcal{E}_{0,11}^{-1}(s) ds \right)^{-1} \right\|_S \leq U$$

(H4)

$$\|1 + u'\|_g \leq N^+$$

$$\|(1 + u')^{-1}\|_g \leq N^-$$

(H5)

$$T\alpha \leq 1/2$$

$$VT\alpha \leq 1/4$$

$\|u\|_g + \delta$ inside the domain
of H_L

Then

If $\varepsilon \leq \varepsilon^*(\sigma, \tau, T, U, \alpha, \delta)$,
 $\cdot 2^4 \delta^{4\tau+4}$

Then,

$\exists u^*$ solng. $F(u^*) = 0$
 $u^* \in A_{\delta/2}$

$$\|u - u^*\|_{\delta/2} \leq C \varepsilon$$

$$\delta^{-2} \delta^{-2\sigma-2}$$

$\square$

Moreover, the function u^* is unique

"up to a change of origin".

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Algorithm

We will need two identities that come from the variational structure

Define $u_\sigma(\theta) = u(\theta + \sigma) + \sigma$

$$F[u_\sigma](\theta) = F[u](\theta + \sigma)$$

Taking derivatives
w.r.t. σ | $\sigma = 0$

$$D F[u] h' = \frac{d}{d\sigma} F[u]$$

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$$\int_0^1 h'(\sigma) F[u](\sigma) d\sigma = 0$$

H

The identity can be derived because $F[u]$ is the variational equation of a "Percival - Mather" variational principle

$$F[u] = \int_L [H_L(h(\sigma), \dots, h(\sigma + L\omega))]$$

$$P[u_\sigma] = P[u]$$

Taking der. w. r. t. σ ,
 $\sigma = 0$,

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The Newton method

Given u , find v

$$(N) \quad D\mathcal{F}[u]v = -\mathcal{F}[u]$$

No easy way in our case.

Following Kozlov, Moser, Levi-Moser

$h'(N)$

$$h'(0) [D\mathcal{F}[u]v](0) = -h'(0)\mathcal{F}[u](0)$$

$\vdash$

(QM)

$$h'(0) [D\mathcal{F}[u]v](0) - v(0) [D\mathcal{F}[u]h'](0) = -\mathcal{F}[u](0)$$

$$h'(0) [D\mathcal{F}[u]v](0) - v(0) \left[\frac{d}{ds} \mathcal{F}[u](0) \right] = -\mathcal{F}[u](0)$$

One can argue
heuristicly that
(QM) is a small change
from (M) because

$$v \approx F[u]$$

$$\frac{d}{ds} F[u](0) \approx F[u]$$

hence the added term
is "quadratic".

—

On the other hand
QM will factorize.

The key point is that (QN) can be factored into several steps

- Multiplication by known functions
- Composition
- Solving constant coefficients coboundary equations

The factorization is
quite elementary
(adding / subtracting
redefining τ_n)

$$v(\theta) = h'(\theta) w(\theta)$$

New variable
unknown W

$$h(\theta) = \theta + u(\theta)$$

$$h^{(i)}(\theta) = \theta + iw + u(\theta + iw)$$

$$\gamma_L(\theta) = (h(\theta), h^{(1)}(\theta), \dots, h^{(L)}(\theta))$$

$$\begin{aligned} \gamma_L^{(i)}(\theta) &= \gamma_L(\theta + iw) \\ &= (h^{(i)}(\theta), \dots, h^{(L+i)}(\theta)) \end{aligned}$$

Notation

The QM equations
for W can be written

$$\sum_L \sum_{j=0}^L \sum_{i=0}^L \partial_j \partial_i H_L \circ \gamma_L^{(-j)} \begin{matrix} h'(\theta) h'(\tau-j) \\ W^{(-j)}(\theta) \end{matrix} \\ = - h'(\theta) F[u](\theta)$$

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Introduce the notation

$$C_{ijL} = \partial_j \partial_i H_L \circ \gamma_L^{-j}(\theta) \begin{matrix} h'(\theta) h'(\tau-j) \\ (\theta) \end{matrix}$$

$$\partial_j \partial_i H_L \circ \gamma_L^{(-j)} h'(\theta) h'_{(\theta)}^{(i-j)} w^{i-j}_{(\theta)}$$

$$+ \partial_i \partial_j H_L \circ \gamma_L^{(-i)} h'(\theta) h'_{(\theta)}^{(j-i)} w^{j-i}_{(\theta)}$$

$$w(\theta)$$

$$w(\theta)$$

$$= \partial_j \partial_i H_L \circ \gamma_L^{(-j)} h'(\theta) h'_{(\theta)}^{(i-j)} [w_{(\theta)}^{(i-j)} - w_{(\theta)}]$$

$$- \partial_j \partial_i H_L \circ \gamma_L^{(-i)} h'(\theta) h'_{(\theta)}^{(j-i)} [w_{(\theta)} - w_{(\theta)}^{(j-i)}]$$

$$= \Delta_{j-i} \partial_j \partial_i H_L \circ \gamma_L^{(-i)} h'(\theta) h'_{(\theta)}^{(i-j)} \Delta_{i-j} w$$

(54) can be rewritten as

$$\sum_L \sum_{\substack{j,i=0 \\ j>i}}^L \Delta_{j-i} \mathcal{E}_{ijL} \Delta_{L-j} \quad \# \\ = h' F(u)$$

$$h' F[u](e) =$$

$$\Delta_1 \left[\mathcal{E}_{011} + \sum_{\substack{L>2 \\ j>i}} \Delta_1^{-1} \Delta_{j-i} \mathcal{E}_{ijL} \Delta_{L-j} \Delta_1^{-1} \right] \Delta_1$$

(separate $\mathcal{E}_{011}$ from the
rest of sum, factor Δ_1
in front and back

Notice that the
non-degeneracy assumptions
are just that

$\mathcal{C}_{0,1}$ is invertible

$\| \sum \Delta_{-1}^{-1} \Delta_{j-1} \mathcal{C}_{ij} \Delta_{i-j} \Delta_{-1}^{-1} \|$
is rather small.

$\Rightarrow$ The term in braces
in the previous is a
~~fast~~ bounded isomorphism.

Now, we start peeling off

1) We solve

$$\Delta_1 \eta = - \int h'(e) F[u](e)$$

Hurrah!

this was the first ~~term~~
easy observation

$$\|\eta\|_{\mathcal{H}^{-\delta}} \leq C \|F[u]\|_{\mathcal{H}^{\delta}} \lesssim \delta^{-\delta}$$

2) Now we determine A in such
a way that

$$\int (\mathcal{L}_{001} + \mathcal{G})^{-1} (\eta + T) = 0$$

The trust
condition!

Then, provided we can compose

$$\|f(u+v)\|_{s-\delta} \leq$$

$$\leq C \|f(u)\|_S^2 \nu^2 \phi^{-2/\sigma-2}$$

Proof

Use Taylor theorem
and add and subtract
(The extra term ~~is~~
~~estimated by the derivative~~
is
estimated by the derivative
and that by Cauchy estimates,

The near approximation
cancels out.)

Iteration of step and convergence.

This is very standard
in KAM theory
→ Choose $S_n = S_{n-1} - S_0 2^{-n} \frac{1}{\epsilon}$
We show by induction

- Error decreases fast
- The non-degeneracy conditions do not change much
- The composition conditions hold