ON THE COHOMOLOGICAL REPRESENTATIONS OF FINITE AUTOMORPHISM GROUPS OF SINGULAR CURVES AND COMPACT COMPLEX SPACES

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Abstract. Let G be a finite group acting tamely on a proper reduced curve C over an algebraically closed field k. We study the G-module structure on the cohomology groups of a G-equivariant locally free sheaf $\mathcal{F}$ on C and give formulas of Chevalley–Weil type, with values in the Grothendieck ring $R_k(G)_{\mathbb{Q}}$ of finitely generated G-modules. We also provide a similar formula for the singular cohomology of compact complex spaces.

Our focus is on the case where C is nodal. Using the Chevalley–Weil formula, we compute the G-invariant part $H^0(C, \omega_C^{\otimes m})^G$ for $m \geq 1$. In turn, we use the formula for $\dim H^0(C, \omega_C^{\otimes 2})^G$ to compute the equivariant deformation space of a stable G-curve C. We also obtain numerical criteria for the presence of any given irreducible representation in $H^0(C, \omega_C \otimes \mathcal{A})$, where $\mathcal{A}$ is an ample locally free G-sheaf on C. Some new phenomena, pathological compared to the smooth curve case, are discussed.

Contents

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1. Introduction

Let X be a projective variety over a field k, let G be a finite group acting on X, and $\mathcal{F}$ a G-equivariant sheaf (also called a G-sheaf) of k-vector spaces. Then G acts on the cohomology groups $H^*(X, \mathcal{F})$, and it is a natural and important problem to determine $H^*(X, \mathcal{F})$ as a G-representation. The formulas of Chevalley–Weil [CW34, Wei35] and Broughton [Bro87] achieve this for the case where X is a complex smooth projective curve and $\mathcal{F}$ is a pluricanonical sheaf $\omega_X^{\otimes m}$ with $n \geq 1$ or the locally constant sheaf $\mathbb{C}_X$ respectively.[1]

The key feature of the Chevalley–Weil formulas in [CW34, Wei35] is that, as G-representations, the cohomology groups $H^0(C, \omega_C^{\otimes m})$ for the pluricanonical sheaves of a complex smooth projective G-curve C can be expressed as a rational multiple of the regular representation $\mathbb{C}[G]$ plus a correction term determined by the inertia locus of the G-action. It is in the same spirit of the holomorphic Lefschetz fixed point theorem, as developed by [AS68] for compact complex manifolds with a finite group action and generalized by [Don69] to the algebraic setting.

This observation was made by Ellingsrud–Lønsted [EL80], who used the idea of equivariant K-theory to set up a fairly general framework for studying the G-module structure on the cohomology groups of G-equivariant sheaves on a projective G-variety X over any algebraically closed base field k. They define the G-equivariant Euler characteristic (called Lefschetz trace in [EL80])

$$\chi_G(\mathcal{E}) = \sum_i (-1)^i [H^i(X, \mathcal{E})]$$

for locally free G-sheaves $\mathcal{E}$, as an element in the Grothendieck ring $R_k(G)$ of G-modules. However, when writing down an explicit formula, one needs to restrict to the case of a smooth projective curve. Later, Kani and Nakajima [Kan86, Nak86b] introduced the notion of ramification module to encode the contribution of the fixed locus. Köck [Koc05] clarified the connection of the Chevalley–Weil formula for locally free G-sheaves on a smooth projective curve to the holomorphic Lefschetz fixed point theorems of [AS68] and [Don69]. There is on-going interest in refining and reproving the Chevalley–Weil formula for smooth curves, or spelling it out more explicitly in specific situations (see, e.g., [Tam51, MV81, Hur92, Bor03, Bor06, Hor12, KK12, KK13, Can18, KFW19, BCK20, CKKK22, Ara22, BW23, Gar23, CL24, LK24]).

Yubo Tong [Ton25] made a first attempt to write down a Chevalley–Weil type formula for the canonical sheaf ω_C of a *nodal* curve C with a faithful tame action of a finite group G. However, his assumption that the quotient curve is smooth is quite restrictive. In fact, if G is cyclic, then the smoothness condition on the quotient curve forces the order $|G|$ to be at most 2. Also, he dealt only with the canonical sheaf ω_C.

In the present work, we treat the general case of locally free G-sheaves $\mathcal{E}$ on a proper reduced singular G-curve C.

Theorem 1.1 (= Theorem 3.10)**.** *Let C be a proper reduced curve over an algebraically closed field k with irreducible components $C_1, \dots, C_n$, and G a finite group*

[1]In retrospect, Broughton's formula for $H^1(X, \mathbb{C})$, where X is a smooth projective curve over $\mathbb{C}$, follows easily from Chevalley–Weil's formula of $H^0(X, \omega_X)$ together with the Hodge decomposition of $H^1(X, \mathbb{C})$.

acting tamely on C (see Definition 2.7). Then for any locally free G-sheaf $\mathcal{E}$ of finite rank r on C, we have the following equality in $R_k(G)_{\mathbb{Q}}$:

$$\chi_G(\mathcal{E}) = \sum_{1\leq i\leq n} \frac{|I_i|}{|G|}\mathrm{Ind}_{I_i}^G \chi_{I_i}(\mathcal{E}|_{C_i}) + \sum_{P\in C}\Gamma_G(\mathcal{E})_P \tag{1.1}$$

where $I_i = \{g \in G \mid g(s) = s, \forall s \in C_i\}$ is the inertia subgroup at C_i, and $\Gamma_G(\mathcal{E})_P$, the virtual ramification module at P is defined in Definitions 3.2 and 3.9.

The key part of the formula (1.1) is to express $\chi_G(\mathcal{E})$ in terms of restrictions $\mathcal{E}|_{C_i}$ and the ramification modules. When the curve C is connected, smooth, and the G-action is faithful, $\Gamma_G(\mathcal{E})_P$ is obtained by twisting the expression of Atiyah–Singer–Köck type

$$-\frac{1}{|G|}\mathrm{Ind}_{G_P}^G \sum_{d=0}^{|G_P|-1} d\theta_P^d \otimes [\mathcal{E}|_P],$$

as discovered in [Koc05], with "the same positive amount of" the regular representation; see Definition 3.2. In general, there are additional contributions from the singularities of C (see Definition 3.9) and from the components of C with nontrivial inertia group.

Let us comment on the difficulties one encountered when dealing with automorphisms of singular curves. One complication coming up in the reducible curve case is that a nontrivial subgroup of $\mathrm{Aut}(C)$ can act trivially on an entire component of C, so the action of a group G can be faithful but not free on a dense open subset of C. In general, the order $|\mathrm{Aut}(C)|$ for a stable curve C of genus g can only be bounded by an exponential function of g ([vOpV07]), in strong contrast to the Hurwitz bound $84(g-1)$ for connected smooth genus g curves ([Hur92]). Recall that a stable curve is a connected nodal curve C, with finite automorphism group $\mathrm{Aut}(C)$, or equivalently, with ample canonical sheaf ω_C.

We were unable to adapt directly the method of [EL80] to the singular case, because the crucial exact sequence (3.1) therein does not hold in the singular case (and in fact neither in higher-dimensional smooth case). Instead, we use the natural idea of comparing the cohomological groups with their pull-back to the normalization $\nu\colon \widetilde{C} \to C$ of C. To fix ideas, take a locally free G-equivariant sheaf $\mathcal{E}$ on C. Then we have

$$0 \to \mathcal{E} \to \nu_*\nu^*\mathcal{E} \to \mathcal{E}|_S \to 0$$

where S is the singular locus of C endowed with the reduced structure. Here we encounter two G-equivariant sheaves $\nu^*\mathcal{E}$ and $\mathcal{E}|_S$ on usually disconnected varieties $\widetilde{C}$ and S respectively. The simple and useful observation is that the cohomology representation of a G-equivariant sheaf $\mathcal{E}$ on a disconnected variety X can be induced from the cohomology representation of $G_{\{X_j\}}$ on $H^*\left(X_j, \mathcal{E}|_{X_j}\right)$, where the X_j's are the connected components of X, and $G_{\{X_j\}} \subseteq G$ is the stabilizer of X_j (see Proposition 2.6).

After writing down the Chevalley–Weil formula, we address the following natural questions for the cohomology of a G-sheaf $\mathcal{F}$ on a G-curve C:

(1) Is there a positive rational number a such that $\chi_G(\mathcal{F}) \geq a[k[G]]$ holds in $R_k(G)_{\mathbb{Q}}$? If G is reductive over k, then this is equivalent to asking whether every irreducible G-module has positive virtual multiplicity in $\chi_G(\mathcal{F})$.
(2) Is $H^*(C, \mathcal{F})$ a faithful representation of G?

(3) What is the G-invariant part $H^*(C,\mathcal{F})^G$?

Of course, the answers depend on $\mathcal{F}$ and C. We focus on the case of nodal G-curves, and the G-sheaves are taken to be related to the canonical sheaf ω_C.

We first compute the G-invariant global sections of the pluricanonical sheaves on a stable pointed G-curve.

Theorem 1.2 (= Corollary 4.14)**.** *Let C be a connected proper nodal curve over an algebraically closed field k, and $G\subset \mathrm{Aut}(C)$ a finite subgroup of automorphisms such that $|G|$ is not divisible by* $\mathrm{char}(k)$. *Let S be the singular locus of C, and T a (possibly empty) finite G-set contained in the smooth locus of C such that $\omega_C(T)$ is ample. Then for any positive integer m, we have*

$$\dim_k H^0(C,\omega_C(T)^{\otimes m})^G = \dim_k H^0(D,\omega_D(\overline{T})^{\otimes m}) + (m-\epsilon_m)\#\overline{S}_2 + \sum_{Q\in D\setminus(\overline{S}\cup\overline{T})}\left\lfloor m\left(1-\frac{1}{e_Q}\right)\right\rfloor$$

where $\pi\colon C\to D=C/G$ is the quotient map, $\overline{S},\overline{T}\subset D$ are images of S,T respectively, $e_Q:=|G_P|$ for $Q\in D\setminus(\overline{S}\cup\overline{T})$ and $P\in\pi^{-1}(Q)$, and $\epsilon_m=1$ if m is odd and 0 if m is even.

In Section 4.4, we apply the obtained formula for $\dim H^0(C,\omega_C^{\otimes 2})^G$ to compute the dimension the G-equivariant deformation space $\mathrm{Def}(C,G)$ of a stable G-curve. This is useful in the study of the singularities of the moduli spaces $\overline{M}_g$ of stable curves. In forthcoming work, we will analyze further the condition for the faithfulness of the action of G on $H^0(C,\omega_C^{\otimes m})$.

If C is nodal and each component of C has trivial inertia group, we obtain numerical criteria for $H^0(C,\omega_C\otimes\mathcal{A})$, where $\mathcal{A}$ is an ample locally free sheaf, to contain a rational multiple of the regular representation (Lemma 4.8 and Corollary 4.12).

It is well known that the dual graph is very useful in describing the structure of a nodal curve C ([ACG11, Chapter X]). We observe that there is an induced action of G on the dual graph $\Gamma:=\Gamma(C)$ of C, viewed as a 1-dimensional CW complex. We use $\chi_G(\Gamma):=[H_0(\Gamma,k)]-[H_1(\Gamma,k)]\in R_k(G)$ to describe the discrepancy between $H^0(C,\omega_C)$ and $H^0(\widetilde{C},\omega_{\widetilde{C}})$, where $\nu\colon\widetilde{C}\to C$ is the normalization (Corollary 4.23):

$$[H^0(C,\omega_C)] = [H^0(\widetilde{C},\omega_{\widetilde{C}})] + [1_G] - \chi_G(\Gamma,k)$$

This can be viewed as the G-equivariant version of the usual relation between the arithmetic genus of C and the genus of $\widetilde{C}$.

We note that Broughton's formula mentioned in the beginning of the introduction may be proved by a straightforward topological argument, and hence can be vastly generalized, as follows.

Theorem 1.3 (= Theorem 5.1)**.** *Let X be a possibly disconnected n-dimensional compact complex space, and $G\subset\mathrm{Aut}(X)$ a finite subgroup of automorphisms. Let $Y=X/G$ be the quotient space and $\pi\colon X\to Y$ the quotient map. For each subgroup $H<G$, denote*

$$X_H := \{x\in X \mid G_x \text{ conjugate to } H\} \text{ and } Y_H:=\pi(X_H).$$

Let $H_1, \dots, H_s$ be a system of representatives of conjugacy classes in G. Then the following equality holds in $R_{\mathbb{C}}(G)_{\mathbb{Q}}$:

$$\sum_{0 \le i \le 2n} (-1)^i [H^i(X, \mathbb{Q})] = \sum_{1 \le j \le s} \chi(Y_{H_j}) \mathrm{Ind}_{H_j}^G [1_{H_j}].$$

where for a complex space Z, $\chi(Z)$ denotes its topological Euler characteristic, and we set $\chi(Y_{H_j}) = 0$ if $Y_{H_j} = \emptyset$.

Specializing Theorem 1.3 to nodal curves, we obtain

Theorem 1.4 (= Corollary 5.3)**.** *Let C be a proper nodal curve over $\mathbb{C}$ and $G \subset \mathrm{Aut}(C)$ a finite subgroup of automorphisms acting freely on a dense open subset of C. Let $\pi \colon C \to D := C/G$ be the quotient map and write $D = \cup_{1 \le j \le m} D_j$ as the union of irreducible components. Let S be the singular locus of C, and set*

$$A := S \cup \{P \in C \mid G_P \text{ is nontrivial}\}$$

Then

$$\sum_{i=0}^{2} (-1)^i [H^i(C, \mathbb{Q})] = \sum_{j=1}^{m} \chi(D_j \setminus \pi(A)) \, [\mathbb{Q}[G]] + \sum_{P \in A'} \mathrm{Ind}_{G_P}^G \, [1_{G_P}] \, .$$

where A' denotes a set of representatives of the G-orbits in A.

Finally, we note that the Chevalley–Weil formula is important for computing of cohomological invariants of the so-called product-quotient varieties $(C_1 \times \cdots \times C_n)/G$, where $C_1, \dots, C_n$ are smooth projective curves and G is a finite group acting diagonally on their product ([CLZ13, Gle16, FGP20]); this class of varieties is very useful for the construction and classification of projective varieties with prescribed invariants. Our Chevalley–Weil type formulas facilitates the computation of cohomological invariants of $(C_1 \times \cdots \times C_n)/G$ when the C_i are nodal curves. These possibly non-normal varieties are in turn useful in the compactification of moduli spaces of varieties of general type; see [vOp06, Liu12] for the surface case.

Notation and Conventions. Let us fix an algebraically closed field k.

- A *variety* in this paper means a reduced scheme of finite type over k, and we only consider the closed points. Thus a variety can be reducible or even disconnected. We will denote by $\mathrm{Aut}(X) := \mathrm{Aut}_k(X)$ the automorphism group of a variety X over k. For a coherent sheaf $\mathcal{F}$ on X and a closed subvariety $Z \subset X$, we denote by $\mathcal{F}|_Z := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ the restriction of $\mathcal{F}$ to Z.
- Let G be a finite group acting on a set X. For any subset Z of X, we denote

  $$G_{\{Z\}} := \{g \in G \mid g(Z) \subseteq Z\}$$

  the *stabilizer or decomposition group* of Z and

  $$G_Z := \{g \in G \mid g(s) = s \text{ for any } s \in Z\}$$

  the *inertia group of Z*. Obviously, we have $G_Z \subseteq G_{\{Z\}}$. For $P \in X$, its orbit $\{g(P) \mid g \in G\}$ is denoted by $G.P$. Conversely, the *inertia locus* or *fixed locus* of (X, G) is the set of $x \in X$ with nontrivial stabilizer G_x.

- Let X be an algebraic variety and G a finite group acting on X by regular automorphisms. Then the inertia locus of (X, G) is a closed subset of X. If G acts freely on a dense open subset of X and if X is quasi-projective, the inertia locus coincides with the ramification locus of the quotient map $X \to X/G$. Note that if G acts trivially on an irreducible component of X, then $X \to X/G$ is étale on this component, while the inertia group of this component is G, hence nontrivial in general.

2. Preliminaries

For the reader's convenience, we recall or give some general facts which will be used later. In particular, we carefully recall facts about the Grothendieck ring of $k[G]$-modules, the Frobenius reciprocity and Brauer character in the representation theory of finite groups, in order to handle the case where the order of the group is divisible by the characteristic of the base field k.

2.1. **Representation theory of finite groups.**

2.1.1. *The Grothendieck ring of finitely generated $k[G]$-modules.* Let G be a finite group and k a field. All $k[G]$-modules in this paper are finitely generated. We write $\mathrm{Irr}_k(G)$ for the set of their irreducible isomorphism classes, 1_G for the trivial representation, and $R_k(G)$ for the Grothendieck group. We use $[V]$ for the class of a $k[G]$-module V in $R_k(G)$. Thus $R_k(G)$ is a free abelian group on the set $\{[V] \mid V \in \mathrm{Irr}_k(G)\}$. We define $R_k(G)_A := R_k(G) \otimes_{\mathbb{Z}} A$ for any commutative ring A with a unit, and use the standard notation Res_H^G and Ind_H^G for restriction and induction. Tensor product makes $R_k(G)$ a ring with unit $[1_G]$. We shall use both Frobenius reciprocity isomorphisms without further comment ([Sch13, Section 2.3]).

For $\alpha \in R_k(G)_{\mathbb{Q}}$, we may write uniquely

$$\alpha = \sum_{V \in \mathrm{Irr}_k(G)} \mu_V(\alpha)[V], \quad \mu_V(\alpha) \in \mathbb{Q}.$$

We call $\mu_V(\alpha)$ the *virtual multiplicity* of V in α. The *virtual degree* of $\alpha \in R_k(G)_{\mathbb{Q}}$ is defined as

$$\deg \alpha := \sum_{V \in \mathrm{Irr}_k(G)} \mu_V(\alpha) \cdot \dim V.$$

We also use the cone

$$R_k(G)_{\mathbb{Q}}^+ := \{\alpha \in R_k(G)_{\mathbb{Q}} \mid \mu_V(\alpha) \geq 0 \text{ for all } V \in \mathrm{Irr}_k(G)\}.$$

It induces a partial order on $R_k(G)_{\mathbb{Q}}$ defined by $\alpha \geq \beta$ if $\alpha - \beta \in R_k(G)_{\mathbb{Q}}^+$.

2.1.2. *Brauer characters.* Let G be a finite group, k an algebraically closed field, and $\mu_\infty(k)$ the group of roots of unity in k. Set

$$G_{\mathrm{reg}} := \begin{cases} G & \text{if } \mathrm{char}(k) = 0 \\ \{g \in G \mid \text{the order of } g \text{ is prime to } p\} & \text{if } \mathrm{char}(k) = p > 0 \end{cases} \tag{2.1}$$

Fix a lift $\varphi\colon \mu_\infty(k) \to \mu_\infty(\mathbb{C})$ (using Teichmüller lifts when $\mathrm{char}(k) > 0$). For a $k[G]$-module V and $g \in G_{\mathrm{reg}}$, with eigenvalues $\lambda_1, \ldots, \lambda_d$, the *Brauer character* is

$$\mathrm{tr}_B(g; V) := \sum_{i=1}^{d} \varphi(\lambda_i) \in \mathbb{C} \tag{2.2}$$

and extends linearly to

$$\phi_\alpha = \mathrm{tr}_B(\cdot;\alpha)\colon G_{\mathrm{reg}} \to \mathbb{C}.$$

where $\alpha \in R_k(G)$. The map $R_k(G)_{\mathbb{C}} \to \mathrm{Cl}(G_{\mathrm{reg}}, \mathbb{C})$, $\alpha \mapsto \phi_\alpha$ is an isomorphism; in particular, a virtual representation is determined by its Brauer character [Ser77, page 149, Theorem 42'].

For a subgroup $H \subset G$, $\alpha \in R_k(H)$, and $g \in G_{\mathrm{reg}}$, we have (cf. [Ser77], Theorem 12 page 30 and Exercise 18.2, page 150]):

$$\phi_{\mathrm{Ind}_H^G \alpha}(g) = \begin{cases} \dfrac{1}{|H|} \displaystyle\sum_{\substack{t \in G \\ t^{-1}gt \in H}} \phi_\alpha(t^{-1}gt) & \text{if } g \in \displaystyle\bigcup_{t\in G} tHt^{-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

In particular, taking $H = \{1\}$, we see that for any $g \in G$,

$$\mathrm{tr}_B(g; k[G]) = \begin{cases} |G| & \text{if } \ g = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

One use the Hermitian form on $\mathrm{Cl}(G_{\mathrm{reg}}, \mathbb{C})$

$$\langle \phi_1, \phi_2 \rangle := \frac{1}{|G|} \sum_{g \in G_{\mathrm{reg}}} \phi_1(g^{-1})\phi_2(g). \tag{2.5}$$

and write $\langle \alpha, \beta \rangle$ for the corresponding pairing of Brauer character.

Let $H \subset G$ be a subgroup. For $\alpha \in R_k(H)$ and $\beta \in R_k(G)$, one has, by the Frobenius reciprocity,

$$\langle \mathrm{Ind}_H^G \alpha, \beta \rangle = \langle \alpha, \mathrm{Res}_H^G \beta \rangle. \tag{2.6}$$

This remains valid when $\mathrm{char}\, k$ divides $|G|$. In fact, since the classes of projective $k[H]$-modules generate $R_k(H)_{\mathbb{C}}$ ([Ser77, §16.1, Cor. 1 of Theorem 35]), we may assume that $\alpha = [V]$ and $\beta = [W]$, where V is a projective $k[H]$-module and W is any $k[G]$-module. Then, $\mathrm{Ind}_H^G V$ is a projective $k[G]$-module, and by the Frobenius reciprocity ([Sch13, Section 2.3]), we have

$$\langle \mathrm{Ind}_H^G \alpha, \beta \rangle = \dim_k \mathrm{Hom}_G(\mathrm{Ind}_H^G V, W) = \dim_k \mathrm{Hom}_H(V, \mathrm{Res}_H^G W) = \langle \alpha, \mathrm{Res}_H^G \beta \rangle.$$

Suppose that $|G|$ is invertible in k. Then every $k[G]$-module is projective, and the classes of the irreducible $k[G]$-modules form an orthonormal basis for $R_k(G)_{\mathbb{Q}}$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$ (cf. [Ser77, page 121] or [Web16, Theorem 10.2.2]). In this case, $\mu_V(\alpha) = \langle [V], \alpha \rangle$ for $V \in \mathrm{Irr}_k(G)$ and $\alpha \in R_k(G)_{\mathbb{Q}}$.

This is not true if p divides $|G|$, as the following example shows.

Example 2.1. Suppose that $\mathrm{char}(k) = p > 0$ and $G = \mathbb{Z}/p\mathbb{Z}$. Then $R_k(G) = \mathbb{Z} \cdot [1_G]$, and $\langle [1_G], [1_G] \rangle = \frac{1}{p} \in \mathbb{Q}$.

Let us finish this recap of representation theory by giving some elementary results on induction and restriction of representations.

Lemma 2.2. *Let $H \to \overline{H} = H/N$ be a surjective group homomorphism.*

(1) Let $\overline{K}$ be a subgroup of $\overline{H}$ and let K be its pre-image in H:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, two heads] & H \arrow[d, two heads] \\ \overline{K} \arrow[r, hook] & \overline{H} \end{tikzcd}$$

Then for any $\overline{K}$-module M we have a canonical isomorphism of H-modules

$$\mathrm{Res}_H^{\overline{H}}(\mathrm{Ind}_{\overline{K}}^{\overline{H}} M) \simeq \mathrm{Ind}_K^H(\mathrm{Res}_K^{\overline{K}} M).$$

(2) We have $\mathrm{Res}_H^{\overline{H}} k[\overline{H}] \simeq \mathrm{Ind}_N^H 1_N$ where 1_N corresponds to the unit representation of N.

(3) Let M be a $k[H]$-module. Then we have

$$\mathrm{Ind}_N^H \mathrm{Res}_N^H(M) \simeq k[H/N] \otimes_k M.$$

Proof. (1) By definition,

$$\mathrm{Ind}_K^H \mathrm{Res}_K^{\overline{K}} M = k[H] \otimes_{k[K]} (k[\overline{K}] \otimes_{k[\overline{K}]} M) = (k[H] \otimes_{k[K]} k[\overline{K}]) \otimes_{k[\overline{K}]} M$$

and

$$\mathrm{Res}_H^{\overline{H}} \mathrm{Ind}_{\overline{K}}^{\overline{H}} M = k[\overline{H}] \otimes_{k[\overline{K}]} M.$$

So it is enough to show that the canonical homomorphism

$$k[H] \otimes_{k[K]} k[\overline{K}] \to k[\overline{H}]$$

is an isomorphism. This can be checked with a basis of $k[H]$ over $k[K]$ and by using the canonical isomorphism $H/K \to \overline{H}/\overline{K}$.

(2)-(3) See [Ser77, §3.3, Example 1, (2) and (5)]. □

Lemma 2.3. *Let G be any finite group. Let M be a $k[G]$-module with $\dim_k M = r$. Then*

$$M \otimes_k k[G] \simeq k[G]^{\oplus r}.$$

Proof. Let M^0 be the k-vector space M endowed with the trivial action of G. Define

$$\phi : M \otimes_k k[G] \to M^0 \otimes_k k[G] \simeq k[G]^{\oplus r}, \quad x \otimes g \mapsto (g^{-1}x) \otimes g.$$

Then ϕ is a G-equivariant isomorphism. □

2.2. **The G-equivariant Euler characteristic and reduction to normal connected varieties.** Let X be a (possibly reducible) proper variety over k. Let G be a finite group acting on X, and $\mathcal{F}$ a G-equivariant coherent sheaf on X (see [Stk25, Tag 03LE] for the definition). Following [EL80], we define the *G-equivariant Euler characteristic* of $\mathcal{F}$ to be the following element

$$\chi_G(\mathcal{F}) := \sum_{q \geq 0} (-1)^q [H^q(X, \mathcal{F})] \in R_k(G) \tag{2.7}$$

in the Grothendieck group of $k[G]$-modules.

Remark 2.4. The correspondence $\mathcal{F} \mapsto \chi_G(\mathcal{F})$ induces a group homomorphism from the Grothendieck group $K_G(X)$ of G-equivariant coherent sheaves on X to $R_k(G)$.

Remark 2.5. If the kernel H of the action homomorphism $G \to \mathrm{Aut}(X)$ acts trivially on the cohomology groups $H^*(X, \mathcal{F}) = \oplus_i H^i(X, \mathcal{F})$, which is the case for sheaves such as $\mathcal{O}_X$ and $\mathbb{C}_X$ that are intrinsic to X, then we may take the image $\overline{G}$ of $G \to \mathrm{Aut}(X)$, and it is clear that

$$\chi_G(\mathcal{F}) = \mathrm{Res}_G^{\overline{G}} \left(\chi_{\overline{G}}(\mathcal{F})\right). \tag{2.8}$$

Thus we may assume that G is a finite subgroup of $\mathrm{Aut}(X)$ and the action of G on X is faithful. This we will do in Sections 4 and 5.

In general, H may act nontrivially on $H^*(X, \mathcal{F})$ and it is more natural to include non-faithful actions of a group G on X. Thus we do not assume the faithfulness of the action in Sections 2 and 3.2, unless otherwise specified. Note that statements of previous works, including [EL80] and [Koc05], assume the action to be faithful.

Next we want to relate $\chi_G(\mathcal{F})$ to the Euler characteristics of the pull-backs of $\mathcal{F}$ to the connected components of the normalization of X. We first quote the following well-known result.

Proposition 2.6. *Let $X_1, \dots, X_n$ be the connected components of X and G_i be the stabilizer of X_i for all $i \le n$. Let $\mathcal{F}$ be a G-equivariant coherent sheaf on X. Then we have*

$$[H^q(X, \mathcal{F})] = \sum_{1 \le i \le n} \frac{|G_i|}{|G|} \mathrm{Ind}_{G_i}^G [H^q(X_i, \mathcal{F}|_{X_i})] \in R_k(G)_{\mathbb{Q}}$$

for all $q \ge 0$, and

$$\chi_G(\mathcal{F}) = \sum_{1 \le i \le n} \frac{|G_i|}{|G|} \mathrm{Ind}_{G_i}^G \chi_{G_i}(\mathcal{F}|_{X_i}).$$

Let $\nu : \widetilde{X} \to X$ be the normalization morphism. Let $\mathcal{E}$ be a G-equivariant coherent locally free sheaf on X. Consider the canonical exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{\widetilde{X}} \to \mathcal{S} \to 0 \tag{2.9}$$

where $\mathrm{Supp}\, \mathcal{S}$ is exactly the non-normal locus of X. Note that the action of G on X lifts to $\widetilde{X}$, so that the exact sequence (2.9) is G-equivariant. Tensoring (2.9) with $\mathcal{E}$, we obtain a G-equivariant exact sequence

$$0 \to \mathcal{E} \to \mathcal{E} \otimes \nu_* \mathcal{O}_{\widetilde{X}} = \nu_*(\nu^* \mathcal{E}) \to \mathcal{E} \otimes \mathcal{S} \to 0. \tag{2.10}$$

By the additivity of χ_G, we obtain from (2.10)

$$\chi_G(\mathcal{E}) = \chi_G(\nu^* \mathcal{E}) - \chi_G(\mathcal{E} \otimes \mathcal{S}). \tag{2.11}$$

In other words, we can compute $\chi_G(\mathcal{E})$ in terms of $\chi_G(\nu^* \mathcal{E})$ and $\chi_G(\mathcal{E} \otimes \mathcal{S})$, which live on the normalization and the non-normal locus of X respectively. We will specialize further to the case when X is a nodal curve, so that $\chi_G(\nu^* \mathcal{E})$ and $\chi_G(\mathcal{E} \otimes \mathcal{S})$ are either known or possible to write down explicitly.

We end this section by giving a definition of the tameness of a G-action. This condition is automatically satisfied when k has characteristic 0, and it is a necessary requirement in almost all situations of the present work when k has positive characteristic.

Definition 2.7 (Tameness)**.** Let C be a proper reduced curve over k, and G a finite group acting on C. We say that the action of G on C is *tame* if the characteristic of k does not divide the order of the inertia group G_Z for any irreducible subvariety $Z \subset C$. If G is a subgroup of $\mathrm{Aut}(C)$, acting tamely on C, then we will say that G is a *tame subgroup of* $\mathrm{Aut}(C)$.

3. The Chevalley–Weil formula for proper reduced curves

Set-up 3.1. Throughout this section, we adopt the following notation.

- C is a proper reduced curve over k, possibly disconnected;
- $C = \bigcup_{1\le i\le n} C_i$ is the irreducible decomposition of C;
- $\nu : \widetilde{C} = \bigsqcup_{1\le i\le n} \widetilde{C}_i \to C$ is the normalization map;
- G is a finite group acting tamely on C (Definition 2.7);
- $\pi\colon C \to D := C/G$ is the quotient morphism. Since the normalization $\nu\colon \widetilde{C} \to C$ is G-equivariant, we have an induced morphism $\bar{\nu}\colon \widetilde{C}/G \to C/G = D$, which is an isomorphism over a dense open subset of D and which fits into the following commutative diagram:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\nu"] \arrow[d, "\tilde{\pi}"'] & C \arrow[d, "\pi"] \\ \widetilde{D} = \widetilde{C}/G \arrow[r, "\bar{\nu}"] & D = C/G \end{tikzcd} \tag{3.1}$$

  where $\tilde{\pi}$ denotes the quotient map. By the universal property of normalization and by the smoothness of $\widetilde{C}/G$, $\bar{\nu}$ is the normalization of D.
- For each $1 \le i \le n$ (see the end of §1),

$$G_i := G_{\{\widetilde{C}_i\}} = G_{\{C_i\}}, \quad I_i := G_{\widetilde{C}_i} = G_{C_i}, \quad \overline{G}_i := G_i/I_i. \tag{3.2}$$

  Note that I_i is a normal subgroup of G_i, so $\overline{G}_i$ is a group.
- For any smooth point $P \in C$, denote by G_P° the group I_i for the unique irreducible component C_i of C passing through P and by $\overline{G}_P = G_P/G_P^\circ$ the quotient group.
- $\mathcal{E}$ is a locally free G-sheaf of rank r on C.

If the action of G on C is free, then [EL80, Theorem 2.4] already gives

$$\chi_G(\mathcal{E}) = \frac{1}{|G|}\chi(\mathcal{E})[k[G]] \in R_k(G)_{\mathbb{Q}}. \tag{3.3}$$

In this section, we will give a formula expressing $\chi_G(\mathcal{E})$ in terms of a "regular part" which looks like $\frac{1}{|G|}\chi(\mathcal{E})[k[G]]$, plus a "ramification part" $\sum_{Z\subseteq C}\Gamma_G(\mathcal{E})_Z$ coming from the inertia loci Z of the G-action; see Theorem 3.10.

3.1. **The Chevalley–Weil formula for smooth curves.**

Let notation be as in Set-up 3.1, so we have a locally free G-sheaf $\mathcal{E}$ of rank r on a curve C endowed with a tame action of G. In this subsection, we assume that the curve C is smooth.

For a point $P \in C$, the Zariski cotangent space T_P^*C of C at P is a 1-dimensional $\overline{G}_P$-representation. We denote by θ_P the class of T_P^*C in $R_k(G_P)$ (it is denoted by χ_P in [Koc05]). It is helpful to notice that we have canonically

$$\theta_P = [\omega_{C,P}|_P] = [\mathcal{O}_C(-P)|_P]. \tag{3.4}$$

Definition 3.2 (Virtual ramification modules on a smooth G-curve)**.** We define the *virtual ramification module* of $\mathcal{E}$ at $P \in C_i$ to be

$$\Gamma_G(\mathcal{E})_P := \frac{|I_i|}{|G|}\mathrm{Ind}_{G_P}^G(\Psi_P \otimes [\mathcal{E}|_P]) \tag{3.5}$$

where

$$\Psi_P := \frac{|\overline{G}_P|-1}{2}[k[\overline{G}_P]] - \sum_{d=0}^{|\overline{G}_P|-1} d\theta_P^d.$$

This is an element of virtual degree 0 in $R_k(G)_{\mathbb{Q}}$. Note that $\Gamma_G(\mathcal{E})_P = 0$ if $\overline{G}_P = \{1\}$ and this happens for all but finitely many P's.

The ramification modules are so defined as to reflect the local contributions of fixed locus of the G-action given by the Atiyah–Singer's holomorphic Lefschetz fixed point formula ([AS68, Don69]).

Lemma 3.3. *Let C be smooth and connected. Let $\mathcal{E}$ be a G-equivariant vector bundle on C. Let $I = G_C$ be the kernel of $G \to \mathrm{Aut}(C)$ (the action of I on $\mathcal{E}$ may be non-trivial).*

(1) *Let $g \in I$ (possibly equal to the identity). Then*
   (a) $\mathrm{tr}_B(g; \frac{|I|}{|G|}\mathrm{Ind}_I^G \chi_I(\mathcal{E})) = \mathrm{tr}_B(g; \chi_I(\mathcal{E}))$*, and*
   (b) $\mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) = 0$ *for any* $P \in C$*.*

(2) *Let $g \in G \setminus I$. Then*
   (a) $\mathrm{tr}_B(g; \mathrm{Ind}_I^G \chi_I(\mathcal{E})) = 0$*, and*
   (b) $\displaystyle\sum_{P\in C} \mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) = \sum_{Q \in C^g} \frac{\mathrm{tr}_B(g; [\mathcal{E}|_Q])}{1 - \mathrm{tr}_B(g; \theta_Q)}.$

Proof. (1.a) As $t^{-1}gt \in I$ for all t, we have

$$\begin{aligned}\mathrm{tr}_B\left(g; \frac{|I|}{|G|}\mathrm{Ind}_I^G \chi_I(\mathcal{E})\right) &= \frac{|I|}{|G|}\frac{1}{|I|}\sum_{t\in G}\mathrm{tr}_B(tgt^{-1}; \chi_I(\mathcal{E})) \\ &= \frac{|I|}{|G|}\frac{1}{|I|}|G|\mathrm{tr}_B(g; \chi_I(\mathcal{E})) = \mathrm{tr}_B(g; \chi_I(\mathcal{E})).\end{aligned}$$

(1.b) Let $P \in C$. As $tgt^{-1} \in I \subseteq G_p$ for all $t \in G$, we have

$$\begin{aligned}\mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) =& \frac{|I|}{|G|}\frac{1}{|G_P|}\sum_{t\in G}\mathrm{tr}_B(tgt^{-1}; \Psi_P)\mathrm{tr}_B(tgt^{-1}; [\mathcal{E}|_P]) \\ =& \frac{|I|}{|G|}\frac{1}{|G_P|}|G|\mathrm{tr}_B(g; \Psi_P)\mathrm{tr}_B(g; [\mathcal{E}|_P]).\end{aligned}$$

As Ψ_P has degree 0 and g acts trivially on C, we have $\mathrm{tr}_B(g; \Psi_P) = \mathrm{tr}_B(1; \Psi_P) = 0$. Hence $\mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) = 0$.

(2.a) There is no $t \in G$ such that $t^{-1}gt \in I$, because $g \notin I$, so $\mathrm{tr}_B(g; \mathrm{Ind}_I^G \chi_I(\mathcal{E})) = 0$.

(2.b) Write $e_P = |\overline{G}_P| - 1$ and $\Psi_P = 2^{-1}(e_P - 1)[k[\overline{G}_P]] - \Lambda_P$ with

$$\Lambda_P = \sum_{d=0}^{e_P-1} d\theta_P^d.$$

Let $H_P = \{t \in G \mid t^{-1}gt \in G_P\} = \{t \in G \mid g \in G_{tP}\}$. For all $t \in H_P$, it follows from Lemma 2.3 and Equality (2.4) that

$$\mathrm{tr}_B(t^{-1}gt; \Psi_P \otimes [\mathcal{E}|_P]) = \mathrm{tr}_B(t^{-1}gt; -\Lambda_P \otimes [\mathcal{E}|_P]),$$

and it is clear that $\mathrm{tr}_B(t^{-1}gt; -\Lambda_P \otimes [\mathcal{E}|_P]) = \mathrm{tr}_B(g; -\Lambda_{tP} \otimes [\mathcal{E}|_{tP}])$. By definition, if we denote $a = |I||G|^{-1}$, then

$$\begin{aligned}\mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) =& a\frac{1}{|G_P|}\sum_{t\in H_P} \mathrm{tr}_B(t^{-1}gt; \Psi_P \otimes [\mathcal{E}|_P]) \\ =& - a\frac{1}{|G_P|}\sum_{t\in H_P} \mathrm{tr}_B(g; \Lambda_{tP} \otimes [\mathcal{E}|_{tP}]).\end{aligned}$$

Let C^g be the locus of fixed points of C under g, then using the surjective map $H_P \to GP \cap C^g$, $t \mapsto tP$, we get

$$\mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) = -a \sum_{Q\in GP\cap C^g} \mathrm{tr}_B(g; \Lambda_Q \otimes [\mathcal{E}|_Q]).$$

Therefore

$$\begin{aligned}\sum_{P\in C} \mathrm{tr}_B(g; \Gamma_G(\mathcal{E})_P) =& - a\sum_{P\in C}\sum_{Q\in GP\cap C^g} \mathrm{tr}_B(g; \Lambda_Q \otimes [\mathcal{E}|_Q]) \\ =& - a\sum_{Q\in C^g}\sum_{P\in GQ} \mathrm{tr}_B(g; \Lambda_Q \otimes [\mathcal{E}|_Q]) \\ =& - a\frac{|G|}{|G_Q|}\sum_{Q\in C^g} \mathrm{tr}_B(g; \Lambda_Q \otimes [\mathcal{E}|_Q]) \\ =& - \frac{1}{e_Q}\sum_{Q\in C^g} \mathrm{tr}_B(g; \Lambda_Q \otimes [\mathcal{E}|_Q]).\end{aligned}$$

As $\mathrm{tr}_B(g; \theta_Q) \in \mu'_\infty(\mathbb{C}) \setminus \{1\}$ (because the image of g in $\mathrm{Aut}(\mathcal{O}_{C,Q})$ is non trivial), we have by [Koc05, Lemma 1.2]

$$\mathrm{tr}_B(g; \Lambda_Q) = \frac{e_Q}{\mathrm{tr}_B(g; \theta_Q) - 1}.$$

This implies the desired equality. □

The following statement is a generalization of [Koc05, Theorem 1] to the case of possibly non-faithful action of a finite group on a possibly disconnected smooth curve. The proof is similar to that of [Koc05], using the algebraic version of the Lefschetz fixed point theorem given by [Don69].

Theorem 3.4. *Let $C = \cup_{i=1}^n C_i$ be a (not necessarily connected) smooth projective curve over k, and G a finite group acting tamely on C. Let $\mathcal{E}$ be a G-equivariant vector bundle on C. Then we have the following equality in $R_k(G)_\mathbb{Q}$:*

$$\chi_G(\mathcal{E}) = \sum_{1\le i\le n} \frac{|I_i|}{|G|}\mathrm{Ind}_{I_i}^G \chi_{I_i}(\mathcal{E}|_{C_i}) + \sum_{P\in C} \Gamma_G(\mathcal{E})_P. \tag{3.6}$$

where $I_i = G_{C_i}$ is the inertia group for each $1 \le i \le n$.

Proof. The proof is immediately reduced to the case where C is connected by using Proposition 2.6. Denote by $I = G_C$ the inertia subgroup of G acting on C. We have to show that

$$\chi_G(\mathcal{E}) = \frac{|I|}{|G|}\mathrm{Ind}_I^G \chi_I(\mathcal{E}) + \sum_{P\in C} \Gamma_G(\mathcal{E})_P. \tag{3.7}$$

By [Ser77, page 149, Corollary 1] or the results recalled in Section 2.1.2, it suffices to show that the Brauer trace of any $g \in G_{\mathrm{reg}}$ on both sides of (3.7) are equal (see (2.2) for the definition of the Brauer trace).

By the Lefschetz fixed point theorem ([AS68, (4.6)] for $k = \mathbb{C}$ and [Don69, Corollary 5.5] for any algebraically closed field k), we have

(3.8)
$$\mathrm{tr}_B(g;\chi_G(\mathcal{E})) = \sum_{P\in Z_0(C^g)} \frac{\mathrm{tr}_B(g;[\mathcal{E}|_P])}{1-\mathrm{tr}_B(g;\theta_P)} + \sum_{E\in \mathrm{Z}_1(C^g)}\sum_{\lambda\in\langle g\rangle^*} \mathrm{tr}_B(g;\lambda)\int_E \mathrm{ch}(\mathcal{E}^\lambda)\mathrm{td}(E)$$

where the notation in the above formula is as follows:

- $Z_0(C^g)$ is the set of isolated points of the fixed locus C^g of g;
- $\mathrm{Z}_1(C^g)$ is the set of the 1-dimensional components of C^g;
- For each character $\lambda \in \langle g\rangle^*$, $\mathcal{E}^\lambda$ is the vector sub-bundle of $\mathcal{E}$ on which g acts via the multiplication by $\lambda(g) \in k^*$;
- $\mathrm{td}(E) \in \mathrm{CH}^*(E)$ denotes the Todd class of the curve E.

As C is irreducible, Equality (3.8) becomes

$$\mathrm{tr}_B(g;\chi_G(\mathcal{E})) = \begin{cases} \sum_{P\in C^g} \frac{\mathrm{tr}_B(g;[\mathcal{E}|_P])}{1-\mathrm{tr}_B(g;\theta_P)} & \text{if } g\notin I \\ \sum_{\lambda\in\langle g\rangle^*} \mathrm{tr}_B(g;\lambda)\int_C \mathrm{ch}(\mathcal{E}^\lambda)\mathrm{td}(C) & \text{if } g\in I. \end{cases} \tag{3.9}$$

If $g \notin I$, the desired equality on $\mathrm{tr}_B(g;-)$ follows from Equality (3.9) above Lemma 3.3(2).

Finally consider the case $g \in I\backslash\{1\}$. For all $\lambda \in \langle g\rangle^*$, Hirzebruch–Riemann–Roch theorem says that

$$\mathrm{tr}_B\left(g;\chi_I(\mathcal{E}^\lambda)\right) = \mathrm{tr}_B(g;\lambda)\int_C \mathrm{ch}(\mathcal{E}^\lambda)\mathrm{td}(C).$$

As $\mathcal{E} = \oplus_\lambda \mathcal{E}^\lambda$, we have

$$\mathrm{tr}_B\left(g;\chi_I(\mathcal{E})\right) = \sum_{\lambda\in\langle g\rangle^*} \mathrm{tr}_B(g;\lambda)\int_C \mathrm{ch}(\mathcal{E}^\lambda)\mathrm{td}(C).$$

Then the desired equality on $\mathrm{tr}_B(g;-)$ follows from Lemma 3.3(1). □

Remark 3.5. For comparison, we show how, if C is connected and the action of G on C is faithful and tame, Theorem 3.4 recovers [Koc05, Theorem 1]. In this case, $I = \{0\}$ and $\chi_I(\mathcal{E}) = \chi(\mathcal{E})[k[G]]$. By Riemann–Hurwitz's theorem we have $|G|\chi(\mathcal{O}_D) = \chi(\mathcal{O}_C) + \frac{1}{2}\deg R$, where $R = \sum_{P\in C}(|G_P|-1)[P]$ is the ramification divisor of the quotient map $\pi\colon C \to D$. As $\chi(\mathcal{E}) = r\chi(\mathcal{O}_C) + \deg\mathcal{E}$, where $r = \mathrm{rk}\,\mathcal{E}$, by Theorem 3.4, $\chi_G(\mathcal{E})$ is equal to

$$\frac{1}{|G|}\chi(\mathcal{E})[k[G]] + \sum_{P\in C}\frac{1}{|G|}\mathrm{Ind}_{G_P}^{G}\left(r\frac{|G_P|-1}{2}[k[G_P]] - \sum_{d=0}^{|G_P|-1} d\theta_P^d \otimes [\mathcal{E}|_P]\right)$$
$$= \left(r\chi(\mathcal{O}_D) + \frac{1}{|G|}\deg\mathcal{E}\right)[k[G]] - \frac{1}{|G|}\sum_{P\in C}\mathrm{Ind}_{G_P}^{G}\left(\sum_{d=0}^{|G_P|-1} d\theta_P^d \otimes [\mathcal{E}|_P]\right).$$

which is [Koc05, Theorem 1].

The next corollary is a generalization of the equality (3.3).

Corollary 3.6. *Keep the hypothesis of Theorem 3.4. Suppose further that G acts freely on an open dense subset of C. Then*

$$\chi_G(\mathcal{E}) = \frac{1}{|G|}\chi(\mathcal{E})[k[G]] + \sum_{P\in C}\Gamma_G(\mathcal{E})_P.$$

Proof. We have $I_i = \{1\}$, $\chi_{I_i}(\mathcal{E}|_{C_i}) = \chi(\mathcal{E}|_{C_i})[1_k]$ and $\sum_i \chi(\mathcal{E}|_{C_i}) = \chi(\mathcal{E})$. □

Example 3.7. Let G be a finite group acting faithfully on a quasi-projective variety X over k. When X is connected, the quotient morphism $X \to X/G$ is étale if and only if G acts freely on X. But this is no longer true when X is not connected. Below we construct an example with $X \to X/G$ étale, X/G smooth and connected, but all points of X have nontrivial stabilizer. Let G be a finite group having a nontrivial subgroup I such that $\cap_{g\in G} gIg^{-1} = \{1\}$ (e.g. $G = S_3$ and I any subgroup of order 2). Let $Y_0 = \mathrm{Spec}(k) = \{y_0\}$ and let

$$X = \mathrm{Ind}_I^G Y_0 := \{(y_0, a) \in Y_0 \times (G/I)\}$$

endowed with the action of G defined by $g * (y_0, a) = (y_0, g * a)$, where $g * a$ is the natural action (on the left) of G on the left cosets G/I. This action is faithful by the hypothesis on I. The quotient morphism is $X \to Y_0$ is étale because it is an isomorphism when restricted to any connected component $X_a := Y_0 \times \{a\}$ of X. Finally if $a = gI$, then the inertia and decomposition groups of X_a are $I_a := gIg^{-1} \neq \{1\}$. Note that for any algebraic variety Y, taking $Y \times X$ with trivial action of G on Y gives similar examples in higher dimension.

For any G-equivariant coherent sheaf $\mathcal{E}$ on X such that I_a acts trivially on $\mathcal{E}|_{X_a}$ for $a \in G/I$, we have

$$\chi_G(\mathcal{E}) = \frac{1}{|G|}\sum_{a\in G/H}\chi(\mathcal{E}|_{X_a})|I_a|\mathrm{Ind}_{I_a}^{G}1_{I_a} = \frac{1}{|G|}\chi(\mathcal{E})(|I|\mathrm{Ind}_I^G[1_I]).$$

But $|I|\mathrm{Ind}_H^G 1_I \neq [k[G]]$ because their characters are different. Therefore, the isomorphism (3.3) is false as soon as $\chi(\mathcal{E}) \neq 0$. In particular, [EL80, Theoreom 2.4] as stated is incorrect; it requires the freeness of the G-action, in order to be valid.

We compute the virtual multiplicity of $[1_G]$ in the G-equivariant Euler characteristic of a pluri-log-canonical sheaf on a log smooth G-curve. In Theorem 4.13, we will use the following proposition.

Proposition 3.8. *Let C and G be as in Theorem 3.4. Let T be a G-invariant finite subset of C, also viewed as a reduced divisor on C. Let $\overline{T} = \pi(T)$, where $\pi\colon C \to D = C/G$ is the quotient map. Then, for any integer m, we have*

$$\langle [1_G], \chi_G(\omega_C(T)^{\otimes m})\rangle = \chi(\omega_D(\overline{T})^{\otimes m}) + \sum_{Q\in D\setminus\overline{T}} \left\lfloor m\left(1-\frac{1}{e_Q}\right)\right\rfloor \tag{3.10}$$

where for $Q \in D\setminus \overline{T}$, $e_Q := e_P = |\overline{G}_P|$ is the ramification index of $C \to D$ at any $P \in \pi^{-1}(Q)$.

Proof. Keep the notation of Set-up 3.1. Denote $\mathcal{E} = (\omega_C(T))^{\otimes m}$. Since I_i acts trivially on $\mathcal{E}|_{C_i}$, we have $\chi_{I_i}(\mathcal{E}|_{C_i}) = \chi(\mathcal{E}|_{C_i})[1_{I_i}]$. By Theorem 3.4, we have

$$\chi_G(\mathcal{E}) = \sum_{1\le i\le n} \frac{|I_i|}{|G|}\operatorname{Ind}_{I_i}^G \chi(\mathcal{E}|_{C_i})[1_{I_i}] + \sum_{P\in C}\Gamma_G(\mathcal{E})_P$$

Let $T_i = T\cap C_i$. Then for $P\in C_i\setminus T_i$, we have

$$[\mathcal{E}|_P] = \left[\omega_C^{\otimes m}|_P\right] = \theta_P^{m_P}$$

where $1\le m_P\le e_P$ satisfies $m_P\equiv m \mod e_P$, and hence

$$\Gamma_G(\mathcal{E})_P = \Gamma_G(\omega_C^{\otimes m})_P = \frac{|I_i|}{|G|}\operatorname{Ind}_{G_P}^G\left(\frac{e_P-1}{2}[k[\overline{G}_P]] - \sum_{d=0}^{e_P-1} d\theta_P^{d+m_P}\right).$$

For any $P\in T_i$, we have (see (3.4))

$$[\mathcal{E}|_P] = \left[\omega_C(T)^{\otimes m}|_P\right] = [1_{G_P}]$$

and hence

$$\Gamma_G(\mathcal{E})_P = \frac{|I_i|}{|G|}\operatorname{Ind}_{G_P}^G\left(\frac{e_P-1}{2}[k[\overline{G}_P]] - \sum_{d=0}^{e_P-1} d\theta_P^{d}\right).$$

Thus, using the Frobenius reciprocity (2.6), the virtual multiplicity of $[1_G]$ in $\chi_G(\mathcal{E})$ is

$$\begin{aligned}\langle [1_G], \chi_G(\mathcal{E})\rangle &= \sum_{1\le i\le n}\frac{|I_i|}{|G|}\chi(\mathcal{E}|_{C_i})\langle [1_{I_i}],[1_{I_i}]\rangle + \sum_{P\in C}\langle [1_G], \Gamma_G(\mathcal{E})_P\rangle \\ &= \sum_{1\le i\le n}\frac{|I_i|}{|G|}\left(\chi(\mathcal{E}|_{C_i}) + \sum_{P\in C_i}\frac{e_P-1}{2} - \sum_{P\in C_i\setminus T_i}(e_P-m_P)\right).\end{aligned}\tag{3.11}$$

By the Riemann–Roch theorem, we have

$$\chi(\mathcal{E}|_{C_i}) = \chi(\mathcal{O}_{C_i}) + \deg \mathcal{E}|_{C_i} \tag{3.12}$$

By the Riemann–Hurwitz formula, we have

$$\chi(\mathcal{O}_{C_i}) = \frac{|G_i|}{|I_i|}\chi(\mathcal{O}_{\pi(C_i)}) - \frac{1}{2}\sum_{P\in C_i}(e_P-1). \tag{3.13}$$

Note that $\mathcal{E}|_{C_i} = \pi^*(\omega_{\pi(C_i)}(\overline{T}_i)^{\otimes m})\otimes \mathcal{O}_{C_i}(\sum_{P\in C_i\setminus T_i}(e_P-1)P)$, where $\overline{T}_i = \pi(T_i)$, so

$$\deg \mathcal{E}|_{C_i} = \frac{|G_i|}{|I_i|}\deg \omega_{\pi(C_i)}(\overline{T}_i)^{\otimes m} + \sum_{P\in C_i\setminus T_i}(e_P-1). \tag{3.14}$$

By construction $e_P \mid m - m_P$. It is then easy to see that

$$m(e_P-1)-(e_P-m_P) = e_P\lfloor m(1-1/e_P)\rfloor. \tag{3.15}$$

Assembling (3.12)–(3.15), we get

$$\chi(\mathcal{E}|_{C_i}) + \sum_{P\in C_i}\frac{e_P-1}{2} - \sum_{P\in C_i\setminus T_i}(e_P-m_P) = \frac{|G_i|}{|I_i|}\chi(\omega_{\pi(C_i)}(\overline{T}_i)^{\otimes m}) + \sum_{P\in C_i\setminus T_i} e_P\lfloor m(1-1/e_P)\rfloor.$$

Now notice that there are $|G|/|G_i|$ components of C lying over $\pi(C_i)$, and $|G_i/I_i|/e_P$ points of C_i lying over $\pi(P)$. We then get the desired equality by using (3.11). $\square$

3.2. **The Chevalley–Weil formula for singular curves.** Let the notation be as in Set-up 3.1. Let $\mathcal{S}$ be the sheaf on C given by the exact sequence (2.9) with $X=C$.

Definition 3.9 (Ramification modules on a singular G-curve)**.** For each $P\in C$, define

$$\Gamma_G(\mathcal{E})_P = \sum_{\widetilde{P}\in\nu^{-1}(P)}\Gamma_G(\nu^*\mathcal{E})_{\widetilde{P}} + \frac{1}{|G|}\mathrm{Ind}_{G_P}^G\left((\dim_k\mathcal{S}_P)[k[G_P]] - |G_P|[\mathcal{S}_P]) \otimes [\mathcal{E}|_P]\right). \tag{3.16}$$

We note that $\Gamma_G(\mathcal{E})_P \neq 0$ for only finitely many points $P\in C$.

Theorem 3.10. *The following equality holds in $R_k(G)_{\mathbb{Q}}$*

$$\chi_G(\mathcal{E}) = \sum_{1\le i\le n}\frac{|I_i|}{|G|}\mathrm{Ind}_{I_i}^G\chi_{I_i}(\mathcal{E}|_{C_i}) + \sum_{P\in C}\Gamma_G(\mathcal{E})_P. \tag{3.17}$$

Proof. By (2.11), we have

$$\chi_G(\mathcal{E}) = \chi_G(\nu_*\nu^*\mathcal{E}) - [\mathcal{E}\otimes\mathcal{S}] = \chi_G(\nu^*\mathcal{E}) - [\mathcal{E}\otimes\mathcal{S}]. \tag{3.18}$$

By Theorem 3.4, we have

$$\chi_G(\nu^*\mathcal{E}) = \sum_{1\le i\le n}\frac{|I_i|}{|G|}\mathrm{Ind}_{I_i}^G\chi_{I_i}((\nu^*\mathcal{E})|_{\widetilde{C}_i}) + \sum_{Q\in\widetilde{C}}\Gamma_G(\nu^*\mathcal{E})_Q. \tag{3.19}$$

For each $i\le n$, we have $(\nu^*\mathcal{E})|_{\widetilde{C}_i} = \nu_i^*(\mathcal{E}|_{C_i})$ where $\nu_i\colon \widetilde{C}_i\to C_i$ is the normalization of C_i, hence as $k[I_i]$-modules,

$$\chi_{I_i}((\nu^*\mathcal{E})|_{\widetilde{C}_i}) = \chi_{I_i}(\mathcal{E}|_{C_i}) - \sum_{P\in C}[\mathcal{E}\otimes\mathcal{S}|_P] = \chi_{I_i}(\mathcal{E}|_{C_i}) - \sum_{P\in C}(\dim_k\mathcal{S}|_P)[\mathcal{E}|_P]. \tag{3.20}$$

Applying Proposition 2.6 to $\mathcal{E}\otimes\mathcal{S}$ over $\mathrm{Supp}\,\mathcal{S}$, we have

$$[\mathcal{E}\otimes\mathcal{S}] = \sum_{P\in S}\frac{|G_P|}{|G|}\mathrm{Ind}_{G_P}^G[(\mathcal{E}\otimes\mathcal{S})_P] \tag{3.21}$$

Plugging (3.19)–(3.21) into (3.18), we obtain the desired expression (3.17). $\square$

Remark 3.11. For an irreducible component C_i of C, we may define:

$$\Gamma_{I_i}(\mathcal{E}|_{C_i}) := \chi_{I_i}(\mathcal{E}|_{C_i}) - \frac{\chi(\mathcal{E}|_{C_i})}{|I_i|}[k[I_i]] \in R_k(I_i)_{\mathbb{Q}}$$

and then the *virtual ramification module* at C_i as follows:

$$\Gamma(\mathcal{E})_{C_i} := \frac{|G_{C_i}|}{|G|}\mathrm{Ind}_{I_i}^G \Gamma_{I_i}(\mathcal{E}|_{C_i}) \in R_k(G)_{\mathbb{Q}}.$$

Then Theorem 3.10 can be reformulated as:

$$\chi_G(\mathcal{E}) = \frac{1}{|G|}\chi(\mathcal{E})[k[G]] + \sum_{Z\subset C} \Gamma_G(\mathcal{E})_Z \tag{3.22}$$

where Z runs through the irreducible closed subvarieties of C. The formula (3.22) better highlights how $\chi_G(\mathcal{E})$ deviates from formula (3.3) in the case of free action.

We may use (3.22) to compute the virtual multiplicity of an irreducible G-module V in $\chi_G(\mathcal{E})$ as follows:

$$\begin{aligned}\langle [V], \chi_G(\mathcal{E})\rangle = \frac{\chi(\mathcal{E})}{|G|}\dim_k V + \sum_{i=1}^{n} \frac{|I_i|}{|G|}\langle \mathrm{Res}_{I_i}^G[V], \Gamma_{I_i}(\mathcal{E}|_{C_i})\rangle \\ + \sum_{P\in C} \frac{|G_P|}{|G|}\langle \mathrm{Res}_{G_P}^G[V], \Gamma_{G_P}(\mathcal{E}|_P)\rangle\end{aligned} \tag{3.23}$$

where $\Gamma_{G_P}(\mathcal{E}|_P) \in R_k(G_P)_{\mathbb{Q}}$ is such that $\Gamma_G(\mathcal{E})_P = \frac{|G_P|}{|G|}\mathrm{Ind}_{G_P}^G\Gamma_{G_P}(\mathcal{E}|_P)$. For example, if C is smooth, then $\Gamma_{G_P}(\mathcal{E}|_P) = \frac{|I_i|}{|G_P|}(\Psi_P \otimes [\mathcal{E}|_P])$, where Ψ_P is as in Definition 3.2. Note that the terms in the summations of (3.23) are computed "locally", that is, they depend only on the behavior of $\mathcal{E}$ at the fixed loci, and the restriction of V to the stabilizers of the fixed loci.

As a consequence of Theorem 3.10, we show in the next corollary that $\chi_G(\mathcal{E})$ for a vector bundle $\mathcal{E}$ on C depends only on its local ramification data and its degree.

Corollary 3.12. *Let C be a (possibly disconnected) proper reduced curve over k, and G be finite group acting tamely on C and freely on a dense open subset of C. Let $\mathcal{E}$ and $\mathcal{F}$ be two locally free G-sheaves of rank r such that $[\mathcal{E}\otimes\mathcal{S}_P] = [\mathcal{F}\otimes\mathcal{S}_P] \in R_k(G_P)$ for all points $P \in C$. Then*

$$\chi_G(\mathcal{E}) - \chi_G(\mathcal{F}) = \frac{1}{|G|}(\deg\mathcal{E} - \deg\mathcal{F})[k[G]].$$

Proof. Under the assumptions of the corollary, we have $\Gamma_G(\mathcal{E})_P = \Gamma_G(\mathcal{F})_P$ for all $P \in C$. Using Equality (3.18) and Corollary 3.6, we have

$$\chi_G(\mathcal{E}) - \chi_G(\mathcal{F}) = \frac{1}{|G|}(\chi(\nu^*\mathcal{E}) - \chi(\nu^*\mathcal{F}))[k[G]].$$

Equality (3.18) with $G = \{1\}$ shows that $\chi(\nu^*\mathcal{E}) - \chi(\nu^*\mathcal{F}) = \chi(\mathcal{E}) - \chi(\mathcal{F})$. Recall that $\deg\mathcal{E}$ is the degree of $\det\mathcal{E}$ viewed as a Cartier divisor on C ([Liu06], Definition 7.3.1). The Riemann-Roch formula $\chi(\mathcal{E}) = \deg\mathcal{E} + r\chi(\mathcal{O}_C)$ (see the proof of [BHPV04, II, Theorem (3.1)] which works for any reduced projective curve over any field) implies the desired equality. $\square$

The following corollaries generalize [EL80, Proposition 4.2, Corollary 4.3].

Corollary 3.13. *Let the assumptions be as in Corollary 3.12. Let $\mathcal{M}$ be a locally free sheaf of rank s on D. Then*

$$\chi_G(\mathcal{F}\otimes\pi^*\mathcal{M}) - s\chi_G(\mathcal{F}) = \frac{r\deg\mathcal{M}}{|G|}[k[G]].$$

In particular, $\chi_G(\pi^*\mathcal{M}) - s\chi_G(\mathcal{O}_C) = \frac{\deg\mathcal{M}}{|G|}[k[G]]$.

Proof. Note that $[(\mathcal{F}\otimes\pi^*\mathcal{M})\otimes\mathcal{S}_P] = [(\mathcal{F}\otimes\mathcal{O}_{C,P}^{\oplus s})\otimes\mathcal{S}_P] = [\mathcal{F}^{\oplus s}\otimes\mathcal{S}_P] \in R_k(G_P)$ for any $P\in C$, and $\deg(\mathcal{F}\otimes\pi^*\mathcal{M}) - \deg(\mathcal{F}^{\oplus s}) = r\deg\mathcal{M}$. Now apply Corollary 3.12. $\square$

3.3. **Equivariant degree of a G-equivariant line bundle.** Let C be smooth and connected. Let G be a finite tame subgroup of $\mathrm{Aut}(C)$. For G-equivariant line bundles $\mathcal{L}$ on C, Bornes [Bor03, Théorème 4.10] established an equivariant Riemann-Roch formula

$$\chi_G(\mathcal{L}) = \deg_G\mathcal{L} + \chi_G(\mathcal{O}_C)$$

for some $\deg_G\mathcal{L}\in R_k(G)_{\mathbb{Q}}$. Comparing with Theorem 3.4, we get

$$\deg_G\mathcal{L} := \frac{1}{|G|}(\deg\mathcal{L})[k[G]] + \sum_{P\in C}(\Gamma_G(\mathcal{L})_P - \Gamma_G(\mathcal{O}_C)_P). \tag{3.24}$$

In particular, when the G-action on C is free, then $\deg_G\mathcal{L} = \frac{1}{|G|}(\deg\mathcal{L})[k[G]]$ and it is additive in $\mathcal{L}$. However, we compute $\deg_G\mathcal{L}$ in two examples below and we will see that $\deg_G$ is not additive in general.

Example 3.14. Let C be a connected smooth projective curve and $G\subseteq\mathrm{Aut}(C)$ a finite tame subgroup with nontrivial G_P for some $P\in C$. For any $m\in\mathbb{Z}$, consider $\mathcal{L}_m := \mathcal{O}_C(m\sum_{g\in G}g(P))$. Then we have $[\mathcal{O}_C(mP)|_P] = \theta_P^{-m}$ (see (3.4)). Denote by $e_P = |G_P|$, and let $1\le m_P\le e_P$ be such that $m_P\equiv m \mod e_P$. Then (using Definition 3.2 and Lemma 2.3)

$$\begin{aligned}
\Gamma_G(\mathcal{L}_m)_P - \Gamma_G(\mathcal{O}_C)_P &= \frac{1}{|G|}\mathrm{Ind}_{G_P}^G(\Psi_P\otimes([\mathcal{O}_C(mP)|_P] - [1_{G_P}]))\\
&= \frac{1}{|G|}\mathrm{Ind}_{G_P}^G\left(\sum_{d=0}^{e_P-1} d\theta_P^d - d\theta_P^{d-m_P}\right)\\
&= \frac{1}{|G|}\mathrm{Ind}_{G_P}^G\left(e_P\sum_{d=e_P-m_P}^{e_P-1}\theta_P^d - m_P\sum_{d=0}^{e_P-1}\theta_P^d\right)\\
&= -\frac{m_P}{|G|}[k[G]] + \frac{e_P}{|G|}\mathrm{Ind}_{G_P}^G\left(\sum_{d=e_P-m_P}^{e_P-1}\theta_P^d\right)
\end{aligned}$$

because $\sum_{d=0}^{e_P-1}\theta_P^d = k[G_P]$. Therefore

$$\begin{aligned}
\deg_G(\mathcal{L}_m) &= \frac{m}{e_P}[k[G]] + \sum_{P'\in G\cdot P}\Gamma_G(\mathcal{L}_m)_{P'} - \Gamma_G(\mathcal{O}_C)_{P'}\\
&= \frac{m-m_P}{e_P}[k[G]] + \mathrm{Ind}_{G_P}^G\left(\sum_{d=e_P-m_P}^{e_P-1}\theta_P^d\right).
\end{aligned}$$

By looking at the multiplicities of $[1_G]$ and $[\theta_P]$ on both sides, it is evident that $\deg_G(\mathcal{L}_m)\neq m\deg_G(\mathcal{L}_1)$ for $m>1$.

Example 3.15. Let C be smooth and connected. Let $T \subset C$ be a finite G-subset. For any $P \in C$, denote $e_P = |G_P|$. For any $m \in \mathbb{Z}$, write $m = re_P + m_P$ with $r \in \mathbb{Z}$ and $1 \leq m_P \leq e_P$. Then we have

$$\deg_G \omega_C(T)^{\otimes m} = \frac{m \deg \omega_C(T)}{|G|}[k[G]] + \frac{1}{|G|} \sum_{P \in C \setminus T} \mathrm{Ind}_{G_P}^G \left(m_P[k[G_P]] - e_P \sum_{d=0}^{m_P - 1} \theta_P^d \right).$$

In particular, if m is divisible by $|G|$, then $\deg_G \omega_C(T)^{\otimes m} = \frac{m}{|G|}(\deg \omega_C(T))[k[G]]$; this is also a consequence of Corollary 3.12, as the action of G_P is then trivial on $\omega_C(T)^{\otimes m}|_P$ for any $P \in C$.

Proof. For $P \in T$, we have $[\omega_C(T)^{\otimes m}|_P] = [1_{G_P}]$, so

$$\Gamma_G(\omega_C(T)^{\otimes m})_P - \Gamma_G(\mathcal{O}_C)_P = 0.$$

For $P \in C \setminus T$, we have $[\omega_C(T)^{\otimes m}|_P] = \theta_P^m$. Similar computations as in Example 3.14 give

$$\begin{aligned}\Gamma(\omega_C(T)^{\otimes m})_P - \Gamma(\mathcal{O}_C)_P &= \frac{1}{|G|} \mathrm{Ind}_{G_P}^G \left(\sum_{d=0}^{e_P - 1} -d\theta_P^{d+m_P} + d\theta_P^d \right) \\ &= \frac{1}{|G|} \mathrm{Ind}_{G_P}^G \left(m_P[k[G_P]] - e_P \sum_{d=0}^{m_P - 1} \theta_P^d \right).\end{aligned}$$

Summing up, we obtain the desired formula for $\deg_G(\omega_C(T)^{\otimes m})$ by (3.24). ☐

4. The case of nodal curves

In this section, we keep the notation of Set-ups 3.1. Additionally, we assume that C has at worst nodes as singularities. This means ([Liu06, Proposition 10.3.7]) that for all singular points P of C, $\nu^{-1}(P)$ consists in two points $\widetilde{P}_1, \widetilde{P}_2$ and that $\dim_{k(P)}(\nu_* \mathcal{O}_{\widetilde{C}} / \mathcal{O}_C)_P = 1$. Equivalently, there exists an isomorphism of k-algebras $\widehat{\mathcal{O}}_{C,P} \simeq k[[x, y]]/(xy)$.

Set-up 4.1. Let S be the singular locus of C. The group G acts on $\widetilde{C}$. Because of the tameness of the G-action, $|G_P|$ is invertible in k for all $P \in S$. For $P \in S$, denote $G_{\widetilde{P}} \subseteq G_P$ the subgroup of the elements acting trivially on $\nu^{-1}(P)$. It is equal to the stabilizer $G_{\widetilde{P}_1} = G_{\widetilde{P}_2}$ of G at $\widetilde{P}_i \in \widetilde{C}$ and it has index at most 2 in G_P. We define two subsets of S:

$$S_1 = \{P \in S \mid G_P = G_{\widetilde{P}}\}, \quad S_2 = S \setminus S_1. \tag{4.1}$$

Geometrically, $P \in S_1$ means that G_P does not swap the local branches of C at P.

Let $\pi : C \to D := C/G$ the quotient map. It is known that D is nodal and that $P \in S_1$ if and only if $\pi(P)$ is singular in D. See *e.g.*, the proof of [Liu06] Proposition 10.3.48(a)-(b); it is helpful to look at the commutative diagram (3.1).

For $P \in S_2$, $G_{\widetilde{P}}$ has index 2 in G_P. The tameness of the G-action implies that $\mathrm{char}(k) \neq 2$. We denote by

$$\chi_P = [k[G_P/G_{\widetilde{P}}]] - [1_{G_P}] \in R_k(G_P). \tag{4.2}$$

This is the class of the G_P-representation of degree 1 given by the character $G_P \to G_P/G_{\widetilde{P}} \simeq \{\pm 1\} \hookrightarrow k^*$

Let $\mathcal{K}_C = \nu_*\mathcal{K}_{\widetilde{C}}$ be the sheaf of rational functions on C (([Liu06, Definition 7.1.13]). The *dualizing (or canonical) sheaf* ω_C *of* C is then an invertible sub-$\mathcal{O}_C$-sheaf of $\Omega_C \otimes \mathcal{K}_C = \nu_*(\Omega_{\widetilde{C}} \otimes \mathcal{K}_{\widetilde{C}})$ such that $\omega_C|_U = \Omega_C|_U$ on the smooth locus $U := C \setminus S$ (see [DM69, §1]). Let $\widetilde{S} = \nu^{-1}S$ viewed as reduced divisors on $\widetilde{C}$. Then ([Liu06], Lemma 10.2.12(b))

$$\nu^*\omega_C = \omega_{\widetilde{C}}(\widetilde{S}). \tag{4.3}$$

Lemma 4.2. *Let C be a nodal curve over k. Let $P \in S$.*

(1) *In the exact sequence (2.9) with $X = C$*

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\widetilde{C}} \to \mathcal{S} \to 0, \tag{4.4}$$

we have $\mathcal{S}_P \simeq \mathcal{S}|_P \simeq k(P)$ as $\mathcal{O}_{C,P}$-modules.

(2) *The normalization map $\widetilde{C} \to C$ is unramified and we have a G_P-equivariant exact sequence of $\mathcal{O}_{C,P}$-modules*

$$0 \to k(P) \to k(\widetilde{P}_1) \oplus k(\widetilde{P}_2) \to \mathcal{S}|_P \to 0. \tag{4.5}$$

(3) *We have a G_P-equivariant exact sequence*

$$0 \to \omega_{C,P}|_P \to \omega_{\widetilde{C}}(\widetilde{P}_1)|_{\widetilde{P}_1} \oplus \omega_{\widetilde{C}}(\widetilde{P}_2)|_{\widetilde{P}_1} \to k(P) \to 0$$

Proof. (1) This follows from the definition of nodal points (see the beginning of §4).

(2) Tensoring (4.4) by $k(P)$ over $\mathcal{O}_{C,P}$, we get an exact sequence

$$k(P) \to \left(\mathcal{O}_{\widetilde{C},\widetilde{P}_1} \otimes_{\mathcal{O}_{C,P}} k(P)\right) \oplus \left(\mathcal{O}_{\widetilde{C},\widetilde{P}_1} \otimes_{\mathcal{O}_{C,P}} k(P)\right) \to \mathcal{S}|_P \to 0.$$

Considering the dimensions on $k(P) = k$, we see that the above exact sequence is equal to (4.5). In particular $\mathcal{O}_{\widetilde{C},\widetilde{P}_i} \otimes_{\mathcal{O}_{C,P}} k(P) = k(\widetilde{P}_i)$, which means that $\widetilde{C} \to C$ is unramified at the $\widetilde{P}_i$'s.

(3) By [DM69, §1, (b)], we have an exact sequence

$$0 \to \omega_{C,P} \to \nu_*(\omega_{\widetilde{C}}(\widetilde{P}_1 + \widetilde{P}_2))_P \to k(P) \to 0$$

where the map at the right takes $\eta \in \Omega_{\widetilde{C}} \otimes \mathcal{K}_{\widetilde{C}}$ to the sum $\mathrm{Res}_{\widetilde{P}_1}(\eta) + \mathrm{Res}_{\widetilde{P}_2}(\eta)$ of its residues. Tensoring it by $k(P)$ over $\mathcal{O}_{C,P}$ and using the fact that $\widetilde{C} \to C$ is unramified, we get the exact sequence

$$\omega_{C,P}|_P \to \omega_{\widetilde{C}}(\widetilde{P}_1)|_{\widetilde{P}_1} \oplus \omega_{\widetilde{C}}(\widetilde{P}_1)|_{\widetilde{P}_2} \to k(P) \to 0.$$

Again, the exactness at the left comes from dimension counting. □

Corollary 4.3. *We have in $R_k(G_P)$:*

$$[\omega_C|_P] = [\mathcal{S}|_P] = \begin{cases} [1_{G_P}] & \textit{if } P \in S_1, \\ \chi_P & \textit{if } P \in S_2. \end{cases}$$

Proof. If $P \in S_1$, then $[k(\widetilde{P}_j)] = [k(P)]$ as $k[G_P]$-modules, thus $[\mathcal{S}|_P] = [k(P)] = [1_{G_P}]$ by Exact sequence (4.5). If $P \in S_2$, then $[k(\widetilde{P}_1) \oplus k(\widetilde{P}_2)] = k[G_P/G_{\widetilde{P}}]$, thus $[\mathcal{S}|_P] = k[G_P/G_{\widetilde{P}}] - [1_{G_P}]$.

Consider now $[\omega_C|_P]$. Let $i = 1, 2$, we have $[\omega_{\widetilde{C}}(\widetilde{P}_i)|_{\widetilde{P}_i}] = [k(\widetilde{P}_i)]$ as $G_{\widetilde{P}}$-modules (see Equality (3.4)). The desired equalities on $[\omega_C|_P]$ is then proved similarly to $[\mathcal{S}|_P]$ using the exact sequence of Lemma 4.2(3). □

4.1. **Computation of $\Gamma_G(\mathcal{E})_P$ at nodes with balanced actions.** Keep the notation as in Set-ups 3.1 and 4.1. Recall that G acts tamely on C. We suppose additionally in this subsection that G acts freely on an open dense subset of C. This is equivalent to the inertia subgroups G_{C_i} being trivial for all irreducible components C_i of C, and to G_P acting faithfully on $\mathcal{O}_{C,P}$ for all $P \in C$. Then for any singular point P of C, $G_{\widetilde{P}} = G_{\widetilde{P}_1} = G_{\widetilde{P}_2}$ is cyclic of order invertible in k, and the action of $G_{\widetilde{P}_i}$ on the cotangent space $\omega_{\widetilde{C}}|_{\widetilde{P}_i} \simeq k$ is given by a cyclotomic character $\chi_i : G_{\widetilde{P}_i} \to k^*$. We say that the action of $G_{\widetilde{P}}$ at P is *balanced* if $\chi_1 = \chi_2^{-1}$. Under this condition, we have a simpler expression for $\Gamma_G(\mathcal{E})_P$ at singular points.

Proposition 4.4. *Keep the above assumption and assume that the action of $G_{\widetilde{P}}$ at P is balanced for all $P \in S$. Then for all G-equivariant vector bundle $\mathcal{E}$ of rank r on C we have*

$$\chi_G(\mathcal{E}) = \frac{1}{|G|}\chi(\mathcal{E})[k[G]] + \sum_{P \in C_{\mathrm{sm}}} \Gamma_G(\mathcal{E})_P + \sum_{P \in S_2} \frac{|G_P|}{2|G|}\mathrm{Ind}_{G_P}^G(([1_{G_P}] - \chi_P) \otimes [\mathcal{E}|_P]).$$

Proof. We first compute $\sum_{i=1,2} \Gamma_G(\nu^*\mathcal{E})_{\widetilde{P}_i}$ for $P \in S$. Denote $e_P = |G_{\widetilde{P}}|$. The balanced condition is $\theta_{\widetilde{P}_1} = \theta_{\widetilde{P}_2}^{-1}$, so

$$\sum_{i=1,2} \sum_{d=0}^{e_P-1} d\theta_{\widetilde{P}_i}^d = \sum_{d=0}^{e_P-1}(d\theta_{\widetilde{P}}^d + d\theta_{\widetilde{P}}^{e_P-d}) = e_P \sum_{d=1}^{e_P-1} \theta_{\widetilde{P}}^d = e_P([k[G_{\widetilde{P}}]] - [1_{G_{\widetilde{P}}}]).$$

Note that $[(\nu^*\mathcal{E})|_{\widetilde{P}_i}] = \mathrm{Res}_{G_{\widetilde{P}}}^{G_P}[\mathcal{E}|_P] =: M_P \in R_k(G_{\widetilde{P}})$, Therefore, by Definition 3.2 and Lemma 2.3,

$$\begin{aligned}
\sum_{i=1,2} \Gamma_G(\nu^*\mathcal{E})_{\widetilde{P}_i} &= -\frac{1}{|G|} \sum_{i=1,2} \mathrm{Ind}_{G_{\widetilde{P}}}^G \sum_{d=0}^{e_P-1} d\theta_{\widetilde{P}_i}^d \otimes M_P \otimes \left([1_{G_{\widetilde{P}}}] - \frac{1}{e_P}[k[G_{\widetilde{P}}]]\right) \\
&= \frac{1}{|G|}\mathrm{Ind}_{G_{\widetilde{P}}}^G e_P([1_{G_{\widetilde{P}}}] - [k[G_{\widetilde{P}}]]) \otimes M_P \otimes \left([1_{G_{\widetilde{P}}}] - \frac{1}{e_P}[k[G_{\widetilde{P}}]]\right) \\
&= \frac{1}{|G|}\mathrm{Ind}_{G_{\widetilde{P}}}^G \left(e_P[1_{G_{\widetilde{P}}}] - [k[G_{\widetilde{P}}]]\right) \otimes M_P \\
&= \frac{e_P}{|G|}\mathrm{Ind}_{G_{\widetilde{P}}}^G M_P - \frac{r}{|G|}[k[G]]
\end{aligned}$$

because $[k[G_{\widetilde{P}}]] \otimes M_P = r[k[G_{\widetilde{P}}]]$ by Lemma 2.3, and $\mathrm{Ind}_{G_{\widetilde{P}}}^G[k[G_{\widetilde{P}}]] = [k[G]]$. On the other hands,

$$\mathrm{Ind}_{G_{\widetilde{P}}}^{G_P} M_P = [k[G_P/G_{\widetilde{P}}]] \otimes [\mathcal{E}|_P]$$

by Lemma 2.2(3). Therefore

$$\sum_{i=1,2} \Gamma_G(\nu^*\mathcal{E})_{\widetilde{P}_i} = \frac{e_P}{|G|}\mathrm{Ind}_{G_P}^G \left([k[G_P/G_{\widetilde{P}}]] \otimes [\mathcal{E}|_P]\right) - \frac{r}{|G|}[k[G]] \tag{4.6}$$

Now we compute $\Gamma_G(\mathcal{E})_P$. By Definition 3.9 and Equality (4.6), we have

$$\begin{aligned}
\Gamma_G(\mathcal{E})_P &= \sum_{\widetilde{P} \in \nu^{-1}(P)} \Gamma_G(\nu^*\mathcal{E})_{\widetilde{P}} + \frac{|G_P|}{|G|}\mathrm{Ind}_{G_P}^G \left(\frac{r}{|G_P|}[k[G_P]] - [(\mathcal{E} \otimes \mathcal{S})|_P]\right) \\
&= \frac{e_P}{|G|}\mathrm{Ind}_{G_P}^G([k[G_P/G_{\widetilde{P}}]] \otimes [\mathcal{E}|_P]) - \frac{|G_P|}{|G|}\mathrm{Ind}_{G_P}^G\left([(\mathcal{E} \otimes \mathcal{S})|_P]\right).
\end{aligned}$$

If $P \in S_1$, then $G_{\widetilde{P}} = G_P$ and $[\mathcal{S}|_P] = [1_{G_P}]$ (Corollary 4.3). This implies that $\Gamma_G(\mathcal{E})_P = 0$. Suppose now $P \in S_2$. Then $e_P = |G_P|/2$, $[k[G_P/G_{\widetilde{P}}]] = \chi_P + [1_{G_P}]$ by definition and $[\mathcal{S}|_P] = \chi_P$ (Corollary 4.3). This implies

$$\Gamma_G(\mathcal{E})_P = \frac{|G_P|}{2|G|}\mathrm{Ind}_{G_P}^G([\mathcal{E}|_P] \otimes ([1_{G_P}] - \chi_P)).$$

The proposition then follows from Theorem 3.10. $\square$

Remark 4.5. When $S_2 = \emptyset$, the balanced condition on the action of G corresponds to the fact that $C \to C/G$ is an admissible cover in the sense of Harris-Mumford [HM82, §4] and balanced in the sense of [ACV03, §2.1.3].

More generally, when S_2 may be non-empty, let us call the action of G *smoothable* at $P \in S$ if $G_{\widetilde{P}}$ is balanced at P and if moreover any element of $G_P \setminus G_{\widetilde{P}}$ acts on the tangent space $T_{C,P} \simeq k^2$ as an involution of determinant -1. When $k = \mathbb{C}$, by [Liu12, Section 2.1], the G-curve C is smoothable at all $P \in S$ if and only if there exists a projective flat family $\mathcal{C} \to \Delta$ over the unit open disc endowed with an action of G (trivial on Δ) such that the following holds:

- the generic fiber of $\mathcal{C} \to \Delta$ is smooth;
- there exist a G-equivariant isomorphism between the central fiber $\mathcal{C}_0$ and C.

For general k, the same arguments show that the similar statement holds with Δ replaced by $\mathrm{Spec} k[[t]]$.

4.2. **Lower bound on $\chi_G(\omega_C \otimes \mathcal{A})$ for ample vector bundles $\mathcal{A}$.** Keep the notation in Set-ups 3.1 and 4.1. So C is a nodal curve with G acting tamely on C. In this subsection, we write $\mathcal{E} = \omega_C \otimes \mathcal{A}$ and try to bound $\chi_G(\mathcal{E})$ from below in terms of the quotient curve D and $\deg \mathcal{A}|_{C_i}$ for each component C_i. Under the conditions that the inertia group I_i for of each irreducible component C_i is trivial, and that $\mathcal{A}$ is sufficiently ample, one can find some positive rational number a such that $\chi_G(\mathcal{E}) \geq a \cdot [k[G]]$ in $R_k(G)_{\mathbb{Q}}$; see Corollary 4.12.

Let us first recall some basic facts on ample vector bundles that we will need in this subsection. The main reference is [Har71].

Facts 4.6. Let X be a projective variety over k. Let $\mathcal{A}$ be a vector bundle of rank $r \geq 1$ on X. By definition ([Har71, §2]), $\mathcal{A}$ is *ample* if for any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{A})$ is generated by its global sections for all n big enough (depending on $\mathcal{F}$). Suppose $\mathcal{A}$ is ample.

(1) The restriction of $\mathcal{A}$ to any closed subvariety of X is ample ([Har71, Proposition 4.1]). More generally for any finite morphism $f : Y \to X$, $f^*\mathcal{A}$ is ample on Y (apply [Har71, Proposition 4.3] to $Y \to f(Y)$ the scheme-theoretical image of f).
(2) The line bundle $\det \mathcal{A}$ on X is ample ([Har71, Corollary 2.6]). In particular if X is an irreducible curve, then $\deg \mathcal{A} > 0$ ([Liu06, Proposition 7.5.5]).
(3) If $X = C$ is a reduced curve, then $H^0(C, \mathcal{A}^\vee) = 0$.

Let us give a proof for the assertion (3). Applying the exact sequence (2.10) to $\mathcal{A}$, we see that it is enough to show $H^0(\widetilde{C}, (\nu^*\mathcal{A})^\vee) = 0$. As $\nu^*\mathcal{A}$ is ample by (1), we are reduced to the case where C is a connected smooth curve. If $H^0(C, \mathcal{A}^\vee) \neq 0$, there exists a non-constant homomorphism $\mathcal{A} \to \mathcal{O}_C$ whose image $\mathcal{L}$ is then a line bundle. By [Har71, Proposition 2.2], $\mathcal{L}$ is ample, hence $\deg \mathcal{L} > 0$. But $\mathcal{L}$ is a sheaf of ideals, contradiction.

Lemma 4.7. *Let $\mathcal{E}$ be a locally free G-sheaf on C and set $\mathcal{A} = \mathcal{E} \otimes \omega_C^{-1}$. Then we have*

$$\chi_G(\mathcal{E}) = \chi_G(\omega_{\widetilde{C}} \otimes \nu^*\mathcal{A}) + [H^0(S, \mathcal{E}|_S)].$$

If $\mathcal{A}$ is ample, then

$$\chi_G(\mathcal{E}) = [H^0(C, \mathcal{E})],\ \chi_G(\omega_{\widetilde{C}} \otimes \nu^*\mathcal{A}) = [H^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \nu^*\mathcal{A})]$$

and $\chi_G(\nu^*\mathcal{E}) = [H^0(\widetilde{C}, \nu^*\mathcal{E})]$.

Proof. Consider the exact sequence ([Liu06, Lemma 10.3.12])

$$0 \to \nu_*\omega_{\widetilde{C}} \to \omega_C \to \omega_C|_S \to 0. \tag{4.7}$$

Tensoring (4.7) with $\mathcal{A}$, we obtain the G-equivariant exact sequence

$$0 \to (\nu_*\omega_{\widetilde{C}}) \otimes \mathcal{A} \to \mathcal{E} \to \mathcal{E}|_S \to 0. \tag{4.8}$$

Note that $\chi_G((\nu_*\omega_{\widetilde{C}}) \otimes \mathcal{A}) = \chi_G(\omega_{\widetilde{C}} \otimes \nu^*\mathcal{A})$. (4.8) implies that the first desired equality in $R_k(G)$ by taking the long cohomology sequence of (4.8).

Suppose that $\mathcal{A}$ is ample. By Serre's duality and Facts 4.6(3), we have $H^1(C, \mathcal{E}) \simeq H^0(C, \mathcal{A}^\vee) = 0$, and $H^1(C, (\nu_*\omega_{\widetilde{C}}) \otimes \mathcal{A}) \simeq H^1(\widetilde{C}, \omega_{\widetilde{C}} \otimes \nu^*\mathcal{A}) \simeq H^0(\widetilde{C}, (\nu^*\mathcal{A})^\vee) = 0$. □

Lemma 4.8. *Suppose that C is smooth, connected and that $G \subseteq \mathrm{Aut}(C)$ is a tame finite subgroup. Let $\mathcal{E}$ be a locally free G-sheaf of rank r on C, and $\mathcal{A} := \omega_C^{-1} \otimes \mathcal{E}$. Then*

$$\chi_G(\mathcal{E}) \geq \left(r\chi(\omega_D) + \frac{1}{|G|} \deg \mathcal{A} \right) [k[G]].$$

Proof. Let R be the ramification divisor of the quotient map $\pi \colon C \to D$. By the Riemann–Hurwitz formula and the tameness hypothesis on the G-action on C, we have

$$\chi(\omega_C) = |G|\chi(\omega_D) + \frac{1}{2} \deg R.$$

By the Riemann–Roch theorem, we have

$$\chi(\mathcal{E}) = r\chi(\mathcal{O}_C) + \deg \mathcal{E} = r\chi(\mathcal{O}_C) + r \deg \omega_C + \deg \mathcal{A} = r\chi(\omega_C) + \deg \mathcal{A}$$

By Theorem 3.4,

$$\begin{aligned} \chi_G(\mathcal{E}) &= \frac{1}{|G|}\chi(\mathcal{E})[k[G]] + \sum_{P \in C} \Gamma(\mathcal{E})_P \\ &= \left(r\chi(\omega_D) + \frac{1}{|G|} \deg \mathcal{A} \right) [k[G]] + \frac{r}{2|G|}(\deg R)[k[G]] + \sum_{P \in C} \Gamma(\mathcal{E})_P. \end{aligned}$$

As $\deg R = \sum_{P \in C}(|G_P| - 1)$, and $r[k[G_P]] = [k[G_P]] \otimes [\mathcal{E}|_P] = \sum_{d=0}^{|G_P|-1} \theta_P^d \otimes [\mathcal{E}|_P]$, we have by Definition 3.2

$$\begin{aligned}
&\frac{r}{2|G|}(\deg R)[k[G]] + \sum_{P \in C} \Gamma(\mathcal{E})_P \\
&= \frac{1}{|G|} \sum_{P \in C} \mathrm{Ind}_{G_P}^G \left(r(|G_P| - 1)[G_P]] - \left(\sum_{d=0}^{|G_P|-1} d\theta_P^d \otimes [\mathcal{E}|_P] \right) \right) \\
&= \frac{1}{|G|} \sum_{d=0}^{|G_P|-1} (|G_P| - 1 - d)\theta_P^d \otimes [\mathcal{E}|_P] \\
&\geq 0
\end{aligned}$$

Thus $\chi_G(\mathcal{E}) \geq \left(r\chi(\omega_D) + \frac{1}{|G|} \deg \mathcal{A} \right) [k[G]]$, as desired. □

As pointed out by the referee, the following corollary is contained in [Kan86, Corollary of Theorem 2].

Corollary 4.9. *Let C be a smooth connected projective curve over k, and $G \subseteq \mathrm{Aut}(C)$ a finite tame subgroup. If $g(C/G) \geq 2$, then*

$$[H^0(C, \omega_C)] \geq [k[G]].$$

In particular, if $|G|$ is not divisible by the characteristic of k, then each $V \in \mathrm{Irr}_k(G)$ has positive multiplicity in the G-module $H^0(C, \omega_C)$.

Proof. Take $\mathcal{E} = \omega_C$ in Lemma 4.8, and note that

$$[H^0(C, \omega_C)] = \chi_G(\omega_C) + [1_G] \geq (g(C/G) - 1)[k[G]] + [1_G].$$

□

The conclusion of Corollary 4.9 does not hold any more in the singular case. We provide in the following two examples for this phenomenon.

Example 4.10. Let $C = C_1 \cup C_2$ be a stable curve consisting of two smooth components such that $C_1 \cap C_2 = \{P\}$, and that $\mathrm{Aut}(C_2)$ contains a cyclic subgroup G of order $n > g(C_2) + 1$. Let G act on C as follows:

(1) G preserves the components C_1 and C_2;
(2) G acts trivially on C_1 and as subgroup of $\mathrm{Aut}(C_2)$ on C_2.

Then $H^0(C, \omega_C) = H^0(C_1, \omega_{C_1}) \oplus H^0(C_2, \omega_{C_2})$ and G acts trivially on the first summand $H^0(C_1, \omega_{C_1})$. Now there are $n - 1$ nontrivial irreducible characters of G, while $\dim H^0(C_2, \omega_{C_2}) = g(C_2) < n - 1$ by assumption, there must be some nontrivial irreducible character of G not present in $H^0(C_2, \omega_{C_2})$ and hence not in $H^0(C, \omega_C)$ either. On the other hand, the arithmetic genus of the quotient curve is

$$p_a(C/G) = g(C_1/G) + g(C_2/G) \geq g(C_1/G) = g(C_1)$$

which can be chosen arbitrarily large.

In the following example the curve C is even irreducible.

Example 4.11. Let $p \geq 5$ be a prime number different from $\mathrm{char}(k)$, and $G = \langle \sigma \rangle \simeq \mathbb{Z}/p\mathbb{Z}$. By the Riemann existence theorem, we may construct a G-cover $\tilde{\pi}\colon \widetilde{C} \to \mathbb{P}^1$ with 4 branch points $Q_j, 1 \leq j \leq 4$, over which the monodromies are given by the sequence

$$\left(\sigma,\, \sigma,\, \sigma,\, \sigma^{-3}\right).$$

Let $\widetilde{P}_j$ be the inverse image of Q_j for $1 \leq j \leq 4$, which are fixed by G. Recall that

$$\theta_{\widetilde{P}_j} = \left[\omega_{\widetilde{C}}|_{\widetilde{P}_j}\right] \in R_k(G), \quad 1 \leq j \leq 4.$$

For $r \in \mathbb{Z}$, define $\chi_r := \theta_{\widetilde{P}_1}^r$. Then the $\{\chi_r\}_{1\leq r\leq p}$ are exactly the classes of the irreducible G-modules, with $\chi_p = [1_G]$. We have

$$\theta_{\widetilde{P}_1} = \theta_{\widetilde{P}_2} = \theta_{\widetilde{P}_3} = \theta_{\widetilde{P}_4}^{-3} = \chi_1$$

One computes easily by the Riemann–Hurwitz formula that $g(\widetilde{C}) = p - 1$. By Theorem 3.4, we have

$$[H^0(\widetilde{C}, \omega_{\widetilde{C}})] = \chi_G(\omega_{\widetilde{C}}) + [1_G] = [1_G] + \frac{p-2}{p}[k[G]] + \frac{1}{p}\sum_{j=1}^{4}\left(\frac{p-1}{2}[k[G]] - \sum_{d=0}^{p-1} d\theta_{\widetilde{P}_j}^{d+1}\right)$$

We have $\langle \chi_r, \theta_{\widetilde{P}_4}^{d+1}\rangle = 1$ if $d+1 \equiv -3r \mod p$ and is zero otherwise. It follows that

$$\mu_{\chi_r} H^0(\widetilde{C}, \omega_{\widetilde{C}}) = \langle \chi_r, [H^0(\widetilde{C}, \omega_{\widetilde{C}})]\rangle = \begin{cases} 2, & \text{if } 1 \leq r \leq \lfloor \frac{p-1}{3} \rfloor, \\ 1, & \text{if } \lceil \frac{p}{3} \rceil \leq r \leq \lfloor \frac{2p-1}{3} \rfloor, \\ 0, & \text{if } \lceil \frac{2p}{3} \rceil \leq r \leq p. \end{cases} \tag{4.9}$$

Let $\nu\colon \widetilde{C} \to C$ be the gluing of the pairs of points $\{\widetilde{P}_1, \widetilde{P}_2\}$ and $\{\widetilde{P}_3, \widetilde{P}_4\}$, so that C is a stable curve with two nodes $P_1 = \nu(\widetilde{P}_1)$ and $P_2 = \nu(\widetilde{P}_3)$. Since the points $\widetilde{P}_j$ are fixed by the G, the action of G descends to C, preserving the local branches at P_1 and P_2. By (4.7), we have

$$[H^0(C, \omega_C)] = \left[H^0(\widetilde{C}, \omega_{\widetilde{C}})\right] + [\omega_C|_S] = \left[H^0(\widetilde{C}, \omega_{\widetilde{C}})\right] + 2[1_G].$$

Thus for $\lceil \frac{2p}{3} \rceil \leq r \leq p-1$, we have

$$\mu_{\chi_r} H^0(C, \omega_C) = \mu_{\chi_r} H^0(\widetilde{C}, \omega_{\widetilde{C}}) = 0.$$

On the other hand, the quotient curve $D = C/G$ is a rational curve with two nodes, and hence $p_a(D) = 2$. However, $H^0(C, \omega_C)$ does not contain any copy of χ_r.

Corollary 4.12. *Let $C = \bigcup_{1\leq i\leq n} C_i$ be a proper nodal curve over k, and $G \subseteq \mathrm{Aut}(C)$ a tame finite subgroup such that the inertia group I_i for each component C_i is trivial. Let $\mathcal{E}$ be a locally free G-sheaf of rank r on C, and denote $\mathcal{A} = \mathcal{E} \otimes \omega_C^{-1}$. Suppose that $\mathcal{A}$ is ample. Then the following holds.*

1. *(1) For any $P \in S$, we have $\chi_G(\mathcal{E}) \geq \mathrm{Ind}_{G_P}^G[\mathcal{E}|_P]$. In particular if G acts freely on the orbit of a singular point, then $\chi_G(\mathcal{E}) \geq r[k[G]]$.*
2. *(2) For all $i \leq n$, we have*

$$\chi_G(\mathcal{E}) \geq \left(r\chi(\omega_{\widetilde{D}_i}) + \frac{1}{|G_i|}\deg(\mathcal{A}|_{C_i})\right)[k[G]]$$

where G_i is the stabilizer of C_i (Set-ups 3.1), and $\widetilde{D}_i = \widetilde{C}_i/G_i$ is the quotient curve. In particular, if $\deg \mathcal{A}|_{C_i} > -r|G_i|\chi(\omega_{\widetilde{D}_i})$ for all i (automatic *if $g(\widetilde{D}_i) > 0$ as $\deg \mathcal{A}|_{C_i} > 0$ by Facts 4.6(1)-(2)), then any irreducible representation of G has positive virtual multiplicity in $\chi_G(\mathcal{E})$.*

(3) *If $\deg \mathcal{A}|_{C_i} \geq |G_i|(-r\chi(\omega_{\widetilde{D}_i})+1)$ (automatic if $g(\widetilde{D}_i) \geq 2$) for some $i \leq n$, then $\chi_G(\mathcal{E}) \geq [k[G]]$.*

Proof. (1) Let $P \in S$. Then $\bigoplus_{P' \in G.P} \mathcal{E}|_{P'}$ is a direct summand of $\mathcal{E}|_S$. By Lemma 4.7,

$$\chi_G(\mathcal{E}) = [H^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes \nu^*\mathcal{A})] + [\mathcal{E}|_S] \geq [\mathcal{E}|_S] \geq \sum_{P' \in G.P} [\mathcal{E}|_{P'}] = \mathrm{Ind}_{G_P}^G [\mathcal{E}|_P].$$

If $G_P = \{1\}$, then the right-hand side is equal to $r[k[G]]$.

(2) First suppose C is smooth. Fix $i \leq n$. Then $\bigoplus_{C_j \subseteq GC_i} H^0(C_j, \mathcal{E}|_{C_j})$ is a direct summand of $H^0(C, \mathcal{E})$, hence

$$\chi_G(\mathcal{E}) = [H^0(C, \mathcal{E})] \geq \left[\bigoplus_{C_j \subseteq GC_i} H^0(C_j, \mathcal{E}|_{C_j}) \right] = \mathrm{Ind}_{G_i}^G [H^0(C_i, \mathcal{E}|_{C_i})].$$

It suffices to apply Lemma 4.8 to $(C_i, \mathcal{E}|_{C_i})$ with the tame subgroup $G_i \subseteq \mathrm{Aut}(C_i)$.

In the general case, $\nu^*\mathcal{A}$ is ample on $\widetilde{C}$ with $\deg \nu^*\mathcal{A}|_{\widetilde{C}_i} = \deg \mathcal{A}|_{C_i}$, $\widetilde{C}_i$ is a connected component of $\widetilde{C}$. So by Lemma 4.7,

$$\chi_G(\mathcal{E}) \geq \chi_G(\omega_{\widetilde{C}} \otimes \nu^*\mathcal{A}) \geq \left(r\chi(\omega_{\widetilde{D}_i}) + \frac{1}{|G_i|} \deg(\mathcal{A}|_{C_i}) \right) [k[G]].$$

(3) is an immediate consequence of (2). $\square$

4.3. **The G-invariant part of $H^0(C, \omega_C(T)^{\otimes m})$.** Let the notation be as in Setups 3.1 and 4.1.

Theorem 4.13. *Assume that G acts faithfully on C. Let T be a finite G-set contained in the smooth locus of C. Then for any integer m, we have*

$$\langle [1_G], \chi_G(\omega_C(T)^{\otimes m}) \rangle = \chi(\omega_D(\overline{T})^{\otimes m}) + (m - \epsilon_m)\#\overline{S}_2 + \sum_{Q \in D \setminus (\overline{S} \cup \overline{T})} \left\lfloor m\left(1 - \frac{1}{e_Q}\right) \right\rfloor$$

where $\overline{S}$, $\overline{S}_2, \overline{T} \subset D$ are the images of S, S_2 and T respectively, for $Q \in D \setminus (\overline{S} \cup \overline{T})$, $e_Q := |G_P|$ for $P \in \pi^{-1}(Q)$, and

$$\epsilon_m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} \tag{4.10}$$

Proof. Tensoring the short exact sequence (4.4) with $\omega_C(T)^{\otimes m}$, we obtain another G-equivariant short exact sequence

$$0 \to \omega_C(T)^{\otimes m} \to \nu_*\nu^*\omega_C(T)^{\otimes m} \to \mathcal{S} \otimes \omega_C(T)^{\otimes m} \to 0$$

Set $\widetilde{S} = \nu^{-1}S$ and $\widetilde{T} = \nu^{-1}T$, viewed also as reduced divisors on $\widetilde{C}$. Recall that $\nu^*\omega_C = \omega_{\widetilde{C}}(\widetilde{S})$ (Isomorphism (4.3)), and we have

$$\begin{aligned} \chi_G(\omega_C(T)^{\otimes m}) &= \chi_G(\nu^*\omega_C(T)^{\otimes m}) - [\mathcal{S} \otimes \omega_C(T)^{\otimes m}] \\ &= \chi_G(\omega_{\widetilde{C}}(\widetilde{S} + \widetilde{T})^{\otimes m}) - [\mathcal{S} \otimes \omega_C(T)^{\otimes m}] \end{aligned} \tag{4.11}$$

Since $S \cap T = \emptyset$, we have $\mathcal{S} \otimes \omega_C(T) = \mathcal{S} \otimes \omega_C$. By Proposition 2.6 and Corollary 4.3,

$$\begin{aligned}[\mathcal{S} \otimes \omega_C(T)^{\otimes m}] &= \sum_{P \in S} \frac{|G_P|}{|G|} \mathrm{Ind}_{G_P}^G [\mathcal{S}_P] \otimes [\omega_C|_P]^{\otimes m} \\ &= \sum_{P \in S_1} \frac{|G_P|}{|G|} \mathrm{Ind}_{G_P}^G [1_{G_P}] + \sum_{P \in S_2} \frac{|G_P|}{|G|} \mathrm{Ind}_{G_P}^G \chi_P^{\otimes m+1}.\end{aligned}$$

By the Frobenius reciprocity (2.6),

$$\langle [1_G], [\mathcal{S} \otimes \omega_C(T)^{\otimes m}] \rangle = \# \overline{S}_1 + \# \overline{S}_2 \cdot \epsilon_m \tag{4.12}$$

where $\overline{S}_1 = \pi(S_1)$.

Let $\nu_D \colon \widetilde{D} \to D$ be the normalization, and let $\tilde{\pi} \colon \widetilde{C} \to \widetilde{D} = \widetilde{C}/G$ be the quotient map. Then $\widetilde{D} \setminus \tilde{\pi}(\widetilde{S} \cup \widetilde{T}) = D \setminus (\overline{S} \cup \overline{T})$. By Proposition 3.8, we have

$$\langle [1_G], \chi_G(\nu^* \omega_C(T)^{\otimes m}) \rangle = \chi(\omega_{\widetilde{D}}(\tilde{\pi}_*(\widetilde{S} + \widetilde{T}))^{\otimes m}) + \sum_{Q \in D \setminus (\overline{S} \cup \overline{T})} \left\lfloor m \left(1 - \frac{1}{e_Q}\right) \right\rfloor. \tag{4.13}$$

Note that, $\omega_{\widetilde{D}}(\tilde{\pi}_*(\widetilde{S} + \widetilde{T}))^{\otimes m} = \nu_D^* \omega_D(\overline{S}_2 + \overline{T})^{\otimes m}$, hence there is a short exact sequence

$$0 \to \omega_D(\overline{S}_2 + \overline{T})^{\otimes m} \to \nu_{D*} \omega_{\widetilde{D}}(\tilde{\pi}_*(\widetilde{S} + \widetilde{T}))^{\otimes m} \to \mathcal{S}_D \to 0$$

where $\mathcal{S}_D = \oplus_{Q \in \overline{S}_1} k_Q$ is the skyscraper sheaf supported on the singular locus $\overline{S}_1$ of D. It follows that

$$\begin{aligned}\chi(\omega_{\widetilde{D}}(\tilde{\pi}_*(\widetilde{S} + \widetilde{T}))^{\otimes m}) &= \chi(\omega_D(\overline{S}_2 + \overline{T})^{\otimes m}) + \# \overline{S}_1 \\ &= \chi(\omega_D(\overline{T})^{\otimes m}) + \# \overline{S}_1 + m \# \overline{S}_2.\end{aligned} \tag{4.14}$$

Combining the equalities (4.11), (4.12), (4.13), and (4.14), we have

$$\begin{aligned}\langle [1_G], \chi_G(\omega_C(T)^{\otimes m}) \rangle &= \langle [1_G], \chi_G(\nu^* \omega_C(T)^{\otimes m}) \rangle - \langle [1_G], [\mathcal{S} \otimes \omega_C(T)^{\otimes m}] \rangle \\ &= \chi(\omega_D(\overline{T})^{\otimes m}) + (m - \epsilon_m) \# \overline{S}_2 + \sum_{Q \in D \setminus (\overline{S} \cup \overline{T})} \left\lfloor m \left(1 - \frac{1}{e_Q}\right) \right\rfloor.\end{aligned}$$

□

Corollary 4.14. *Keep the hypothesis of Theorem 4.13. Suppose further that C is stable and $G \subseteq \mathrm{Aut}(C)$ is a subgroup of order non divisible by the characteristic of k. Then for any positive integer m such that $m + |T| \geq 2$, we have*

$$\begin{aligned}\dim_k H^0(C, \omega_C(T)^{\otimes m})^G = \chi(\omega_D(\overline{T})^{\otimes m}) + (m - \epsilon_m) \# \overline{S}_2 \\ + \sum_{Q \in D \setminus (\overline{S} \cup \overline{T})} \left\lfloor m \left(1 - \frac{1}{e_Q}\right) \right\rfloor\end{aligned} \tag{4.15}$$

with $\epsilon_m = 1$ if m is odd and 0 if m is even.

Proof. Since $|G| \in k^*$, the isomorphism classes of G-modules V are determined by their classes $[V]$ in $R_k(G)$. Also, we have $\# S_2 = 0$ if $\mathrm{char}\, k = 2$. Since $m + |T| \geq 2$, we have either $m \geq 2$ or $|T| > 0$. In both cases, we have by Serre duality $H^1(C, \omega_C(T)^{\otimes m}) = H^0(C, (\omega_C^{\otimes (m-1)}(mT))^\vee) = 0$, where the second equality holds when $m \geq 2$ because C is stable and hence ω_C is ample, and when $m = 1$ because

C is connected. With this vanishing, the equality (4.15) follows from Theorem 4.13. $\square$

4.4. **The equivariant deformation space of a stable G-curve.** Let the notation be as in Set-ups 3.1 and 4.1. Suppose further that C is stable and $|G| \in k^*$. If G acts freely on a dense open subset of C, then the equivariant deformation of G-equivariant stable curves (C, G) is unobstructed, and the tangent space of the G-equivariant deformation space $\mathrm{Def}(C, G)$ is $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G$, where Ω_C denotes the sheaf of Kähler differentials on C ([Tuf93]). The more general case, where the action is possibly not free on a dense open subset, has been studied in [Mau06, Cat12, Li21]. See also [Kon07] for some information when $|G|$ is divisible by $\mathrm{char}(k)$.

Based on Theorem 4.13, the following theorem computes $\dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G$.

Theorem 4.15. *Let C be a stable curve endowed with the tame action of a finite group G. Let $S_3 \subseteq S$ be the subset of singular points that are smoothable with respect to the G-action, as in Remark 4.5. Let $B \subset D_{\mathrm{sm}}$ be the set of branch points of $C_{\mathrm{sm}} \to \pi(C_{\mathrm{sm}})$, $\overline{S}_i \subset D$ be the image of S_i for $1 \le i \le 3$. Then*

$$\dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G = 3(p_a(D) - 1) + \#B + 2\#\overline{S}_2 - \#\overline{S} + \#\overline{S}_3. \tag{4.16}$$

We first establish some preliminary results.

Lemma 4.16. *Let C be a nodal curve over k as in Set-ups 3.1 and 4.1. Endow $S \subset C$ with the reduced structure.*

(1) *We have a canonical G-equivariant exact sequence*

$$0 \to \Omega_{C,\mathrm{tor}} \to \Omega_C \xrightarrow{\phi} \omega_C \to \omega_C|_S \to 0, \tag{4.17}$$

where $\Omega_{C,\mathrm{tor}}$ is the sheaf of differential forms annihilated (locally) by regular elements in $\mathcal{O}_C$, and we have $\Omega_{C,\mathrm{tor}} \simeq \mathcal{O}_S$ as $\mathcal{O}_C$-modules (not G-equivariant in general).

(2) *We have a G-equivariant k-linear isomorphism*

$$\Omega_{C,\mathrm{tor}} \simeq \det(\Omega_C|_S).$$

(3) *Let $P \in S$. The action of G_P on $(\Omega_{C,\mathrm{tor}})_P \otimes \omega_C$ is trivial if and only if (C, G) is smoothable at P.*

Proof. (1)-(2) Let $\mathcal{I} \subset \mathcal{O}_C$ be the ideal defining S in C. Note that $\mathcal{I}\nu_*\mathcal{O}_{\widetilde{C}} \subseteq \mathcal{O}_C$ by Lemma 4.2(1). We have canonical maps

$$\phi\colon \Omega_C \to \nu^*\Omega_C \to \omega_{\widetilde{C}} = \mathcal{I}\nu^*(\omega_C) \simeq \mathcal{I}\omega_C$$

(see Isomorphism (4.3)). This gives (4.17) as a complex. To check its exactness, it is enough to check the exactness at the $P \in S$. Let $U \to C$ be a morphism étale at some point $Q \in U$ lying over P (the pair (U, Q) is a *pointed étale neighborhood of* (C, P)). Then Q is a node because $\widehat{\mathcal{O}}_{U,Q} \simeq \widehat{\mathcal{O}}_{C,P}$. As all the items in (4.17) commute with étale base changes on C, the sequence is exact at P if and only if the corresponding complex for U is étale at Q.

Let $V = \mathrm{Spec}(k[x, y]/(xy))$ and $R \in V$ the origin. As $\widehat{\mathcal{O}}_{C,P} \simeq \widehat{\mathcal{O}}_{V,R}$, there exists a common pointed étale neighborhood $(U, Q) \to (C, P)$ and $(U, Q) \to (V, R)$ ([Stk25, Tag 0CAV]). Therefore it is enough to check the exactness of (4.17) for V at R. In this case $\mathcal{I}_R = (x, y)$, $\Omega_{V,R} = \mathcal{O}_{V,R}dx + \mathcal{O}_{V,R}dy$ and $\omega_{V,R} = \mathcal{O}_{V,R}dx/x$

with $dx/x = -dy/y$. Finally $(\omega_{V,\mathrm{tor}})_R = k.xdy = k.ydx$. Then (1) follows from straightforward computations.

(2) Let $d\colon \Omega_C \to \Omega_C \wedge \Omega_C$ be the k-linear exterior derivation [Stk25, Tag 0FKL]. Consider the composition map

$$\psi\colon \Omega_{C,\mathrm{tor}} \to \Omega_C \xrightarrow{d} \Omega_C \wedge \Omega_C \to \Omega_C|_S \wedge \Omega_C|_S = \det(\Omega_C|_S).$$

(for all $P \in S$, $\dim_k \Omega_C|_P = \dim_k \Omega_V|_R = 2$ with the notation of (1).) It is clearly G-equivariant, and as in (1), compatible with étale base changes. So it is enough to show that it is an isomorphism for the pointed curve (V, R) as in (1). As $\Omega_V|_R = kdx \oplus kdy$ and $\psi_R(xdy) = dx \wedge dy$, ψ_R is an isomorphism.

(3) Let $P \in S$. By (1), the canonical G_P-equivariant k-linear map

$$\Omega_C|_P \to (\nu_* \omega_{\widetilde{C}})|_P = \omega_{\widetilde{C}}|_{\widetilde{P}_1} \oplus \omega_{\widetilde{C}}|_{\widetilde{P}_2} \tag{4.18}$$

is surjective, hence an isomorphism for dimension reasons. It follows that (C, G) is smoothable at P if and only $\det[\Omega_C|_P] = [1_{G_P}]$ if $P \in S_1$ and $\det[\Omega_C|_P] = \chi_P$ (see Equality (4.2) for the notation) if $P \in S_2$. The conclusion then follows from Corollary 4.3. □

Lemma 4.17. *Let C be a projective nodal curve over k.*

(1) *Let $\mathcal{F}$ be a coherent sheaf on C. Then we have a canonical isomorphism*
$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_C) \simeq H^0(C, \mathcal{F} \otimes \omega_C)^\vee.$$
(2) *Let G be a finite group of order $|G| \in k^*$ acting on C. Suppose that $\mathcal{F}$ is a skyscraper G-sheaf. Then*
$$\dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_C)^G = \sum_{P \in C} \frac{|G_P|}{|G|} \dim(\mathcal{F}_P \otimes \omega_C)^{G_P}.$$
(3) *For any invertible sheaf $\mathcal{L}$ on C and any $i > 1$, we have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{L}) = 0$.*

Proof. (1) As ω_C is invertible, we have $\mathrm{Ext}^1(\mathcal{F}, \mathcal{O}_C) \simeq \mathrm{Ext}^1(\mathcal{F} \otimes \omega_C, \omega_C)$ ([Har71, Prop. III.6.7]). Then apply Serre's duality theorem ([Har71, Theorem III.7.6]).

(2) Apply (1) and Proposition 2.6.

(3) We have $\mathrm{Ext}^i(\mathcal{F}, \mathcal{L}) \simeq \mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}^\vee \otimes \omega_C, \omega_C)$. So it is enough to show that $\mathrm{Ext}^i(\mathcal{F}, \omega_C) = 0$ for all coherent sheaves $\mathcal{F}$ and for all $i > 1$.

For all $q \gg 0$ (depending only on C) and all $i > 0$, we have

$$\mathrm{Ext}^i(\mathcal{O}_C(-q), \omega_C) \simeq \mathrm{Ext}^i(\mathcal{O}_C, \omega_C(q)) = H^i(C, \omega_C(q)) = 0.$$

Consider the contravariant δ-function $(T^i)_{i \geq 0}$ where $T^i = H^{1-i}(C, .)^\vee$ if $i = 0, 1$ and $T^i \equiv 0$ if $i \geq 2$. We can proceed exactly as for the proof of Serre's duality ([Har71, Theorem III.7.6]) to show that the δ-functor $(\mathrm{Ext}^i(., \omega_C))_{i \geq 0}$ coincides with $(T^i)_{i \geq 0}$. Therefore $\mathrm{Ext}^i(\mathcal{F}, \omega_C) = 0$ if $i > 1$. □

Proof of Theorem 4.15. We compute the dimension of $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G$ by comparing it with the dimension of $\mathrm{Ext}^1(\omega_C, \mathcal{O}_C)^G$. Let $\mathcal{I} \subset \mathcal{O}_C$ be the ideal defining the reduced subvariety $S \subset C$. By Lemma 4.16, we have an exact sequence.

$$0 \to \Omega_{C,\mathrm{tor}} \to \Omega_C \to \mathcal{I}\omega_C \to 0.$$

By [DM69, Lemma 1.4], $\mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) = 0$. Moreover $\Omega_{C,\mathrm{tor}}$ has finite support, hence $\mathrm{Ext}^0(\Omega_{C,\mathrm{tor}}, \mathcal{O}_C) = 0$. Thus, taking the $\mathrm{Ext}^i(\cdot, \mathcal{O}_C)$-groups we obtain

$\mathrm{Ext}^0(\mathcal{I}\omega_C, \mathcal{O}_C) = 0$ and the exact sequence

$$0 \to \mathrm{Ext}^1(\mathcal{I}\omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\Omega_{C,\mathrm{tor}}, \mathcal{O}_C) \to 0. \tag{4.19}$$

Similarly, taking $\mathrm{Ext}^i(., \mathcal{O}_C)$ in the exact sequence $0 \to \mathcal{I}\omega_C \to \omega_C \to \omega_C|_S \to 0$ we get the exact sequence

$$0 \to \mathrm{Ext}^1(\omega_C|_S, \mathcal{O}_C) \to \mathrm{Ext}^1(\omega_C, \mathcal{O}_C) \to \mathrm{Ext}^1(\mathcal{I}\omega_C, \mathcal{O}_C) \to 0. \tag{4.20}$$

Combining Exact sequences (4.19) and (4.20), and taking G-invariants (which is exact as $|G| \in k^*$), we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1(\omega_C|_S, \mathcal{O}_C)^G \to \mathrm{Ext}^1(\omega_C, \mathcal{O}_C)^G \to \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G \to \\ \to \mathrm{Ext}^1(\Omega_{C,\mathrm{tor}}, \mathcal{O}_C)^G \to 0. \end{aligned}$$

It follows that

$$\begin{aligned} \dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G = \dim \mathrm{Ext}^1(\omega_C, \mathcal{O}_C)^G - \dim \mathrm{Ext}^1(\omega_C|_S, \mathcal{O}_C)^G \\ + \dim \mathrm{Ext}^1(\Omega_{C,\mathrm{tor}}, \mathcal{O}_C)^G. \end{aligned} \tag{4.21}$$

By Lemma 4.17(1) and Theorem 4.13, we have

$$\dim \mathrm{Ext}^1(\omega_C, \mathcal{O}_C)^G = \dim \left(H^0(C, \omega_C^{\otimes 2})^\vee\right)^G = 3(p_a(D) - 1) + \#B + 2\#\overline{S}_2.$$

For each $P \in S$, by Lemma 4.17(2), we have

$$\dim \mathrm{Ext}^1(\omega_C|_S, \mathcal{O}_C)^G = \sum_{P \in S} \frac{|G_P|}{|G|} \dim(\omega_C^{\otimes 2}|_P)^{G_P} = \#\overline{S}.$$

Similarly, the third term of (4.21) is

$$\dim \mathrm{Ext}^1(\Omega_{C,\mathrm{tor}}, \mathcal{O}_C)^G = \sum_{P \in S} \frac{|G_P|}{|G|} \dim((\Omega_{C,\mathrm{tor}})_P \otimes \omega_C)^{G_P}$$

is exactly the number $\#\overline{S}_3$ of G-orbits $G.P$ of smoothable nodal points P by Lemma 4.16 (4) (Note that $(\Omega_{C,\mathrm{tor}})_P \otimes \omega_C$ has dimension 1.)

Plugging these observations into (4.21), we obtain the desired formula (4.16) for $\dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G$. □

Remark 4.18. If $S = S_3$, that is, if (C, G) is smoothable, then (4.16) becomes a formula similar to the smooth case:

$$\dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)^G = 3(p_a(D) - 1) + \#B + 2\#\overline{S}_2.$$

4.5. **Relations between $H^1(C, \mathbb{C})$ and $H^0(C, \omega_C)$.** Let the notation be as in Setup 4.1. In this subsection, we assume that the base field $k = \mathbb{C}$. If C is smooth, then the Hodge decomposition gives the desired relation between $H^1(C, \mathbb{C})$ and $H^0(C, \omega_C)$, as G-modules:

$$H^1(C, \mathbb{C}) \simeq H^0(C, \omega_C) \oplus H^1(C, \mathcal{O}_C) \simeq H^0(C, \omega_C) \oplus H^0(C, \omega_C)^\vee. \tag{4.22}$$

We explore relations between $H^1(C, \mathbb{C})$ and $H^0(C, \omega_C)$ when C has at most nodal singularities. There are two G-equivariant short exact sequences of sheaves of $\mathbb{C}$-vector spaces:

$$0 \to \mathbb{C}_C \to \nu_* \mathbb{C}_{\widetilde{C}} \to \oplus_{P \in S} \mathcal{S}_P \to 0 \tag{4.23}$$

and

$$0 \to \nu_* \omega_{\widetilde{C}} \to \omega_C \to \oplus_{P \in S} \omega_C|_P \to 0 \tag{4.24}$$

where S is the set of singular points of C.

Proposition 4.19. *We have*

$$\left[H^1(C,\mathbb{C})\right]-\left[H^0(C,\omega_C)\right]=\left[H^1(\widetilde{C},\mathbb{C})\right]-\left[H^0(\widetilde{C},\omega_{\widetilde{C}})\right]=\left[H^1(\widetilde{C},\mathcal{O}_{\widetilde{C}})\right]$$

Proof. By (4.23), we have

$$\begin{aligned}\left[H^0(C,\mathbb{C})\right]-\left[H^1(C,\mathbb{C})\right]+\left[H^2(C,\mathbb{C})\right]\\ =\left[H^0(\widetilde{C},\mathbb{C})\right]-\left[H^1(\widetilde{C},\mathbb{C})\right]+\left[H^2(\widetilde{C},\mathbb{C})\right]-\left[\bigoplus_{P\in S}\mathcal{S}_P\right]\end{aligned}\tag{4.25}$$

By (4.24), we obtain

$$\begin{aligned}\left[H^0(C,\omega_C)\right]-\left[H^1(C,\omega_C)\right]\\ =\left[H^0(\widetilde{C},\omega_{\widetilde{C}})\right]-\left[H^1(\widetilde{C},\omega_{\widetilde{C}})\right]+\left[\bigoplus_{P\in S}\omega_C|_P\right]\end{aligned}\tag{4.26}$$

By Serre duality and Corollary 4.3,

$$\begin{aligned}\left[H^2(C,\mathbb{C})\right]=\left[H^2(\widetilde{C},\mathbb{C})\right],\quad \left[H^0(C,\mathbb{C})\right]=\left[H^1(C,\omega_C)\right],\\ \left[H^0(\widetilde{C},\mathbb{C})\right]=\left[H^1(\widetilde{C},\omega_{\widetilde{C}})\right],\quad \left[\bigoplus_{P\in S}\mathcal{S}_P\right]=\left[\bigoplus_{P\in S}\omega_C|_P\right]\end{aligned}\tag{4.27}$$

Combining (4.25), (4.26), and (4.27), we obtain the desired equalities

$$\left[H^1(C,\mathbb{C})\right]-\left[H^0(C,\omega_C)\right]=\left[H^1(\widetilde{C},\mathbb{C})\right]-\left[H^0(\widetilde{C},\omega_{\widetilde{C}})\right]=\left[H^1(\widetilde{C},\mathcal{O}_{\widetilde{C}})\right].$$

□

Corollary 4.20. *If the components of C are all rational, then $H^1(C,\mathbb{C})=H^0(C,\omega_C)$ as G-modules.*

Remark 4.21. For each $i\geq 0$, we have $H^i(C,\mathbb{C})\simeq H^i(C,\mathbb{C})^\vee$ as G-modules, that is, $H^i(C,\mathbb{C})$ is a self-dual G-representation. This is due to the fact that G acts on $H^i(C,\mathbb{Z})$, and $H^i(C,\mathbb{C})=H^i(C,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$, and real G-representations are self-dual ([CR62, Theorem 31.21 and Definition 43.7] and [FH91, Section 3.5]).

4.6. **The induced action on the dual graph of a nodal G-curve.** Resume the notation from Set-up 4.1. Let C be a connected nodal curve over k, and $G\subseteq \mathrm{Aut}(C)$ a finite subgroup of automorphisms. One can define its dual graph $\Gamma=(V,E)$:

(1) Each irreducible component C_i of C corresponds to a vertex, denoted by $\langle C_i\rangle$.
(2) Each node P of C corresponds to an open edge, denoted by $\langle P\rangle$, connecting the vertices C_i and C_j such that $P\in C_i\cap C_j$.

We view Γ as a cell complex with the S as the set of the 0-cells, and $\{C_i \mid 1\leq i\leq n\}$ as the set of 1-cells. The cellular chain complex of Γ reads

$$C_\bullet(\Gamma): 0\to C_1(\Gamma)\to C_0(\Gamma)\to 0$$

with $C_1(\Gamma)=\oplus_P\mathbb{Z}\langle P\rangle$ and $C_0(\Gamma)=\oplus_i\mathbb{Z}\langle C_i\rangle$. Note that there is an induced action of $\mathrm{Aut}(C)$ on Γ and $C_\bullet(\Gamma)$. For a node $P\in C$, the stabilizer $G_{\langle P\rangle}$ of the edge $\langle P\rangle$ is the same as the stabilizer G_P of the point P, and for $g\in G_P$, we have

$$g(\langle P\rangle)=\begin{cases}\langle P\rangle, & \text{if } G_P \text{ preserves the local branches of } C\ni P\\ -\langle P\rangle, & \text{if } G_P \text{ interchanges the local branches of } C\ni P\end{cases}$$

Here, as usual in algebraic topology, $-\langle P\rangle$ can be understood as the path traversing in the opposite direction of P. This action does not make Γ into a G-CW complex in the strict sense, because the points of the edges (or 1-cell) can have different stabilizers. Nevertheless, by the additivity of the Euler characteristic, we have an equality in the Grothendieck group $R_k(G)$:

$$[C_0(\Gamma)\otimes_{\mathbb{Z}}k]-[C_1(\Gamma)\otimes_{\mathbb{Z}}k]=[H_0(\Gamma,k)]-[H_1(\Gamma,k)]. \tag{4.28}$$

Lemma 4.22. *Let $\nu:\widetilde{C}\to C$ be the normalization. Then there are identifications of G-representations:*

$$C_0(\Gamma)\otimes_{\mathbb{Z}}k\simeq H^1(\widetilde{C},\omega_{\widetilde{C}}),\quad C_1(\Gamma)\otimes_{\mathbb{Z}}k\simeq\bigoplus_{P\in S}\omega_C|_P \tag{4.29}$$

Proof. Let $\{[C_{i_j}]\mid 1\le j\le l\}$ be a system of representatives of the G-orbits in the set of components $\{[C_i]\mid 1\le i\le n\}$. Then we have isomorphisms of G-representations

$$H^1(\widetilde{C},\omega_{\widetilde{C}})=\bigoplus_i H^1(\widetilde{C}_i,\omega_{\widetilde{C}_i})\simeq\bigoplus_{1\le j\le l}\mathrm{Ind}_{G_{[C_{i_j}]}}^G 1_{G_{[C_{i_j}]}}\simeq C_0(\Gamma)\otimes_{\mathbb{Z}}k.$$

Let $\{P_j\mid 1\le j\le s\}$ be a system of representatives of the G-orbits in S. Then we have isomorphisms of G-representations

$$\bigoplus_{P\in S}\omega_C|_P\simeq\bigoplus_{j=1}^m\mathrm{Ind}_{G_{P_j}}^G\omega_C|_{P_j}\simeq C_1(\Gamma)\otimes_{\mathbb{Z}}k$$

□

We may interpret the difference between $[H^0(C,\omega_C)]$ and $[H^0(\widetilde{C},\omega_{\widetilde{C}})]$ using the action of G on the dual graph Γ of C.

Corollary 4.23. *Let C be a connected nodal curve and $G\subseteq\mathrm{Aut}(C)$ a finite subgroup. Let $\nu\colon\widetilde{C}\to C$ be the normalization. Then we have the following equality in $R_k(G)$:*

$$[H^0(C,\omega_C)]=[H^0(\widetilde{C},\omega_{\widetilde{C}})]+[1_G]-\chi_G(\Gamma,k)$$

where $\chi_G(\Gamma,k):=[H_0(\Gamma,k)]-[H_1(\Gamma,k)]\in R_k(G)$.

Proof. Taking the long exact sequence of cohomology groups associated to the short exact sequence (4.7) and noting that $[H^1(C,\omega_C)]=[1_G]$, we get

$$[H^0(C,\omega_C)]=[H^0(\widetilde{C},\omega_{\widetilde{C}})]+[1_G]-[H^1(\widetilde{C},\omega_{\widetilde{C}})]+[\omega_C|_S].$$

By (4.28) and (4.29) we have $[H^1(\widetilde{C},\omega_{\widetilde{C}})]-[\omega_C|_S]=\chi_G(\Gamma,k)$. This proves the corollary. □

5. G-MODULE STRUCTURE ON THE SINGULAR COHOMOLOGY OF G-CW COMPLEXES

Theorem 5.1. *Let X be an n-dimensional compact complex space, and $G \subseteq \operatorname{Aut}(X)$ a finite subgroup of automorphisms. Let $Y = X/G$ be the quotient space and $\pi\colon X \to Y$ the quotient map. For each subgroup $H \subseteq G$, denote*

$$X_H := \{x \in X \mid G_x \text{ conjugate to } H\} \text{ and } Y_H := \pi(X_H).$$

Let $H_1, \ldots, H_s$ be a system of representatives of conjugacy classes of subgroups in G. Then the following equality holds in $R_{\mathbb{Q}}(G)$:

$$\sum_{0\le i\le 2n} (-1)^i [H^i(X, \mathbb{Q})] = \sum_{1\le j\le s} \chi(Y_{H_j}) \operatorname{Ind}_{H_j}^G [1_{H_j}]. \tag{5.1}$$

where for a complex space Z, $\chi(Z)$ denotes its topological Euler characteristic; if $Y_{H_j} = \emptyset$, we set $\chi(Y_{H_j}) = 0$.

Proof. As in the proof of [CL18, Theorem 2.3], there is a finite G-equivariant triangulation $\mathcal{T}$ of X. In particular, for each simplex $\Delta \in \mathcal{T}$ and for all $x \in \Delta^0$, we have

$$G_x = G_{\Delta^0} = G_{\{\Delta\}},$$

where Δ^0 denotes the relative interior of Δ. Then, X_{H_j} is a union of relatively open simplices of the triangulation $\mathcal{T}$ for each $1 \le j \le s$.

For each $0 \le i \le 2n$, let $\mathcal{T}_i \subseteq \mathcal{T}$ be the set of i-dimensional simplices, and

$$C_i(\mathcal{T}) = \bigoplus_{\Delta\in\mathcal{T}_i} \mathbb{Q}[\Delta]$$

the i-th cellular chain group. Then, by the G-equivariance of $\mathcal{T}$, there is an induced action of G on $C_i(\mathcal{T})$ for each i, such that $C_i(\mathcal{T}) \to C_{i-1}(\mathcal{T})$ is G-equivariant. It follows that there is an equality in $R_{\mathbb{Q}}(G)$:

$$\sum_{0\le i\le 2n} (-1)^i [H^i(X, \mathbb{Q})] = \sum_{0\le i\le 2n} (-1)^i [C_i(\mathcal{T})]. \tag{5.2}$$

For each orbit of a simplex $\Delta \in \mathcal{T}$, we obtain on the right-hand side of (5.2) a direct summand $\operatorname{Ind}_{H_j}^G [1_{H_j}]$, where $H_j \subseteq G$ is the subgroup conjugate to $G_{\{\Delta\}}$. Summing over all the orbits of G in $\mathcal{T}$, we obtain the desired equality (5.1):

$$\sum_{0\le i\le 2n} (-1)^i [C_i(\mathcal{T})] = \sum_{1\le j\le s} \chi(Y_{H_j}) \operatorname{Ind}_{H_j}^G [1_{H_j}].$$

□

Corollary 5.2. *Let $C = \bigcup_{1\le i\le n} C_i$ be a compact nodal curve over $\mathbb{C}$, and $G \subseteq \operatorname{Aut}(C)$ a finite subgroup of automorphisms. Let $\pi\colon C \to D := C/G$ be the quotient map. Let S be the singular locus of C. Set $T := \{P \in C \setminus S \mid G_P \supsetneq I_i \text{ for } C_i \ni P\}$ and $A := S \cup T$. Write $D = \bigcup_{1\le j\le m} D_j$ as the union of irreducible components. Then we have*

$$\sum_{i=0}^{2} (-1)^i [H^i(C, \mathbb{Q})] = \sum_{j=1}^{m} \chi(D_j \setminus \pi(A)) \operatorname{Ind}_{I_i}^G [1_{I_i}] + \sum_{P\in A'} \operatorname{Ind}_{G_P}^G [1_{G_P}], \tag{5.3}$$

where A' denotes a set of representatives of the G-orbits in A.

The following corollary is a direct generalization of [Bro87, Proposition 2], which deals with the case when C is a connected smooth curve.

Corollary 5.3. *Let the notation be as in Corollary 5.2. Suppose that I_i is trivial for each component C_i. Then*

$$\sum_{i=0}^{2}(-1)^i[H^i(C,\mathbb{Q})] = \sum_{j=1}^{m}\chi(D_j\setminus\pi(A))\,[\mathbb{Q}[G]] + \sum_{P\in A'}\mathrm{Ind}_{G_P}^{G}\,[1_{G_P}]\,.$$

In particular, if C is smooth and connected, then

$$[H^1(X,\mathbb{Q})] = (2g(D)-2+\#\pi(A))[\mathbb{Q}[G]] + \sum_{P\in A'}\mathrm{Ind}_{G_P}^{G}\,[1_{G_P}]$$

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