**Lemma 1.22.** *Let* $B = A[T_0, \dots, T_d]$. *With the notation of* Example 1.19, *we have* $\mathcal{O}_X(n)(X) = B_n$ *if* $n \geq 0$ *and* $\mathcal{O}_X(n) = 0$ *if* $n < 0$. *In other words,* $\oplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)(X) = B$.

**Proof** We may suppose $d \geq 1$. Let $f \in \mathcal{O}_X(n)(X)$. We have $f|_{D_+(T_i)} = T_i^n P_i / T_i^m$ with $m \geq 0$ and $P_i \in B_m$. Multiplying $P_i$ and $T_i^m$ by a suitable power of $T_i$ if necessary, we can suppose that $m \geq n$ and is independent of $i$. On $D_+(T_0 T_1)$ we then have the equality

$$f|_{D_+(T_0T_1)} = T_0^n P_0 / T_0^m = T_1^n P_1 / T_1^m \in B_{(T_0 T_1)}.$$

As $T_0, T_1$ are regular elements in $B$, this implies that $T_0^{m-n} P_1 = T_1^{m-n} P_0$ in $B$. Hence $T_0^{m-n}$ divides $P_0$ in $B$ and $f = P_0 / T_0^{m-n}$ is a homogeneous polynomial of degree $\deg P - (m-n) = n$. so $f \in B_n$ if $n \geq 0$ and $f = 0$ otherwise.

Conversely, it can immediately be verified that $B_n \subseteq \mathcal{O}_X(n)(X)$. □

**Definition 1.23.** Let $X = \mathbb{P}^d_A$. For any integer $n \in \mathbb{Z}$ and any quasi-coherent sheaf $\mathcal{F}$ on $X$, we denote the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ by $\mathcal{F}(n)$. The $\mathcal{F}(n)$ are called the *twists* of $\mathcal{F}$. For any affine open subset $U$ of $X$, we have $\mathcal{F}(n)(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(n)(U)$ (Proposition 1.12(b)).

**Definition 1.24.** Let $X$ be a scheme. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by* $s_0, \dots, s_d \in \mathcal{F}(X)$ if $\mathcal{F}_x = \sum_i (s_i)_x \mathcal{O}_{X,x}$ for every $x \in X$ (see Definition 1.2). Let $\mathcal{L}$ be an invertible sheaf. Let $s \in \mathcal{L}(X)$. Let us set

$$X_s := \{x \in X \mid \mathcal{L}_x = s_x \mathcal{O}_{X,x}\}.$$

This is an open subset of $X$. When $\mathcal{L} = \mathcal{O}_X$, $X_s$ coincides with Definition 2.3.11.

The following lemma generalizes Proposition 1.6 by replacing $\mathcal{O}_X$ by an invertible sheaf $\mathcal{L}$. For simplicity, we denote the tensor power $\mathcal{L}^{\otimes n}$ by $\mathcal{L}^n$ and the tensor power $s^{\otimes n}$, $s \in \mathcal{L}(X)$, by $s^n$.

**Lemma 1.25.** *Let* $X$ *be a Noetherian or separated and quasi-compact scheme, let* $\mathcal{F}$ *be a quasi-coherent sheaf on* $X$, *and* $\mathcal{L}$ *an invertible sheaf on* $X$. *Let us fix sections* $f \in \mathcal{F}(X)$ *and* $s \in \mathcal{L}(X)$.

(a) *If* $f|_{X_s} = 0$, *then there exists an* $n \geq 1$ *such that* $f \otimes s^n = 0$ *in* $(\mathcal{F} \otimes \mathcal{L}^n)(X)$.

(b) *Let* $g \in \mathcal{F}(X_s)$. *Then there exists an* $n_0 \geq 1$ *such that* $g \otimes (s^n|_{X_s})$ *lifts to a section of* $(\mathcal{F} \otimes \mathcal{L}^n)(X)$ *for all* $n \geq n_0$.

**Proof** (See also Exercise 1.18.) We cover $X$ by a finite number of affine open subsets $X_1, \dots, X_r$ such that $\mathcal{L}|_{X_i}$ is free for every $i \leq r$. Let $e_i$ be a generator of $\mathcal{L}|_{X_i}$, and let $h_i \in \mathcal{O}_X(X_i)$ be such that $s|_{X_i} = e_i h_i$. Then $X_s \cap X_i$ is the principal open subset $D(h_i)$ of $X_i$.

(a) The $f|_{X_s} = 0$ hypothesis implies that $f|_{X_s \cap X_i} = 0$. Hence there exists an integer $n \geq 1$ such that $f|_{X_i} h_i^n = 0$. We can take $n$ sufficiently large so that it is independent of $i$. On $X_i$, the isomorphism $\mathcal{O}_{X_i} \to \mathcal{L}^n|_{X_i}$ defined as the