Massey products in Galois cohomology via rational points

Abstract: the Milnor conjecture identifies the cohomology ring \( H^*(\text{Gal}(\bar{k}/k),\mathbb{Z}/2) \) with the tensor algebra of \( k^* \) mod the ideal generated by \( x \otimes (1 - x) \) for \( x \in k - \{0,1\} \) mod 2. In particular, \( x \cup (1 - x) \) vanishes, where \( x \in k^* \) is identified with an element of \( H^1 \). We show that order \( n \) Massey products of \( n - 1 \) factors of \( x \) and one factor of \( 1 - x \) vanish by embedding \( \mathbb{P}^1 - \{0,1,\infty\} \) into its Picard variety and constructing \( \text{Gal}(\bar{k}/k) \) equivariant maps from \( \pi^1_{\text{et}} \) applied to this embedding to unipotent matrix groups. This also identifies Massey products of the form \( \langle 1 - x, x, \ldots, x, 1 - x \rangle \) with \( f \cup (1 - x) \), where \( f \) is a certain cohomology class which arises in the description of the action of \( \text{Gal}(\bar{k}/k) \) on \( \pi^1_{\text{et}}(\mathbb{P}^1 - \{0,1,\infty\}) \).