a closed subset, which does not contain the generic point, by the hypothesis that $H^0(X, \omega_{X/S}) \neq 0$. Let $E$ be an exceptional divisor. By Proposition 3.10, $\deg \omega_{X/S}|_E < 0$. It follows that $H^0(E, \omega_{X/S}|_E) = 0$. It immediately follows from this that $E \subseteq \mathcal{B}$. As $\mathcal{B}$ is a proper closed subset of $X$, the lemma is proven.

Let us now suppose $H^0(X, \omega_{X/S}) = 0$. Let $X \to S' \to S$ be the decomposition as in Proposition 8.3.8. It suffices to show that the fibers of $X \to S'$ are irreducible, except for a finite number of them. By hypothesis, $H^0(X_K, \omega_{X_K/K}) = H^0(X, \omega_{X/S}) \otimes_{\mathcal{O}_K} K = 0$. By duality (Remark 6.4.21), we have $H^1(X_K, \mathcal{O}_{X_K}) = 0$. Let $L = K(S')$. By Proposition 3.16(c), $X_K$ is smooth over $L$ or purely inseparable over $\mathbb{P}^1_L$. The smooth case was seen at the beginning of the proof. Let us therefore suppose that there exists a finite purely inseparable morphism $\pi_L : X_K \to \mathbb{P}^1_L$. Replacing $S$ by a dense open subscheme if necessary, $\pi_L$ extends to a finite purely inseparable morphism $\pi : X \to \mathbb{P}^1_{\mathcal{O}_L}$. In particular, it is a homeomorphism (Exercise 5.3.9(a)). Therefore the fibers of $X \to \operatorname{Spec} \mathcal{O}_L$ are irreducible. □

**Remark 3.18.** We can also show Lemma 3.17 with the help of the following result: *Let $f : X \to Y$ be a morphism of finite type of locally Noetherian irreducible schemes. Let us suppose that the generic fiber $X_\eta$ is non-empty and geometrically irreducible. Then $X_y$ is (geometrically) irreducible for every point $y$ of a dense open subscheme of $Y$* ([42], Proposition IV.9.7.8).

**Proposition 3.19.** *Let $f : X \to S$ be an arithmetic surface. Then there exists a birational morphism $X \to Y$ of arithmetic surfaces over $S$, with $Y$ relatively minimal.*

**Proof** Let $X_0 = X \to X_1 \to \dots \to X_n \to \dots$ be a sequence of contractions of exceptional divisors. We must show that the sequence is necessarily finite. Let $B_n$ be the (finite) set of points $s \in S$ such that $(X_n)_s$ contains an exceptional divisor. Then $B_{n+1} \subseteq B_n$. Moreover, the total number of irreducible components contained in the fibers $(X_n)_s$, $s \in B_n$, decreases strictly with $n$. Therefore the sequence is finite. □

### 9.3.3 Existence of the minimal regular model

We are going to show the existence of minimal models (Definition 3.14) for arithmetic surfaces whose generic fibers have arithmetic genus $p_a \geq 1$. We will also show that the minimal model is compatible with étale base change (Proposition 3.28).

**Lemma 3.20.** *Let $X \to S$ be an arithmetic surface.*

(1) *Suppose that two distinct exceptional divisors $E_1, E_2$ on $X$ meet each other. Then $p_a(X_\eta) \leq 0$ and $E_1 \cup E_2$ is a connected component of a closed fiber $X_s$.*

(2) *Let $f : Z \dashrightarrow X$ be a birational map of arithmetic surfaces over $S$. Let $x_0 \in X$ be a closed point where $f^{-1}$ is not defined. Then the total*

*transform* (Definition 8.3.21) *of* $x_0$ *by* $f^{-1}$ *contains either an exceptional divisor or a connected component of* $Z_s$*. In the second case,* $p_a(X_\eta) \leq 0$.

**Proof** (1) We first reduce to the case when $X \to S$ has connected fibers. As usual, we can decompose $\pi : X \to S$ into $X \to T \to S$ where $T = \operatorname{Spec} \pi_* \mathcal{O}_X$ is a Dedekind scheme of dimension 1, finite and flat over $S$, and $X \to T$ has connected fibers. Let $s$ be the image of $E_i$ in $S$. Then $X_s$ is the disjoint union of the connected fibers $X_t$, $t \in T \times_S \operatorname{Spec} k(s)$. Let $\xi$ (resp. $\eta$) be the generic point of $T$ (resp. $S$). Then $X_\eta = X_\xi$, and $p_a(X_\eta) \leq 0$ (as $k(\eta)$-scheme) if and only if $H^1(X_\eta, \mathcal{O}_{X_\eta}) = 0$, which is equivalent to $p_a(X_\xi) \leq 0$. Finally, if $E$ is a vertical prime divisor on $X$, it does not make difference for $E$ being an exceptional divisor on $X$ as $S$-scheme or as $T$-scheme (use Castelnuovo's criterion 3.8). Therefore we can suppose that $T = S$ and $X_s$ is connected.

Let $k = k(s)$, $x \in E_1 \cap E_2$ and let $k_i = H^0(E_i, \mathcal{O}_{E_i})$. Then $k_i \subseteq k(x)$, and

$$(E_1 + E_2)^2 = -[k_1 : k] - [k_2 : k] + 2E_1E_2 \geq -[k_1 : k] - [k_2 : k] + 2[k(x) : k] \geq 0.$$

By Theorem 1.23, this implies that $E_1 \cup E_2 = X_s$. Let $K_{X/S}$ be a canonical divisor on $X$ and let $d_i$ be the multiplicity of $E_i$ in $X_s$. Then

$$2p_a(X_\eta) - 2 = K_{X/S} \cdot X_s = d_1 K_{X/S} \cdot E_1 + d_2 K_{X/S} \cdot E_2 \leq -d_1 - d_2 \leq -2$$

(Propositions 1.35 and 3.10(a)). Hence $p_a(X_\eta) \leq 0$.

(2) By Theorem 2.7, there exists a morphism $g : \widetilde{Z} \to Z$ made of a finite sequence of blowing-ups of closed points

$$g : \widetilde{Z} = Z_n \to Z_{n-1} \to \cdots \to Z_0 = Z$$

and a morphism $h : \widetilde{Z} \to X$ with the commutative diagram

$$\begin{array}{ccc} \widetilde{Z} & & \\ {\scriptstyle g}\downarrow & \searrow^{h} & \\ Z & \overset{f}{\dashrightarrow} & X \end{array}$$

As $f^{-1}$ is not defined at $x_0$, $h^{-1}(x_0)$ has dimension 1. As $h$ is an isomorphism above $V \setminus \{x_0\}$ for some open neighborhood $V$ of $x_0$, it is easy to see that $h^{-1}(x_0)$ contains an exceptional divisor $E$ on $\widetilde{Z}$. Let $\Gamma_i \subset Z_i$ be the exceptional locus of $Z_i \to Z_{i-1}$. This is an exceptional divisor by definition of $Z_i \to Z_{i-1}$. If $\Gamma_n \cap E = \emptyset$, then the image of $E$ in $Z_{n-1}$ is an exceptional divisor that we still denote by $E$. Let $m \leq n$ be the smallest positive integer such that $E \cap \Gamma_n = \cdots = E \cap \Gamma_m = \emptyset$. If $m = 1$, then $E$ is an exceptional divisor on $Z$ contained in the total transform $g(h^{-1}(x_0))$ of $x_0$. If $m \geq 2$, as $E$ and $\Gamma_{m-1}$ are exceptional divisors on $Z_{m-1}$ with non-empty intersection, it follows from (1) that $E \cap \Gamma_{m-1}$ is a connected component of $(Z_{m-1})_s$ and $p_a(X_\eta) \leq 0$. The image of $E$ in $Z_{m-2}$ is a connected component of $(Z_{m-2})_s$. The same is then true in $Z_s$. This proves (2). □

**Theorem 3.21.** *Let $X \to S$ be an arithmetic surface with generic fiber of genus $p_a(X_\eta) \geq 1$. Then $X$ admits a unique minimal model over $S$, up to unique isomorphism.*

**Proof** The uniqueness of a minimal model (up to unique isomorphism) follows from the definition. We already know that $X$ admits relatively minimal models (Proposition 3.19). The existence of the minimal model is equivalent to saying that two relatively minimal models $X_1, X_2$ of $X$ are isomorphic. Let us suppose that this is not the case. Then $X_1 \dashrightarrow X_2$ is not defined at some closed point of $X_1$. By Lemma 3.20, this implies that $X_2$ contains an exceptional divisor. Contradiction. □

**Remark 3.22.** This theorem was first proven by Lichtenbaum and Shafarevich. See the references in Remark 3.9.

**Remark 3.23.** Theorem 3.21 is false without the $p_a(X_\eta) \geq 1$ hypothesis. Indeed, and let us take $X_1 = \mathbb{P}^1_S$, let $X$ be the blowing-up of $X_1$ with center a closed point $x \in X_1(k(s))$. In $X_s$, the strict transform $E$ of $(X_1)_s$ is an exceptional divisor. Let $X \to X_2$ be the contraction of $E$. Then the models $X_1$ and $X_2$ of $X$ are relatively minimal, but not isomorphic as models of $X$ (more precisely, the birational map $X_1 \dashrightarrow X_2$ induced by the identity on the generic fiber does not extend to a morphism because the generic points of the fibers $(X_1)_s$ and $(X_2)_s$ induce distinct valuations in $K(X)$, even if, abstractly, we have $X_1 \simeq X_2 \simeq \mathbb{P}^1_S$). See also Exercise 3.1.

**Corollary 3.24.** *Let $X \to S$ be a relatively minimal arithmetic surface, with generic fiber $X_\eta$ verifying $p_a(X_\eta) \geq 1$. Then $X$ is minimal.*

**Proof** This is an immediate consequence of Theorem 3.21 and of the definition of a relatively minimal surface. □

**Definition 3.25.** Let $D$ be a divisor on a regular fibered surface $X \to S$. We say that $D$ is *numerically effective* if $D \cdot C \geq 0$ for every vertical prime divisor $C$. For example, an ample divisor is numerically effective by Proposition 7.5.5 and the fact that the restriction of an ample divisor to a closed subscheme remains ample.

**Corollary 3.26.** *Let $X \to S$ be an arithmetic surface with $p_a(X_\eta) \geq 1$. Let $K_{X/S}$ be a canonical divisor. Then $X \to S$ is minimal if and only if $K_{X/S}$ is numerically effective.*

**Proof** Indeed, by Proposition 3.10(b), $K_{X/S}$ is numerically effective if and only if $X$ is relatively minimal, which, in turn, is equivalent to $X$ being minimal by Corollary 3.24. □

**Corollary 3.27.** *Let $\pi : X \to S$ be a minimal arithmetic surface whose generic fiber is an elliptic curve. Then $\pi_*\omega_{X/S}$ is an invertible sheaf on $S$, and the canonical homomorphism $\pi^*\pi_*\omega_{X/S} \to \omega_{X/S}$ is an isomorphism.*