Figure 5. A non-geometrically irreducible integral conic.

Example 2.12. Let K be a non-trivial finite extension of k, and let $X = \operatorname{Spec} K$. If K/k is purely inseparable, then X is reduced but not geometrically reduced, because $X_K = \operatorname{Spec}(K \otimes_k K)$, and hence $X_{\overline{k}}$, is not reduced. If K/k is separable, then X is integral but not geometrically integral, nor even geometrically connected, because $X_{\overline{k}}$ is made up of $[K : k]$ isolated points.

Example 2.13. Let k be a field of characteristic different from 2, and let $a \in k$ which is not a square. Let us consider the projective variety

$$X = \operatorname{Proj} k[u, v, w]/(u^2 - av^2).$$

Let $\alpha \in \overline{k}$ be a square root of a and $K = k[\alpha]$. Then we easily verify that X is integral, while $X_K = \operatorname{Proj} K[u, v, w]/(u - \alpha v)(u + \alpha v)$ is not. See Figure 5 below.

Corollary 2.14. *Let X be an integral algebraic variety over k, with function field $K(X)$.*

(a) *Let us suppose that* $\operatorname{char}(k) = p > 0$. *Let* $L := k^{p^{-\infty}}$ *be the perfect closure of k. Then X is geometrically reduced if and only if X_L is reduced.*

(b) *Let k^s be the separable closure of k. Then X is geometrically connected (resp. irreducible) if and only if X_{k^s} is connected (resp. irreducible).*

(c) *The variety X is geometrically integral if and only if $K(X)$ and $\overline{k}$ are linearly disjoint over k*[1]*. Moreover, in that case we have $K(X_{\overline{k}}) = K(X) \otimes_k \overline{k}$.*

(d) *X is geometrically irreducible if and only if $K(X) \cap k^s = k$.*

Proof (a) and (b) result from Proposition 2.7(b) (indeed, $\overline{k}$ is separable over $k^{p^{-\infty}}$) and (c).

(c) Let us suppose $K(X)$ and $\overline{k}$ are linearly disjoint over k. Then $K(X) \otimes_k \overline{k}$ is an integral domain. For any affine open subset U of X, we have $\mathcal{O}_{X_{\overline{k}}}(U_{\overline{k}}) = \mathcal{O}_X(U) \otimes_k \overline{k} \subseteq K(X) \otimes_k \overline{k}$. Therefore $U_{\overline{k}}$ is integral. Taking an affine open covering of X, we see that $X_{\overline{k}}$ is reduced. This implies that $X_{\overline{k}}$ is integral by (d). Conversely, let us suppose $X_{\overline{k}}$ is integral. Let $U = \operatorname{Spec} A$ be an affine open subset of X. Then $K(X) \otimes_k \overline{k} = \operatorname{Frac}(A) \otimes_k \overline{k}$ is a localization of $A \otimes_k \overline{k}$, and is therefore an integral domain. Moreover, it is integral over $K(X)$ because $\overline{k}$

[1] Two k-extensions E, F endowed with k-homomorphisms $i : E \to \Omega, j : F \to \Omega$ in a k-extension Ω are said to be *linearly disjoint* when the canonical homomorphism $E \otimes_k F \to \Omega$ is injective. When E or F is algebraic over k, this is equivalent to $E \otimes_k F$ being a field (which is then isomorphic to the compositum of $i(E)$ and $j(F)$ in Ω). In particular, in this case, linear disjunction is independent on Ω and i, j.