NÉRON MODELS OF ELLIPTIC CURVES

QING LIU, DINO LORENZINI

This is a private note. Let S be a noetherian regular scheme of dimension 1, with function field $K = K(S)$. Let E_K be an elliptic curve over K. Than E_K has two natural models over S. On the one hand, as a smooth projective curve of genus > 0, it has a minimal regular model E over S. On the other hand, as an abelian variety over K, it admits a Néron model N. Let E_{sm} be the biggest smooth open subscheme of E. By the universal property of N, the identity $E_K \to N_K$ extends uniquely to a morphism $f : E_{\mathrm{sm}} \to N$.

Proposition 1. *The morphism $f : E_{\mathrm{sm}} \to N$ is an isomorphism.*

This proposition is well-known. Cf. [2], Proposition 1.5/1; [1], Proposition 1.15 (the end of the proof has to be modified if $k(s)$ is not perfect for some closed point $s \in S$); and [3], Theorem 10.2.14 (where a technical part, Lemma 10.2.12(c), is left to the reader in Exercise 10.2.5 when $k(s)$ is not perfect). All these proofs consist in extending the group law of E_K to E_{sm}. Admitting the existence of the Néron model N, we present below an alternative approach. Of course, the real hard part in this story is to prove the existence of N (see [2]).

Proof. To see that f is an isomorphism, we can localize S at closed points. Moreover, since the constructions of E, E_{sm}, N all commute with étale base changes, we can suppose that S is local, and that the residue field $k := k(s)$ at the closed point is separably closed.

(1) *There is a birational morphism $N \to E$.* As N is separated, it can be embedded into a proper scheme X over S. Remplacing X by the schematic closure of N in X, we can suppose that X is an integral flat proper curve over S, with smooth generic fiber E_K. Let $\pi : X' \to X$ be its minimal desingularization, then N is isomorphic to $\pi^{-1}(N)$ because it is already regular. We identify N with $\pi^{-1}(N)$. Since E is minimal, there exists a unique birational morphism $g : X' \to E$ extending the identity map $E_K \to E_K$.

(2) *The map g is injective on a 'big' open subset N' of N.* Let $N' = g^{-1}(E_{\mathrm{sm}}) \cap N$. Let $x \in N_k(k)$ be a rational point. Since N is smooth, it lifts to a rational point P in $N_K(K) = E_K(K)$. The

Date: September 30, 2003, second version.

image $g(x)$ in E_k is the specialization of P in E_k. Since E is regular, $g(x) \in E_{\text{sm}}$. Hence $N_k(k) \subseteq N'$. Note that N being smooth and k separably closed, $N_k(k)$, and thus N'_k, is everywhere dense in N_k. The composition of the maps

$$N' \xrightarrow{g} E_{\text{sm}} \xrightarrow{f} N$$

coincides with the open immersion $N' \to N$ by the uniqueness of $N' \to N$ extending the identity map $E_K \to E_K$. In particular, $g|_{N'}$ is injective.

(3) *We have* $N = N'$. Let Γ be an exceptional divisor on X'. Then $\Gamma \cap N' = \emptyset$ because otherwise g maps all (so at least two) points of $\Gamma \cap N'$ to a single point. Since N'_k is everywhere dense in N_k, we have $\Gamma \cap N = \emptyset$. So the contraction of Γ is an isomorphism on a neighborhood of N. By successive contractions of exceptional divisors, we are reduced to the case when X' is minimal, hence isomorphic to E. Thus $N = N'$.

(4) *Final step.* Since $f \circ g|_N : N \to N$ is identity, $g|_N : N \to E_{\text{sm}}$ is a birational closed immersion. Therefore $g|_N$, and hence f, is an isomorphism.

Remark 2 When k is algebraically closed, then the proof above is simpler. Actually, step (2) already shows that $N' = N$, so step (3) can be dropped.

References

[1] M. Artin, *Néron models*, in Arithmetical Geometry, ed. Cornell and Silverman, Springer Verlag (1986), 213-230.

[2] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron Models*, Ergebnisse der Math., **3**. Folge, Bd. 21, Springer 1990.

[3] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Grad. Texts in Math., **6**, Oxford Univ. Press, 2002.