Congruences of models of elliptic curves

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Abstract
Let $O_K$ be a discrete valuation ring with field of fractions $K$ and perfect residue field. Let $E$ be an elliptic curve over $K$, let $L/K$ be a finite Galois extension and let $O_L$ be the integral closure of $O_K$ in $L$. Denote by $\mathcal{X}'$ the minimal regular model of $E_L$ over $O_L$. We show that the special fibers of the minimal Weierstrass model and the minimal regular model of $E$ over $O_K$ are determined by the infinitesimal fiber $X'_m$ together with the action of $\text{Gal}(L/K)$, when $m$ is big enough (depending on the minimal discriminant of $E$ and the different of $L/K$).

1. Introduction
Let $O_K$ be a discrete valuation ring with field of fractions $K$ and perfect residue field. Let $E$ be an elliptic curve over $K$. The minimal (projective) regular model $X$ of $E$ over $O_K$ encodes interesting arithmetical invariants of $E$ (e.g. the conductor of $E$, and the smooth locus of $X$ is the Néron model of $E$). It is then important to be able to determine this model. Let $L/K$ be a finite Galois extension with Galois group $G$ and let $O_L$ be the integral closure of $O_K$ in $L$ (this is a semilocal Dedekind domain). Let $X'$ be the minimal regular model of $E_L$ over $O_L$. By the uniqueness of minimal regular models, $G$ acts on the $O_K$-scheme $X'$. It is well known that there exists $L$ as above such that $X'$ is semi-stable. When $L/K$ is moreover tamely ramified and $K$ is complete, Viehweg [29] (for curves of any genus $\geq 1$) showed that the type of the special fiber $X_0$ of $X'$ is determined by the action of $G$ on the special fiber of $X'$.

In the present work, we consider wildly ramified extensions $L/K$ for elliptic curves. We will not suppose $E_L$ has semi-stable reduction, even though this is probably the most interesting situation. For any $O_K$-scheme $Z$ and for any integer $N \geq 0$, we will denote as usual

$$Z_N := Z \times_{\text{Spec} O_K} \text{Spec}(O_K/\pi^{N+1}O_K)$$

where $\pi$ is a uniformizing element of $O_K$. For any $O_L$-scheme $Z'$, the infinitesimal fiber $Z'_N$ is by definition

$$Z'_N := Z' \times_{\text{Spec} O_K} \text{Spec}(O_K/\pi^{N+1}O_K).$$

In §2, Examples 2.1 and 2.2, we exhibit for any positive integer $l$, two elliptic curves over $K$ having isomorphic $(X'_l, G)$ but with non-isomorphic special fibers $X_0$. Hence Viehweg’s result can not be extended directly to the wild ramification case. A natural question, attributed to B. Mazur and pointed out to us by W. McCallum, is whether $X_0$ is determined by $(X'_l, G)$ for $l$ big enough. We give a positive answer in the present work:

**Theorem 7.3** Let $N \geq 0$. Let $\Delta$ be the minimal discriminant of $E$. Let $\mathfrak{D}_{L/K}$ be the different of $L/K$. Then the scheme $X_N$ is determined by $X'_N$ and the action of $G$ on $X'_N$ for $\ell = 2v_{K}(\Delta) + 12[v_{K}(\mathfrak{D}_{L/K})] + 18$.

If the reduction type of $E$ is neither $I^*_r$ nor $I_r$ with $r > 0$ (e.g. if $E$ has potentially good reduction and the residue characteristic of $K$ is different from 2), we can find such $l$ depending

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only on \([v_K(\mathcal{D}_{L/K})]\). Note that \([v_K(\mathcal{D}_{L/K})]\) is bounded by a constant depending only on the absolute ramification index of \(K\) and on the degree \([L : K]\) when \(\text{char}(K) = 0\).

At this stage, let us make precise the meaning of \(\sim_X\) is determined by \(\mathcal{X}_N\) and the action of \(G\) on \(\mathcal{X}_N\). Let \(E_o\) be an elliptic curve over \(K_o\), let \(L_o\) be a finite Galois extension of \(K_o\) of the same Galois group \(G\). Let \(X_0, X'_0\) be the respective minimal regular models of \(E_o\) over \(\mathcal{O}_{K_o}\) and of \((E_o)L_o\) over \(\mathcal{O}_{L_o}\). We say that

\[
\mathcal{X}_N \text{ is determined by } \mathcal{X}'_N \text{ and the action of } G \text{ on } \mathcal{X}'_N \text{ if the existence of an isomorphism } \psi_{N+\ell} : \mathcal{O}_K/\pi^{N+\ell+1}\mathcal{O}_K \simeq \mathcal{O}_{K_o}/\pi_o^{N+\ell+1}\mathcal{O}_{K_o} \text{ and of } G\text{-equivariant isomorphisms }
\]

\[
\mathcal{O}_L/\pi^{N+\ell+1}\mathcal{O}_L \simeq \mathcal{O}_{L_o}/\pi_o^{N+\ell+1}\mathcal{O}_{L_o}, \quad \mathcal{X}'_{N+\ell} \simeq \mathcal{X}'_{o,N+\ell}
\]

implies the existence of an isomorphism \(\mathcal{X}_N \simeq \mathcal{X}_{o,N}\) compatible with the isomorphism \(\mathcal{O}_K/\pi^{N+1}\mathcal{O}_K \simeq \mathcal{O}_{K_o}/\pi_o^{N+1}\mathcal{O}_{K_o}\) induced by \(\psi_{N+\ell}\). We define similar notion for minimal Weierstrass models.

Let us present the organization of this paper. In §2 we construct the examples mentioned above. Section 3 is a technical preliminary work. We study the invariants of an \(\mathcal{O}_L\)-module under a semi-linear action. In §4 and 5, we study the minimal Weierstrass model \(W\) of \(E\) over \(\mathcal{O}_K\), as well as the fibers \(W_N\) in relation with the action of \(G\) on the minimal Weierstrass model of \(E_L\) over \(\mathcal{O}_L\).

In §6, we study the relation between \(W_{N+\ell}\) and \(\mathcal{X}_N\). It is known that \(\mathcal{X}\) is obtained by a sequence of blowing-ups starting with \(W\). We show in Theorem 6.4 that if \(\mathcal{Y} \rightarrow \mathcal{Y}\) is the blowing-up morphism along a closed point in a scheme \(\mathcal{Y}\) over \(\mathcal{O}_K\), there exists an explicit integer \(\ell \geq 0\) such that \(\mathcal{Y}_{N+\ell}\) determines \((\mathcal{Y})_N\) for all \(N \geq 0\). We then apply this result to \(W\) and show that \(W_{N+\ell}\) determines \(\mathcal{X}_N\) for some explicit constant \(\ell\) (Corollary 6.7).

The main result Theorem 7.3 is proved in §7 using the connection between \(\mathcal{X}\) and the minimal Weierstrass model \(W\) of \(E\). The proof can be divided into three steps:

1. Let \(W'\) be the minimal Weierstrass model of \(E_L\) over \(\mathcal{O}_L\). We show that the \(G\)-action on \(\mathcal{X}_{N+\ell_1}\) determines the \(G\)-action on \(W'_{N+\ell_1}\) in Proposition 7.2.

2. We prove that the \(G\)-action on \(W'_{N+\ell_1}\) determines \(W_{N+\ell_2}\) if \(\ell_2 \ll \ell_1\) (Theorem 5.5). This is the crucial part. We can choose \(\ell_1 - \ell_2\) such that it depends only on the valuation of the different of \(L/K\).

3. Finally, as we described above, \(W_{N+\ell_2}\) determines \(\mathcal{X}_N\) for some \(\ell_2 \geq 0\) (Corollary 6.7).

As we always work with pointed schemes \(W'_{N}, \mathcal{X}'_{N}\), in the last section, we show that a Galois invariant section of such a fiber lifts to a Galois invariant section over Spec \(\mathcal{O}_L\) when \(N \gg 0\) (Proposition 8.1). If we use Néron models, then we get an explicit bound on \(N\) (Proposition 8.3).

We should mention that the present work is similar to (and inspired by) Chai-Yu and Chai’s articles [6], [5] where they dealt with Néron models of semi-abelian varieties, though we use a more down-to-earth method. It is shown in [5], Theorem 7.6, that for any semi-abelian variety \(A\) over \(K\), the infinitesimal fiber \(\mathcal{A}_N\) of the finite type Néron model \(A\) of \(A\) over \(\mathcal{O}_K\) is determined by the \(G\)-action on \(\mathcal{A}'_{N+\ell}\) (where \(\mathcal{A}'\) is the Néron model of \(A_L\) over \(\mathcal{O}_{L_L}\)) for \(\ell\) big enough and depending on \(\mathcal{A}'\). Related to his work is the computation, in case of elliptic curves, of the base change conductor (4.11).

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The smooth model $\mathcal{O}_K$ is the valuation ring of $K$, $\pi$ denotes a uniformizing element of $K$ and $v_K$ is the normalized valuation $(v_K(\pi) = 1)$, $E$ is an elliptic curve over $K$ and $\mathcal{X}$ (resp. $W$) denotes its minimal projective regular (resp. minimal Weierstrass) model over $\mathcal{O}_K$.

We will suppose the residue field $k$ is perfect starting §5.1.

We will denote by $L/K$ a finite Galois extension with Galois group $G$ and by $\mathcal{O}_L$ the integral closure of $\mathcal{O}_K$ in $L$. As usual, the different of $L/K$ will be denoted by $\mathcal{D}_L/K$. The ramification index of $L/K$ at a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_L$ will be denoted by $e_{L/K}$. The exponent of $\mathcal{D}_L/K$ at $\mathfrak{p}$ will be denoted by $v_L(\mathcal{D}_L/K)$. As $L/K$ is Galois, these invariants are independent on the choice of $\mathfrak{p}$. Sometimes it is convenient to write $v_K(\mathcal{D}_L/K) := v_L(\mathcal{D}_L/K)/e_{L/K} \in \mathbb{Q}$.

2. Two Examples

In this section, we give two examples. The first one shows that, contrary to the tamely ramified case, for any $l \geq 0$, there exist $K$ and $E/K$ such that the special fiber $\mathcal{X}_0$ (resp. $W_1$) is not determined by the $G$-action on $\mathcal{X}'_0$ (resp. the infinitesimal fiber $W'_1$). The second example, of similar nature, shows that in equal characteristic case, there is no bound on $l$ independent on $E$ for which Theorem 5.8 holds.

**Example 2.1.** Let $d \geq 3$ be an odd integer divisible by 3. Let $K = W(\mathbb{F}_2)(\pi)$ with $\pi^d = 2$ where $W(\mathbb{F}_2)$ is the Witt ring of $\mathbb{F}_2$. Let $E$ be the elliptic curve defined by the equation

$$y^2 = x^3 + \pi^3.$$ 

Then $E$ has good reduction over $L = K(\sqrt{\pi})$, with

$$G = \text{Gal}(L/K) = \langle \sigma \rangle, \quad \sigma(\sqrt{\pi}) = -\sqrt{\pi}, \quad \mathcal{D}_L/K = 2\sqrt{\pi}\mathcal{O}_L.$$ 

The smooth model $\mathcal{X}'$ of $E_L$ over $\mathcal{O}_L$ is defined by the equation

$$v^3 + v = u^3,$$

where $x = \pi\sqrt{4u}$ (note that $\sqrt{4} \in K$) and $y = \pi^{3/2}(1 + 2v)$. Hence the action of $G$ on $\mathcal{X}'$ is given by $\sigma(u) = u$, $\sigma(v) = -1 - v$.

Now let $\mathcal{X}_0'$ be the elliptic curve over the same $K$ defined by the equation:

$$y_0^2 = x_0^3 + (1 + \pi).$$

Then $\mathcal{X}_0'$ has good reduction over $L_0 = K(\sqrt{1 + \pi})$, with

$$\text{Gal}(L_0/K) = \langle \sigma \rangle, \quad \sigma(\sqrt{1 + \pi}) = -\sqrt{1 + \pi}, \quad \mathcal{D}_{L_0/K} = 2\mathcal{O}_{L_0}.$$ 

The smooth model $\mathcal{X}'_0$ of $\mathcal{X}_0'$ over $L_0$ is defined by the equation:

$$v_0^2 + v_0 = u_0^3,$$

where $x_0 = (4(1 + \pi))^{1/3}u_0$ with $(1 + \pi)^{1/3} \in K$ and $y_0 = \sqrt{1 + \pi}(1 + 2v_0)$. The action of $G$ on $\mathcal{X}'_0$ is then given by $\sigma(u_0) = u_0$, $\sigma(v_0) = -1 - v_0$.

It is easy to see that we have an isomorphism

$$\mathcal{O}_L/(2) = \mathcal{O}_L/(\pi^d) \cong \mathcal{O}_{L_0}/(\pi^d)$$

which sends $\sqrt{\pi}$ to $\sqrt{1 + \pi} - 1$ and which is compatible with the $G$-action, because modulo 2, the image of $\sigma(\sqrt{\pi}) = -\sqrt{\pi}$ is $-(\sqrt{1 + \pi} - 1) = \sigma(\sqrt{1 + \pi} - 1) + 2 \equiv \sigma(\sqrt{1 + \pi} - 1)$. Note

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1We thank Ivan Fesenko for encouraging us to remove the original hypothesis $k$ algebraically closed.
that $v_K(D_{L/K}) = d + 1/2 \neq v_K(D_{L_o/K}) = d$, hence by Lemma 5.1, $O_L/(\pi^{d+1}) \not\cong O_{L_o}/(\pi^{d+1})$.

We also have an isomorphism

$$\mathcal{X}_{d-1}' \cong \mathcal{X}_{o,d-1}'$$

which sends $u_o$ (resp. $v_o$) to $u$ (resp. $v$) and which is compatible with the $G$-action. However, the special fibers of the minimal regular models of $E$ and $E_o$ over $O_K$ have different Kodaira types: the first curve has type $I'_1$ by Tate’s Algorithm, and the second one has type II. Note that this example doesn’t contradict the conclusion of Theorem 7.3.

Let $W$ (resp. $W_o$) be the minimal Weierstrass model of $E$ (resp. $E_o$) over $O_K$. We have that $\mathcal{X}', \mathcal{X}'_o$ are the respective minimal Weierstrass models over $O_L, O_{L_o}$. Clearly the special fibers of $W, W_o$ are isomorphic, but using Lemma 5.3, we can show that $W_1 \not\cong W_{0,1}$.

**Example 2.2.** Fix $m \geq 1$ and let $r = 1, 3$. Let $k$ be an algebraically closed field of characteristic 2 and $K = k((t))$. Consider the elliptic curve $E$:

$$y^2 + t^{3m} y = x^3 + t^r,$$

whose $j$-invariant is 0 and whose discriminant has valuation equal to $12m$. The above equation defines a minimal Weierstrass model $\mathcal{X}_r$ of $E$ over $k[[t]]$. Let $\alpha_r$ be a root of the polynomial $X^2 + t^{3m}X + t^r$ in $K$ and let $L_r = K[\alpha_r]$. Then $E_{L_r}$ has a smooth model $r, \mathcal{X}'$ over $O_L$, defined by the equation:

$$y'^2 + y' = x^3,$$

where $x = t^{2m}x'$ and $y = t^{3m}y' + \alpha_r$. Hence

$$G = \text{Gal}(L_r/K) = \langle \sigma \rangle, \quad \sigma(\alpha_r) = t^{3m} + \alpha_r; \quad \sigma(x') = x', \quad \sigma(y') = y' + 1.$$

For $r = 1$, the model $1 \mathcal{X}$ is regular, hence $1 E$ has reduction type II. For $r = 3$, the curve $E$ has reduction type $I'_0$ by Tate’s Algorithm. Let $d = 3m - 2$. We have $G$-equivariant isomorphisms

$$O_{L_1}/(t^{3m-1}) \cong O_{L_3}/(t^{3m-1}), \quad (1 \mathcal{X}_1)'_d \cong (3 \mathcal{X}_1)'_d.$$

Hence the $l$ in Theorem 5.8 must be bigger than $3m - 1$ and it tends to infinity if $m$ does.


Let $L/K$ be a finite Galois extension with Galois group $G$. Denote the integral closure of $O_K$ in $L$ by $O_L$. The aim of this section is, given a semi-linear $O_L[G]$-module $M$, to compare $M^G/\pi^{N+1}M^G$ with the image of $(M/\pi^{N+1}M)^G$ in $(M/\pi^{N+1}M^G)$ (Proposition 3.8). The result is used in §4 and §5.

**Definition 3.1.** A **semi-linear $O_L[G]$-module** is an $O_L$-module $M$ (not necessarily finitely generated) endowed with an action of $G$ such that

(i) $g(x_1 + x_2) = g(x_1) + g(x_2)$ for all $x_1, x_2 \in M$ and $g \in G$,

(ii) $g(ax) = g(a)g(x)$ for all $a \in O_L, x \in M$ and $g \in G$.

A morphism $\phi$ between two semi-linear $O_L[G]$-modules $M$ and $N$ is an $O_L$-morphism which is $G$-equivariant (i.e. $\phi(gx) = g\phi(x), \forall g \in G, \forall x \in M$).

Let us recall the following well-known lemma (see for instance [23], Proposition 1(a) for finite dimensional vector spaces; the general situation follows easily):
LEMMA 3.2. (Speiser’s lemma) Let $V$ be a semi-linear $L[G]$-vector space. Then the canonical morphism of semi-linear $L[G]$-vector spaces

$$L \otimes_K V^G \to V$$

is an isomorphism.

PROPOSITION 3.3. Let $M$ be a semi-linear $O_L[G]$-module. Let

$$\varphi: O_L \otimes_K M^G \to M$$

be the natural morphism of semi-linear $O_L[G]$-modules.

1. If $M$ is flat over $O_L$, then $\varphi$ is injective and $\text{rank}_{O_K} M^G = \text{rank}_{O_L} M$.
2. The cokernel of $\varphi$ is killed by the different ideal $D_{L/K}$ of $O_L$ over $O_K$.

Proof. (1) comes from 3.2 by tensoring $\varphi$ by $L$.

(2) We first reduce to the case when $K$ is complete. Let $\hat{O}_L := O_L \otimes_{O_K} \hat{O}_K$ and let $\hat{\varphi}$ be the canonical map

$$\hat{\varphi}: \hat{O}_L \otimes_{O_K} M^G \to \hat{O}_L \otimes_{O_L} M.$$  \hfill (3.1)

As $\hat{O}_L/O_L$ is faithfully flat, to prove (2) it is enough to show $D_{L/K} \text{coker} \hat{\varphi} = 0$. Let $p_1, \ldots, p_n$ be the maximal ideals of $O_L$. Let $D_i$ be the decomposition group of $G$ at $p_i$. Then $\hat{O}_L = \oplus_{1 \leq i \leq n} \hat{O}_{L,p_i}$ and $\hat{O}_{L,p_i}/\hat{O}_K$ is Galois of group $D_i$. Let $M_i = \hat{O}_{L,p_i} \otimes_{O_L} M$. Then (3.1) can be identified with the direct sum of the maps

$$\hat{O}_{L,p_i} \otimes_{O_K} M_{D_i} \to M_i.$$

As $D_{L/K} \hat{O}_L = \oplus_i D_{L,p_i}/\hat{K}$, it is enough to show (2) for the extension $\hat{O}_{L,p_i}/\hat{O}_K$ and the module $M_i$. Hence we can and do suppose $K$ is complete.

Now we proceed by induction on $|G|$. Suppose that $H \subseteq G$ is a normal subgroup and the proposition holds for $L/L^H$ and $L^H/K$. Let $E = L^H$. Then

$$D_{L/E} \subseteq O_L M^E, \quad D_{E/K} M^E \subseteq O_E (M^H)^{G/H} = O_E M^G.$$

As $D_{L/K} = D_{L/E}(D_{E/K} O_L)$, the proposition also holds for $L/K$. As $O_K$ is complete, $O_L/O_K$ can be decomposed into successive Galois monogeneous (cyclic) extensions (see for instance the explanations in [9], proof of Theorem 4.1). Thus we are reduced to the case $O_L$ is monogeneous over $O_K$.

Let $O_L = O_K[\theta]$ for some $\theta \in O_L$. Let $P(X) \in O_K[X]$ be the monic minimal polynomial of $\theta$. Then $O_L \simeq O_K[X]/(P(X))$. We have a decomposition in $O_L[X]$:

$$P(X) = (X - \theta)f(X), \quad \text{with } f(X) = b_{n-1}X^{n-1} + b_{n-2}X^{n-2} + \cdots + b_0 \in O_L[X].$$

For any $v \in M$, let

$$g(v) = \sum_{0 \leq i \leq n-1} b_i \sum_{\sigma \in G} \sigma(\theta^iv) \in O_L \otimes M^G.$$  

We have

$$g(v) = \sum_{\sigma \in G} \left( \sum_{i} b_i \sigma(\theta^i) \right) \sigma(v) = \sum_{\sigma \in G} f(\sigma(\theta)) \sigma(v) = f(\theta)v = P'(\theta)v.$$  

So $P'(\theta)M \subseteq O_L \otimes_{O_K} M^G$ and the proposition is proved because $D_{L/K} = P'(\theta)O_L$. \qed
**Remark 3.4.** Proposition 3.3 is sharp for monogeneous Galois extensions \(O_L = O_K[\theta]\).

Indeed, let \(M = O_L[G]\) with the natural structure of semi-linear \(O_L[G]\)-module:

\[
\sigma(\sum_{\tau \in G} \lambda_\tau \tau) := \sum_{\tau \in G} \sigma(\lambda_\tau \tau)(\sigma\tau).
\]

This is a right \(O_L\)-module by

\[
\left(\sum_{\tau \in G} \lambda_\tau \tau\right) * \mu := \sum_{\tau \in G} \lambda_\tau \tau(\mu) \tau.
\]

Let \(b \in O_L\) be such that \(bM \subseteq O_L \otimes M^G\). Let us show that \(b \in O_{L/K}\). Let \(t = \sum_{\tau \in G} 1 \tau \in M\).

The vectors \(t * \theta^i \in M, 0 \leq i \leq n - 1\), where \(n\) is the degree of the minimal polynomial \(P(T)\) of \(\theta\), generate the left \(O_L\)-module \(O_L \otimes M^G\). So, if \(e\) denotes the unit of \(G\),

\[
b.e = \lambda_0 t * e + \lambda_1 t * \theta + \cdots + \lambda_{n-1} t * (\theta^{n-1}), \quad \lambda_i \in O_L.
\]

By expanding \(t * \theta^i\), we see that for all \(\sigma \neq e\),

\[
\lambda_0 + \lambda_1 \sigma(\theta) + \cdots + \lambda_{n-1} \sigma(\theta)^{n-1} = 0.
\]

So \(F(T) = \lambda_0 + \lambda_1 T + \cdots + \lambda_{n-1} T^{n-1} \in O_L[T]\) is divisible by \(f(T) := P(T)/(T - \theta)\), and

\[
b = \lambda_0 + \lambda_1 \theta + \cdots + \lambda_{n-1} \theta^{n-1} = F(\theta) \in f(\theta)O_L = O_{L/K}.
\]

**Definition 3.5.** Let \(H\) be an \(O_K\)-module. We define the exponent \(\varepsilon(H)\) of \(H\) to be, when it exists, the smallest non-negative integer \(e\) such that \(\pi^e H = 0\). Note that for any \(O_L\)-module \(M\), \(\varepsilon(M)\) is defined to be its exponent as \(O_K\)-module.

**Proposition 3.6.** Let \(M\) be a semi-linear \(O_L[G]\)-module, flat over \(O_L\).

(1) Suppose \(\text{char}(K) = 0\). Let \(I\) be the inertia group of \(G\) at a maximal ideal of \(O_L\). Then \(\varepsilon(H^1(G, M)) \leq v_K(|I|)\) (exponent as \(O_K\)-module).

(2) In general, we have

\[
\varepsilon(H^1(G, M)) \leq 2v_K(O_{L/K}).
\]

**Proof.** (1) Let \(\hat{O}_K\) be the completion of \(O_K\). Since \(O_K \to \hat{O}_K\) is flat, we have

\[
\varepsilon(H^1(G, M)) = \varepsilon(H^1(G, M) \otimes_{O_K} \hat{O}_K) = \varepsilon(H^1(G, \hat{O}_K \otimes_{O_K} M)).
\]

Let \(p_1, \ldots, p_r\) be the maximal ideals of \(O_L\). Let \(D\) be the decomposition group of \(p := p_1\). Then

\[
\hat{M} := \hat{O}_K \otimes_{O_K} M = \oplus_{1 \leq i \leq r}(\hat{O}_{L,p_i} \otimes_{O_L} M) \simeq \text{Ind}_D^G(M),
\]

where \(M_i = \hat{O}_{L,p_i} \otimes_{O_L} M\). By Shapiro’s lemma \(H^1(G, \hat{M}) \simeq H^1(D, M_i)\). Let \(I\) be the inertia group at \(p\), let \(O_F = (\hat{O}_{L,p})^I\). Then \(O_F/\hat{O}_K\) is étale of Galois group \(D/I\). The inflation-restriction exact sequence

\[
\begin{align*}
0 \to H^1(D/I, M_i) & \to H^1(D, M_i) \to H^1(I, M_i) \\
\end{align*}
\]

implies that \(\varepsilon(H^1(G, M)) \leq \varepsilon(H^1(I, M_i))\). As \(I\) is finite, \(|I|\) kills \(H^1(I, M_i)\) ([24], VIII.1, Corollary 1). This implies the desired inequality.

Note that during this reduction step, we didn’t change the valuations of the differents:

\[
v_L(O_{L/F}) = v_L(O_{L/K}).
\]

(2) As we saw above, we can suppose that \(O_K\) is complete and \(G\) equal to its inertia group. Consider the \(G\)-equivariant exact sequence of \(O_L\)-modules:

\[
0 \to O_L \otimes_{O_K} M^G \to M \to M/(O_L \otimes_{O_K} M^G) \to 0.
\]
Let the exact sequence
\[ H^1(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} M^G) \to H^1(G, M) \to H^1(G, M/(\mathcal{O}_L \otimes_{\mathcal{O}_K} M^G)). \]

Let \( \mathcal{D} = \mathcal{D}_{L/K} \) and \( e = e_{L/K} \). By Proposition 3.3, we have \( \mathcal{D}.(M/(M^G \otimes_{\mathcal{O}_K} \mathcal{O}_L)) = 0 \). Hence \( \mathcal{D}.H^1(G, M/(\mathcal{O}_L \otimes_{\mathcal{O}_K} M^G)) = 0 \). It remains to find the annihilator of \( H^1(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} M^G) \).

If \( N \) is a free \( \mathcal{O}_K \)-module (with trivial \( G \)-action), then the canonical map \( H^i(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} N) \to H^i(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} M^G) \) is an isomorphism for all \( i \geq 0 \). As \( M^G \) is flat over \( \mathcal{O}_K \), it is an increasing union of free \( \mathcal{O}_K \)-modules. This implies easily that \( H^1(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} M^G) \simeq H^1(G, \mathcal{O}_L) \otimes_{\mathcal{O}_K} M^G \), hence
\[ \text{Ann}(H^1(G, \mathcal{O}_L \otimes_{\mathcal{O}_K} M^G)) = \text{Ann}(H^1(G, \mathcal{O}_L) \otimes_{\mathcal{O}_K} M^G) = \text{Ann}(H^1(G, \mathcal{O}_L)). \]

Let \( \pi_L \) be a uniformizing element of \( L \). Let us show that
\[ \pi_L^{[v_L(\mathcal{D})/p]}H^1(G, \mathcal{O}_L) = 0 \tag{3.2} \]
where \( p \) is the residue characteristic of \( K \) (the case \( p = 0 \) is considered in Part (1)). Let \( H \) be a normal subgroup of \( G \). Again using the inflation-restriction exact sequence
\[ 0 \to H^1(G/H, \mathcal{O}_{L^H}) \to H^1(G, \mathcal{O}_L) \to H^1(H, \mathcal{O}_L) \]
it is easy to show that Equality (3.2) holds if it holds for \( L^H/K \) and for \( L/L^H \). Therefore, similarly to the the proof of Proposition 3.3 (2), we are reduced to the case when \( G \) is cyclic. Using Herbrand’s quotient as in [22], Remark, pp. 38-39 or [17], p. 508, lines 4-9, we have
\[ \text{length}_{\mathcal{O}_K} H^1(G, \mathcal{O}_L) = \text{length}_{\mathcal{O}_K} \mathcal{O}_K/\text{Tr}(\mathcal{O}_L) = [v_K(\mathcal{D})]. \]

Let \( f \) be the degree of the residue extension of \( L/K \). Then
\[ \text{length}_{\mathcal{O}_L} H^1(G, \mathcal{O}_L) \leq [v_L(\mathcal{D})/e]/f = [v_L(\mathcal{D})]/ef. \]
As we can restrict ourselves to non-trivial wild ramified extensions, we have \( ef = [L : K] \geq p \). Hence \( \pi_L^{[v_L(\mathcal{D})/p]} \) kills \( H^1(G, \mathcal{O}_L) \). This implies that \( \pi_L^{[v_L(\mathcal{D})+v_L(\mathcal{D})/p]}H^1(G, M) = 0 \) and the exponent of \( H^1(G, M) \) is bounded by the smallest integer bigger than or equal to \( (v_L(\mathcal{D}) + [v_L(\mathcal{D})/2])/e \). The only case this might fail is when \( v_L(\mathcal{D}) = e + r \) with \( 0 \leq r \leq e - 1 \). As \( L/K \) is wildly ramified, this implies that \( L/K \) has no non-trivial intermediate extensions, hence \( G \) is cyclic (of prime order) and \( [v_L(\mathcal{D})/p] = 1 \). But then \( (v_L(\mathcal{D}) + [v_L(\mathcal{D})/p])/e \leq 2 = 2[v_K(\mathcal{D})]. \)

\[ \square \]

**Remark 3.7.** When \( \text{char}(K) = 0 \), one has ([22], Theorem 3)
\[ [v_K(M)] \leq [v_K(p)]/(p - 1). \]

For any \( \mathcal{O}_K \)-module \( F \) and any \( N \geq 0 \), we will denote \( F_N = F/\pi^{N+1}F \). Let \( M \) be a semi-linear \( \mathcal{O}_L[G] \)-module flat over \( \mathcal{O}_L \). Let \( N \geq 0 \). We want to compare \( (M_N)^G \) with \( (M^G)_N \). For all \( m \geq N \), we have canonical morphisms of \( \mathcal{O}_K \)-modules
\[
\begin{array}{ccc}
(M^G)_m & \xrightarrow{f_m} & (M_m)^G \\
\downarrow & & \downarrow \text{f.}\text{m.} \text{N} \\
(M^G)_N & \xrightarrow{f_N} & (M_N)^G.
\end{array}
\]

**Proposition 3.8.** Let \( M \) be a semi-linear \( \mathcal{O}_L[G] \)-module, flat over \( \mathcal{O}_L \). Consider the integer \( h = 2[v_K(\mathcal{D}_{L/K})] \geq 0 \). Then for all \( N \geq 0 \) and for all \( m \geq N + h \), \( (M^G)_N \) is determined by the
**G-module** $M_m$. More precisely, $f_N$ induces an isomorphism 

$$(M^G)_N \to \text{Im } f_{m,N} \simeq (M_m)^G / (\pi^{N+1}).$$

**Proof.** The above diagram implies that $\text{Im } f_N \subseteq \text{Im } f_{m,N}$. It remains to show the inverse inclusion. Let us consider the following commutative diagram with horizontal exact sequences:

$$
\begin{array}{cccccc}
0 & \to & M & \xrightarrow{-\pi^{m+1}} & M & \to & M_m & \to & 0 \\
\downarrow & & \downarrow & & \downarrow id & & \downarrow & & \downarrow id \\
0 & \to & M & \to & M & \to & M_N & \to & 0 
\end{array}
$$

where $\cdot$ means the multiplication and the maps in the rows are the canonical surjection. Then we have the following diagram of long exact sequences by taking group cohomology:

$$
\begin{array}{cccccc}
(M^G)_m & \xrightarrow{f_m} & (M^G)_N & \xrightarrow{\Delta_m} & H^1(G, M) & \xrightarrow{-\pi^{m+1}} & H^1(G, M) \\
\downarrow id & & \downarrow f_{m,N} & & \downarrow id & & \downarrow id \\
(M^G)_N & \xrightarrow{f_N} & (M^G)_N & \xrightarrow{\Delta_N} & H^1(G, M) & \xrightarrow{-\pi^{N+1}} & H^1(G, M) 
\end{array}
$$

We see that $\text{Im } f_{m,N} \subseteq \text{Im } f_N$ if and only if $\Delta_N f_{m,N} = -\pi^{m-N} \Delta_m = 0$. This happens when $m - N \geq \varepsilon(H^1(G, M))$. The latter inequality is true by Proposition 3.6. \qed

**Remark 3.9.** (1) If $L/K$ is tamely ramified, then $h = 0$ and we get the well-known equality $(M_N)^G = (M^G)_N$.

(2) Suppose $\text{char}(K) = 0$ and $p > 0$. Then the ramification filtration of $G$ has length at most $v_L(p)/(p - 1)$ ([24], IV.2, Exercise 3c). Hence $v_L(D_{L/K}) \leq |G|v_L(p)/(p - 1)$ by [24], IV.1, Proposition 4, and we get

$$v_K(D_{L/K}) \leq |G|v_K(p)/(p - 1).$$

When $p^2 | |G|$, a better bound is given by Lenstra ([14], 4.1.1):

$$v_K(D_{L/K}) \leq v_K(|G|) + (e_{L/K} - 1)/e_{L/K}.$$ 

**Corollary 3.10.** Under the hypothesis of Proposition 3.8, the canonical homomorphism

$$\lim_{\leftarrow n}(M^G)_n \to \lim_{\leftarrow n}(M_n)^G = \left(\lim_{\leftarrow n} M_n\right)^G$$

is an isomorphism.

### 4. Minimal Weierstrass models

We compare in this section the minimal Weierstrass model of $E$ over $\mathcal{O}_K$ with that of $E_L$ over $\mathcal{O}_L$.

**Definition 4.1.** Let $E$ be an elliptic curve over the field of fractions of a principal ideal domain $R$ (e.g. $R = \mathcal{O}_K$ or $\mathcal{O}_L$). A Weierstrass model of $E$ over $R$ is a triplet consisting in

(1) a scheme $W$ proper and flat over $R$ with geometrically integral fibers;

(2) an isomorphism $W_n \to E$ where $W_n$ denotes the generic fiber of $W$; and

(3) a section $\epsilon \in W(R)$ in the smooth locus of $W$ and whose generic fiber is mapped to the origin of $E$ by the isomorphism of (2). To lighten the notation, we will often omit the isomorphism $W_n \to E$. 
Let us recall the correspondence between Weierstrass models and Weierstrass equations ([7], §1 or [16], §9.4.4). Let \( W \) be a Weierstrass model of \( E \) over \( R \). Consider the invertible sheaf \( \mathcal{O}_W(ne) \) on \( W \). Let \( L(ne) = \Gamma(W, \mathcal{O}_W(ne)) \). For all \( n \geq 1 \), \( L(ne) \) is free of rank \( n \) and \( L((n + 1)e)/L(ne) \) is free of rank 1. Let \( \{1, x\} \), \( \{1, x, y\} \) be respective bases of \( L(2e) \) and \( L(3e) \). Scaling \( x, y \) by suitable units of \( R \), we get a relation

\[
y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6
\]

which (after homogenization) defines \( W \) as a closed subscheme of \( \mathbb{P}^2_R \). We will call such a triplet \( \{1, x, y\} \) a Weierstrass basis of \( W \). Denote by \( \Delta(W) \in R \setminus \{0\} \) the discriminant of the above equation.

**Definition 4.2.** If the ideal \( \Delta(W)R \) is the biggest possible among all Weierstrass models of \( E \), then \( W \) will be called a minimal Weierstrass model of \( E \). Such a model exists and is unique up to isomorphism ([16], 9.4.35 or [25], VII.1.3).

The notion of Weierstrass model depends a priori on the origin of \( E \). However, the next proposition says that, up to isomorphism, the choice of the origin does not matter. See also discussions in §8.

**Proposition 4.3.** Let \( (E, e) \) be an elliptic curve over \( K \) with minimal Weierstrass model \( W \) over \( \mathcal{O}_K \).

1. Let \( q \in E(K) \) and let \( Z \) be the minimal Weierstrass model of \( (E, q) \) over \( \mathcal{O}_K \). Then there exists an isomorphism \( W \simeq Z \) which maps \( e \) to \( q \).

2. Let \( N \geq 0 \) and let \( e_N \) be the section of \( W_N \) induced by \( e \). Let \( \bar{q} \in W_N \) be a section contained in the smooth locus. Then there exists an isomorphism (not unique) \( W_N \rightarrow W_N \) which maps \( \bar{q} \) to \( e_N \).

**Proof.** (1) Let \( t : E \rightarrow E \) be an isomorphism which maps \( e \) to \( q \). Then \( Z \) endowed with the isomorphism \( \mathcal{Z}_q \simeq E \xrightarrow{t} E \) is a minimal Weierstrass model of \( (E, e) \). By the uniqueness property, we get an isomorphism \( W \rightarrow Z \) as desired.

(2) As \( \mathcal{O}_K \) and \( \mathcal{O}_K \) coincide modulo \( \pi^{N+1} \), we can suppose \( \mathcal{O}_K \) is complete. We can lift \( q \) to a rational point \( q \in E(K) \). Let \( t : E \rightarrow E \) be as in (1). Let \( E^0 \) be the identity component of the Néron model \( E \) of \( E \). It is equal to the smooth locus of \( W \). By the universal property of \( E, t \) extends to a morphism \( E^0 \rightarrow E. \) As \( t(\xi) \in E^0 \), \( t \) is actually a morphism \( E^0 \rightarrow E \). As \( W \setminus E^0 \) has codimension \( \geq 2 \) in \( W \), \( t \) extends to a finite birational morphism \( W \rightarrow W \). It is an isomorphism because \( W \) is normal.

From now on \( W \) will denote the minimal Weierstrass model of \( E \) over \( \mathcal{O}_K \). The next lemma can be proved by direct computations using [25], VII.1.3(d) and [7], (1.6).

**Lemma 4.4.** Fix a Weierstrass basis \( \{1, x, y\} \) of \( W \).

1. Let \( w, z \in L(3e) \). Then \( \{1, w, z\} \) is a Weierstrass basis of \( W \) if and only if \( \{1, w\} \) is a basis of \( L(2e) \), \( z \in L(3e) \setminus L(2e) \) and \( z^2 - w^3 \in L(5e) \).

2. The set \( \{1, w, z\} \) is a Weierstrass basis of some Weierstrass model \( Z \) if and only if \( w \in L(2e) \setminus \mathcal{O}_K, z \in L(3e) \setminus L(2e), z^2 - w^3 \in L(5e) \) and \( z \in \mathcal{O}_K + \mathcal{O}_K, w + \mathcal{O}_K, y \).

3. Under the above condition, \( w = u^2x + r, z = u^3y + u^2sx + t \) for some \( u, r, s, t \in \mathcal{O}_K \) and we have \( \Delta(Z) = u^2 \Delta(W) \).
Let $L/K$ be a finite Galois extension of $K$ with Galois group $G$. Let $W'$ be the minimal Weierstrass model of $E_L$ over $K$, and let $e' \subset W'(\mathcal{O}_L)$ be the closure in $W'$ of the origin of $E_L$. As we saw before, for all $n \geq 1$, $L(ne')$ is free of rank $n$ and $L((n+1)e')/L(ne')$ is free of rank $1$ over $\mathcal{O}_L$. Similarly, $L(ne')^G$ is free of rank $n$ over $\mathcal{O}_K$ and the quotient $L((n+1)e')^G/(L(ne')^G)$ is free of rank $1$ over $\mathcal{O}_K$.

**Lemma 4.5.** The Galois group $G$ acts on the $\mathcal{O}_K$-scheme $W'$ and induces a semi-linear $G$-action (cf. §2) on $L(ne')$ for all $n \in \mathbb{N}$. Moreover, a subset $\{1, w, z\} \subset L(3e')$ is a Weierstrass basis of some Weierstrass model of $E$ over $\mathcal{O}_K$ if and only if

$$w \in L(2e')^G \setminus \mathcal{O}_K, \quad z \in L(3e')^G \setminus L(2e')^G$$

and

$$z^2 - w^3 \in L(5e'), \quad z \in \mathcal{O}_L + \mathcal{O}_Lw + \mathcal{O}_Ly'.

**Proof.** The group $G$ acts on $E_L$ and fixes the generic fiber of $e'$. By the uniqueness of the minimal Weierstrass model over $\mathcal{O}_L$, $G$ acts on $W'$ and semi-linearly on $L(ne')$ for all $n \geq 1$. The remaining part of the lemma results easily from Lemma 4.4(2).

**Theorem 4.6.** Let $v_L$ denote the normalized valuation on $L$ associated to a maximal ideal of $\mathcal{O}_L$. Let $W$, $W'$ be the respective minimal Weierstrass models of $E$ and $E_L$. Then

$$0 \leq v_L(\Delta(W)) - v_L(\Delta(W')) \leq 12(v_L(\mathfrak{D}_{L/K}) + e_{L/K} - 1).$$

**Proof.** The filtration $\mathcal{O}_K \subset L(2e')^G \subset L(3e')^G$ has successive quotients free of rank $1$ over $\mathcal{O}_K$. This implies the existence of a basis $\{1, w_0, z_0\}$ of $L(3e')^G$ over $\mathcal{O}_K$ such that $\{1, w_0\}$ is a basis of $L(2e')^G$ over $\mathcal{O}_K$. Let $\{1, x', y'\}$ be a Weierstrass basis of $W'$. Then there exist $a_1, \ldots, a_5 \in \mathcal{O}_L$, such that $w_0 = a_1x' + a_2$, $z_0 = a_3y' + a_4x' + a_5$ and $a_1, a_3 \neq 0$. There exist $\alpha_1, \alpha_3 \in K^*$ such that $\alpha_1w_0, \alpha_3z_0 \in L(3e) \otimes K$ define a Weierstrass equation of $E$ over $K$. This implies that $t := a_3^2/a_1^2 = a_3^2/a_5^2 \in K$.

Now let us construct a Weierstrass model $Z$ of $E$ over $\mathcal{O}_K$. If $t \in \mathcal{O}_K$, set

$$\beta_1 = \pi^{2n}t, \quad \beta_3 = \pi^{3n}t \in \mathcal{O}_K$$

where $n$ is the smallest integer such that $a_1 | \pi^n$. If $t^{-1} \in \mathcal{O}_K$, set

$$\beta_1 = \pi^{2m}t^{-2}, \quad \beta_3 = \pi^{3m}t^{-2} \in \mathcal{O}_K$$

where $m$ is the smallest non-negative integer such that $a_1 | \pi^m t^{-1}$.

Consider $w = \beta_1w_0 \in L(2e')^G$ and $z = \beta_3z_0 \in L(3e')^G$. We have

$$w = \beta_1a_1x' + \beta_1a_2, \quad z = \beta_3a_3y' + \beta_3a_4x' + \beta_3a_5.

We can check that $(\beta_1a_1)^3 = (\beta_3a_3)^2$, and $\beta_3/(\beta_1a_1) = a^{-1}_1(\pi^n t) \in \mathcal{O}_K$ if $t \in \mathcal{O}_K$, and that $\beta_3/(\beta_1a_1) = a^{-1}_1(\pi^m t^{-1}) \in \mathcal{O}_K$ otherwise. Thus $\beta_3a_4 \in \beta_1a_1\mathcal{O}_L$, and $z \in \mathcal{O}_L + \mathcal{O}_Lw + \mathcal{O}_Ly'$.

By Lemma 4.5. $\{1, w, z\}$ is a Weierstrass basis of some Weierstrass model $Z$ of $E$ over $\mathcal{O}_K$. In particular $v_K(\Delta(Z)) \geq v_K(\Delta(W))$.

Now let us compute $v_L(\beta_1a_1)$. We have

$$\beta_1a_1 = \begin{cases} (a_1^{-1} \pi^n)^2 a_2^2 & \text{if } t \in \mathcal{O}_K \\ (a_1^{-1} \pi^m t^{-1})^2 a_3^2 & \text{otherwise}. \end{cases}$$

By Proposition 3.3 (2),

$$\mathfrak{D}_{L/K}.L(3e')^G \mathcal{O}_L = \mathcal{O}_L + \mathcal{O}_Lw_0 + \mathcal{O}_Lz_0.$$
Hence \( v_L(a_3) \leq v_L(\mathfrak{D}_{L/K}) \). Therefore
\[
v_L(\Delta(Z)) - v_L(\Delta(W)) = 6v_L(\beta_1 a_1) \leq 12(v_L(\mathfrak{D}_{L/K}) + e_{L/K} - 1).
\]

**COROLLARY 4.7.** Keep the notation of Theorem 4.6. Let \( \{1, x, y\} \) be a Weierstrass basis of \( W \) and let \( \{1, x', y'\} \) be a Weierstrass basis of \( W' \).

(1) Then
\[
x = b_1 x' + b_2, \quad y = b_3 y' + b_4 x' + b_5, \quad b_i \in \mathcal{O}_L
\]
with
\[
v_L(b_1) \leq 2(v_L(\mathfrak{D}_{L/K}) + e_{L/K} - 1), \quad v_L(b_3) \leq 3(v_L(\mathfrak{D}_{L/K}) + e_{L/K} - 1).
\]
(2) The \( \mathcal{O}_K \)-modules \( L(2c')^G/L(2c) \) and \( L(3c')^G/L(3c) \) are annihilated by \( \pi^{2(v_L(\mathfrak{D}_{L/K})+3} \) and \( \pi^{2(v_L(\mathfrak{D}_{L/K})+4} \) respectively.

**Proof.** (1) As \( x \in L(2c') \) and \( y \in L(3c') \), we can write \( x = b_1 x' + b_2, y = b_3 y' + b_4 x' + b_5 \) for \( b_i \in \mathcal{O}_L \). By Lemma 4.4, we have \( b_1^2 = b_2^2 \) and \( v_L(\Delta(W)) - v_L(\Delta(W')) = 6v_L(b_1) \). The bounds on \( v_L(b_1) \) and \( v_L(b_3) \) are then a consequence of Theorem 4.6 and of the relation \( b_1^2 = b_2^2 \).

(2) Keep the notation in the proof of 4.6. As \( \beta_1 v_0, \beta_2 v_0 \in L(3c) \), it is enough to bound \( v_K(\beta_1) \) and \( v_K(\beta_3) \). The computations in the proof of Theorem 4.6 imply that \( v_L(\beta_1) \leq v_L(\beta_1 a_1) \leq 2v_L(\mathfrak{D}_{L/K}) + 2e_{L/K} - 2 \) and \( \beta_3 = \beta_1 a_1 \gamma_1 \) with \( \gamma_1 = \pi^{m/1} / a_1 \) if \( t \in \mathcal{O}_K \) and \( \gamma = (\pi^{m/1}) / a_1 \) otherwise. Hence \( v_L(\beta_3) \leq v_L(\beta_1) + e_{L/K} - 1 \). Dividing by the ramification index \( e_{L/K} \) we then get (2).

**COROLLARY 4.8.** Suppose \( E \) has good reduction over some Galois extension \( L \). Then
\[
v_K(\Delta(W)) \leq [12(v_L(\mathfrak{D}_{L/K}) - 1)/e_{L/K}] + 12.
\]

**REMARK 4.9.** (Absolute bound for the minimal discriminant) Let \( f \) be the conductor of \( E \) and let \( m \) be the number of geometric irreducible components of the special fiber of the minimal regular model \( \mathcal{X} \) of \( E \) over \( \mathcal{O}_K \). Then Ogg-Saito’s formula ([21], Corollary 2, [26], §IV.11) is
\[
v_K(\Delta(W)) = f + m - 1.
\]
Suppose that \( E \) has potentially good reduction and \( K \) is a finite extension of \( \mathbb{Q}_p \), by [4], Theorem 6.2, \( f \leq 2 + 6v_K(2) + 3v_K(3) \). Hence
\[
v_K(\Delta(W)) \leq 10 + 6v_K(2) + 3v_K(3).
\]
To be more precise, [4] gives a bound on the Artin conductor \( f(V, L/K) \) for any \( G \)-module \( V \) ([4], §5), and the bound on \( f \) is then deduced using the \( G \)-module of the \( \ell \)-torsions \( E[\ell] \) for a prime number \( \ell \neq p \). Our Corollary 4.8 is of different nature because it gives a bound of \( v_K(\Delta(W)) \) in terms of the Artin conductor for the representation \( r_G - I_G \) (regular representation minus unit representation). It is better when \( v_L(\mathfrak{D}_{L/K}) \) is small with respect to \( \max\{v_L(2), v_L(3)\} \).

**REMARK 4.10.** Suppose \( E \) has potentially multiplicative reduction. Consider \( c_4(W) \) the invariant \( c_4 \) associated to Equation (4.1). Then \( j(E) = c_4(W^3) / \Delta(W) \). Similarly to
Theorem 4.6, we have $0 \leq v_L(c_4(W)) - v_L(c_4(W')) \leq 4(v_L(\mathcal{D}_{L/K}) + e_{L/K} - 1)$. Hence

$$v_K(c_4(W)) \leq \frac{4}{4}(v_L(\mathcal{D}_{L/K}) - 1)/e_{L/K} + 4$$

for any (quadratic) extension $L/K$ such that $E_L$ has multiplicative reduction.

### 4.11. Base change conductor

Suppose $K$ henselian. In [6] and [5], C.-L. Chai and J.-K. Yu introduced the notion of base change conductor for algebraic tori and abelian varieties over $K$. Let $L/K$ be a Galois extension such that $E_L$ has semi-stable reduction. Consider the Néron model $E$ (resp. $E'$) of $E$ (resp. $E_L$) over $\mathcal{O}_K$ (resp. $\mathcal{O}_L$). Let $\mathcal{O}_{E/K}$ (resp. $\mathcal{O}_{E'/O_L}$) be the module of the translation invariant differential forms on $E$ (resp. $E'$) over $\mathcal{O}_K$ (resp. $\mathcal{O}_L$). By definition, the base change conductor $c(E) \in \mathbb{Q}$ of $E$ is the length of $(\mathcal{O}_{E/K} \otimes \mathcal{O}_L)/\mathcal{O}_{E'/O_L}$ as $\mathcal{O}_L$-modules divided by the ramification index $e_{L/K}$ of $L/K$.

**Proposition 4.12.** Suppose $K$ henselian. Let $W$ be the minimal Weierstrass model of $E$ over $\mathcal{O}_K$. Then the base change conductor $c(E)$ is given by

$$c(E) = \min\left\{\frac{1}{12}v_K(\Delta(W)), \frac{1}{4}v_K(c_4(W))\right\}.$$

**Proof.** It is known that $\mathcal{O}_{E/K}$ (resp. $\mathcal{O}_{E'/O_L}$) is a free $\mathcal{O}_K$-module (resp. $\mathcal{O}_L$-module) generated by some canonical differential form $\omega = dx/(2y + a_1x)$ (resp. $\omega'$) ([16], Proposition 9.4.35). By Lemma 4.4, there exists $u \in O_L$ such that $\omega' = u\omega$ and $\Delta(W) = u^{12}\Delta(W')$, $c_4(W) = u^c_4(W')$. Hence $c(E) = v_L(u)/e_{L/K}$. If $E$ has potentially good reduction, then $v_L(\Delta(W')) = 0$, $c(E) = v_K(\Delta(W))/12$ and

$$\frac{1}{4}v_K(c_4(W)) - \frac{1}{12}v_K(\Delta(W)) = \frac{1}{12}v_K(j) \geq 0.$$

Similarly, if $E$ has potentially multiplicative reduction, then we have $v_L(c_4(W')) = 0$, $c(E) = v_K(c_4(W))/4$ and $v_K(c_4(W))/4 < v_K(\Delta(W))$ because $v_K(j) < 0$. \qed

**Corollary 4.13.** Let $L/K$ be a finite Galois extension such that $E_L$ has semi-stable reduction. Then

$$c(E) < v_K(\mathcal{D}_{L/K}) + 1.$$ 

In particular, if $\text{char}(K) = 0$, then $c(E) < 24v_K(p)/(p - 1) + 1$.

**Proof.** The first part comes from Corollary 4.8 and Remark 4.10. For the second part, note that $E$ has semi-stable reduction over an extension $L/K$ of degree dividing 24 and then use Remark 3.9. \qed

### 5. Congruences of minimal Weierstrass models

From now on, we will suppose that $K$ has perfect residue field. For any scheme $Z$ over $\mathcal{O}_K$ (including $\mathcal{O}_L$-schemes), recall that

$$Z_N := Z \times_{\text{Spec} \mathcal{O}_K} \text{Spec}(\mathcal{O}_K/\pi^{N+1} \mathcal{O}_K)$$

for any non-negative integer $N$. Similarly, recall that for any $\mathcal{O}_K$ or $\mathcal{O}_L$-module $M$, we denote

$$M_N = M/\pi^{N+1}M.$$
For any morphism $f$ of schemes over $\mathcal{O}_K$, $f_N$ denotes the canonical morphism obtained by base change to $\mathcal{O}_K/\pi^{N+1}\mathcal{O}_K$.

We keep the notation of §4. In particular, $W$ and $W'$ are the respective minimal Weierstrass model of $E$ and $E_L$ over $\mathcal{O}_K$ and $\mathcal{O}_L$, and $\epsilon$ is the closure in $W'$ of the origin of $E_L$. We will also work with another discrete valuation field $K_0$ with perfect residue field, a Galois extension $L_0/K_0$ of group $G$ and an elliptic curve $E_0$ over $K_0$. We will denote the analogous construction by the same notation with a subscript $o$. We will say $W_N$ is determined by the $G$-action on $W_m$ (or by $(W'_m, G)$ for short) for some $m \geq N$ if the existence of compatible $G$-equivariant isomorphisms:

$$\left\{(\text{Iso}_m) \right\} \begin{cases} \mathcal{O}_{K,m} \cong \mathcal{O}_{K_0,m}, \\ \theta_m : \mathcal{O}_{L,m} \cong \mathcal{O}_{L_0,m}, \\ W'_m \cong W'_0,m, \\ \epsilon'_m \mapsto \epsilon'_o,m \end{cases}$$

implies $W_N \cong W_{o,N}$. Let us stress that $W'_m \rightarrow W'_{o,m}$ is supposed to map $\epsilon'_m$ to $\epsilon'_o,m$ (the Weierstrass models should be regarded as pointed schemes). The aim of this section is to show that $W_N$ is determined by $(W_m,G)$ when $m \gg 0$ (Theorem 5.5).

**Lemma 5.1.** Let $N > \nu_K(\mathcal{D}_{L/K}) - 1$. If $\mathcal{O}_{L,N} \cong \mathcal{O}_{L_0,N}$ as $G$-modules, then $v_{L_0}(\mathcal{D}_{L_0/K_0}) = v_L(\mathcal{D}_{L/K})$ and $e_{L/K} = e_{L_0/K_0}$.

**Proof.** The maximal ideals $p_1, \ldots, p_n$ of $\mathcal{O}_L$ correspond to the maximal ideals of the semilocal ring $\mathcal{O}_{L,N}$. Hence there is a one-one correspondence between the maximal ideals of $\mathcal{O}_L$ and that of $\mathcal{O}_{L_0}$. We have

$$\mathcal{D}_{L/K} \mathcal{O}_{L,p_i} = \mathcal{D}_{L,p_i}/\mathcal{O}_K$$

It is therefore enough to deal with the case when $K$ and $K_0$ are complete.

The isomorphism $\mathcal{O}_L/\pi \mathcal{O}_L \cong \mathcal{O}_{L_0}/\pi \mathcal{O}_{L_0}$ implies the equality of ramification indexes $e_{L/K} = e_{L_0/K_0}$. Let $G = G_0 \geq G_1 \geq G_2 \geq \ldots \geq G_r = \{1\}$ be the ramification filtration of $G$ acting on $\mathcal{O}_L$ (with $G_{r-1}$ non-trivial). Then

$$v_L(\mathcal{D}_{L/K}) = \left(\sum_{i \geq 0} |(G_i) - 1| \right) \geq r$$

([24], IV.1, Proposition 4). As $v_L(\sigma(\pi_L) - \pi_L) \leq r$ for all $\sigma \in G \setminus \{1\}$, the same property holds for $\pi_{L_0}$ thanks to the isomorphism $\theta_N$ and because $(N + 1)e_{L/K} > v_L(\mathcal{D}_{L/K}) \geq r$. Thus the ramification filtration of $G$ acting on $L_0$ is the same as that of $G$ acting on $L$, and $v_{L_0}(\mathcal{D}_{L_0/K_0}) = v_L(\mathcal{D}_{L_0/K_0})$. \[\square\]

Fix $N \geq 0$ and let $m \geq N$. Suppose we are given isomorphisms $(\text{Iso}_m)$ as above. Then they induce canonically isomorphisms $(\text{Iso}_i)$ for all integers $0 \leq i \leq m$. The isomorphisms $W'_i \rightarrow W'_{o,i}$ ($0 \leq i \leq m$) induce $G$-invariant isomorphisms

$$\varphi_i : L(6\epsilon'_i) \cong L(6\epsilon'), \quad i \leq m$$

which respect the filtration by the order of the pole and are compatible with the multiplications $L(u\epsilon'_i) \times L(r\epsilon'_i) \rightarrow L((n + r)\epsilon'_i)$. For any element $z \in M$ in some $\mathcal{O}_K$-module, we denote by $\tilde{z}$ its image in $M$, if no confusion is possible.

**Lemma 5.2.** Let $\{1, x'_o, y'_o\}$ be a Weierstrass basis of $W'_o$. Then $\{1, \varphi_m(x'_o), \varphi_m(y'_o)\} \subset L(3\epsilon'_m)$ lifts to a Weierstrass basis of $W'$.

---

1Here the hypothesis $k$ perfect is used. We don’t know whether it is really necessary.
Proof. Lift arbitrarily \( \varphi_m(x'_o), \varphi_m(y'_o) \) to \( w' \in \mathbb{L}(2c') \) and \( z' \in \mathbb{L}(3c') \). There exist \( \lambda_i \in \mathcal{O}_L \) such that \( w' = \lambda_1 x' + \lambda_2, z' = \lambda_3 y' + \lambda_4 w' + \lambda_5 \) (recall that \( \{1, x', y'\} \) is a Weierstrass basis of \( W' \)). As \( \varphi_m \) is an isomorphism, \( \lambda_1, \lambda_2 \in \mathcal{O}_L^* \). The relation \( (y'_o)^2 - (x'_o)^3 \in \mathbb{L}(5c') \) implies that \( \lambda_3^2 - \lambda_1^3 = 0 \) in \( \mathcal{O}_{L,m} \). Therefore \( \lambda = \lambda_3^2/\lambda_1^3 \in 1 + \pi^{m+1} \mathcal{O}_L \). Replacing \( w' \) (resp. \( z' \)) by \( \lambda w' \) (resp. \( \lambda z' \)), we find new liftings \( w, z \) such that \( z^2 - w^3 \in \mathbb{L}(5c') \) and \( \{1, w, z\} \) is a basis of \( \mathbb{L}(3c') \). This implies that \( \{1, w, z\} \) is a Weierstrass basis of \( W' \).

The next lemma is used in Example 2.1 and in Proposition 7.7.

**Lemma 5.3.** Let \( N \geq 0 \), let \( W, W_o \) be respective Weierstrass models of \( E, E_o \) over \( \mathcal{O}_K \) and let \( \{1, x, y\}, \{1, x_o, y_o\} \) be corresponding Weierstrass basis. Suppose there exists an isomorphism \( \varphi_N : W_N \cong W_o.N \). Then the following properties are true:

1. there exist \( u, s, r \in \mathcal{O}_{K,N} \) such that \( u \in \mathcal{O}_{K,N}^* \) and
   \[ \varphi_N(x_o) = u^2 x + r, \quad \varphi_N(y_o) = u^3 y + w^2 s x + t. \]
2. If \( N \geq 5 \), then \( W \) is minimal if and only if \( W_o \) is minimal.

**Proof.** (1) is an immediate consequence of Lemma 5.2 and of Lemma 4.4(3).

(2) First the minimality of \( W \) can be checked over the strict henselization of \( \mathcal{O}_K \). As \( k \) is perfect, we can suppose \( k \) is algebraically closed. By Tate’s algorithm [27], §7-8, \( W \) is not minimal if and only if there exist \( r, s, t \in \mathcal{O}_K \) such that \( v(a'_i) \geq i \). This condition is checked modulo \( \pi^i \), so can be detected in \( W_\delta \). □

**Lemma 5.4.** Let \( 0 \to M \to H \to T \to 0 \) be an exact sequence of \( \mathcal{O}_K \)-modules such that \( \pi^i T = 0 \) for some \( i \geq 1 \). Then for all \( m \geq r \), \( \ker(M_m \to H_m) \subseteq \ker(M_m \to M_{m-r}) \).

**Proof.** Use the Snake Lemma. □

**Theorem 5.5.** Let \( N \geq 0 \). If
\[ m \geq N + 12[v_K(\mathfrak{D}_{L/K})] + 19, \]
then \( W_N \) is determined by the \( G \)-action on \( W'_m \).

**Proof.** By hypothesis, we have isomorphisms \( (\text{Isom}_m) \) (hence \( (\text{Isom}_i) \) for all \( i \leq m \)). Denote by \( \rho_i \) the canonical maps \( L(3c)_i \to (L(3c')^G)_i \) and by \( \rho_{o,i} \) the analogue maps for \( E_o \). We have a commutative diagram
\[
\begin{array}{c}
L(3c)_m \xrightarrow{\rho_m} (L(3c')^G)_m & \cong & (L(3c'_o)_m)^G \xrightarrow{\varphi_m} L(3c'_o)_m \\
\downarrow \varphi_m & & \downarrow \varphi_m \\
L(3c)_m & \cong & (L(3c')^G)_m \xrightarrow{\rho_m} (L(3c'_o)_m)^G
\end{array}
\]
where the vertical arrows are isomorphisms. We want to complete this diagram with an isomorphism \( L(3c_o)_m \cong L(3c'_o) \) (for some \( m_2 \leq m \)) sending a Weierstrass basis to a Weierstrass basis. Let \( \{1, x_o, y_o\} \) be a Weierstrass basis of \( W_o \).

**Step 1.** Let \( \mathfrak{D} = \mathfrak{D}_{L/K} \). Let \( m_1 = m - 2[v_K(\mathfrak{D})] \). We first construct images of \( x_o, y_o \) in \( (L(3c')^G)_m \). According to Proposition 3.8, the above commutative diagram induces a new
commutative diagram at level $m_1$ with isomorphic vertical arrows

$$
\begin{array}{ccc}
L(3e_o)_{m_1} & \xrightarrow{\rho_{m_1}} & (L(3e'_o)^G)_{m_1} \\
n & \phi_{m_1} & n^G \\
L(3e)_{m_1} & \xrightarrow{\rho_{m_1}} & (L(3e'_o)^G)_{m_1} \\
\end{array}
$$

Let $w \in L(2e')^G, z \in L(3e')^G$ be liftings of $\phi_{m_1}(\bar{x}_o)$ and $\phi_{m_1}(\bar{y}_o)$.

**Step 2.** Now we modify $w, z$ to $x, y$ so that $\{1, x, y\}$ is a Weierstrass basis of a Weierstrass model of $E$ over $\mathcal{O}_K$. We can write

$$
x_o = b_{0,1} x'_o + b_{0,2}, \quad y_o = b_{0,3} y'_o + b_{0,4} x'_o + b_{0,5}, \quad b_{0,i} \in \mathcal{O}_{L_o}
$$

where $\{1, x'_o, y'_o\}$ is a Weierstrass basis of $W'$. By Corollary 4.7 (1), Lemmas 5.1 and 4.4 (3), we have

$$
v_{L_o}(b_{0,1}) \leq 2(v_{L_o}(\mathcal{O}_{L_o/K_o}) + c_{L_o/K_o} - 1) \leq m_1, \quad v_{L_o}(b_{0,1}) \leq v_{L_o}(b_{0,4}).
$$

Let $\{1, x', y'\}$ be a Weierstrass basis of $W'$ lifting $\{1, \phi_{m_1}(\bar{x}_o), \phi_{m_1}(\bar{y}_o)\}$ (Lemma 5.2). Then

$$
w = c_1 x' + c_2, \quad z = c_3 y' + c_4 x' + c_5, \quad c_i \in \mathcal{O}_L.
$$

For all $i \leq 5, c_i = \theta_{m_1}(b_{0,i}) \in \mathcal{O}_{L,m_1}$. Therefore

$$
v_{L}(c_1) \leq 2(v_{L}(\mathcal{D}) + c_{L/K} - 1) \leq m_1, \quad v_{L}(c_1) \leq v_{L}(c_4).\quad (5.1)
$$

We have

$$
c_2^2 - c_1^2 = \theta_{m_1}(b_{0,3}^2) - \theta_{m_1}(b_{0,1}^3) = 0 \in \mathcal{O}_{L,m_1}.
$$

(Lemma 4.4(3)). Writing $w, z$ in a Weierstrass basis of $E$ (with coefficients in $K$), we see that $\lambda := c_2^2/c_1^3 \in K$. Moreover, the inequality on $v_{L}(c_1)$ implies that $v_{L}(c_1^3)/\lambda < 6[v_{K}(\mathcal{D})] + 12,\quad$ hence

$$
\lambda \in 1 + \pi^{m_1+4-6[v_{K}(\mathcal{D})]}\mathcal{O}_K.
$$

Let $x = \lambda w$ and $y = \lambda z$ and let $m_2 = m_1 - 6[v_{K}(\mathcal{D})] - 11$. Then $y^2 - x^3 \in L(5e')$ and $x, y \in L(3e')^G$ coincide with $w, z$ in $(L(3e'_o)^G)_{m_2}$. Multiplying Equation (5.1) above by $\lambda$, and using the second inequality of Equation (5.2) and Lemma 4.5, we see that $\{1, x, y\}$ is a Weierstrass basis of a Weierstrass model $Z$ of $E$ over $\mathcal{O}_K$. This implies that $x \in L(2e), y \in L(3e)$.

**Step 3.** Let us show that $\{1, x, y\}$ is a Weierstrass basis of $W$. The above construction shows that we have a canonical commutative diagram

$$
\begin{array}{ccc}
(L(3e'_o)^G)_{m_2} & \xrightarrow{\phi_{m_2}} & (L(3e'_o)^G)_{m_2} \\
\text{Im}(\rho_{m_2}) & \xrightarrow{\phi_{m_2}} & \text{Im}(\rho_{m_2}) \\
L(3e)_{m_2} & \xrightarrow{\rho_{m_2}} & L(3e)_{m_2} \\
\end{array}
$$

which implies that $\phi_{m_2} \text{Im}(\rho_{m_2}) = \text{Im}(\rho_{m_2})$. Thus $\{1, x, y\}$ generate $L(3e)$ in $(L(3e')^G)_{m_2}$. As $m_2 \geq 2(v_{K}(\mathcal{D})) + 4$, it follows from Corollary 4.7(2) that

$$
\ker(L(3e)) \to (L(3e')^G)_{m_2} \subseteq \pi^{m_2+4} L(3e')^G = \pi^{m_2} L(3e')^G \subseteq \pi L(3e).
$$

By Nakayama’s lemma, $\{1, x, y\}$ generate, hence is a basis of, $L(3e)$. Therefore $\{1, x, y\}$ is a Weierstrass basis of $W$.  


Last step: Let’s show \( W_N \simeq W_{o,N} \). Let
\[
y^2 + (a_{o,1}x + a_{o,3})y_0 = x_o^3 + a_{o,2}x_o^2 + a_{o,4}x_o + a_{o,6}
\]
be an equation of \( W_o \). Let \( a_i \in \mathcal{O}_K \) be such that \( \bar{a}_i = \varphi_{m_2}(\bar{a}_{o,i}) \). Then
\[
y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6
\]
holds in \( (L(6\epsilon')^G)_{m_2} \). By Lemma 5.4 and Corollary 4.7(2), the same relation holds in \( L(6\epsilon)_{m_3} \), where \( m_3 = m_2 - (4[v_K(\mathcal{D})] + 8) \). Therefore \( W_{m_3} \simeq W_{o,m_3} \). As \( m_3 \geq N \), we have \( W_N \simeq W_{o,N} \).

\[ \square \]

Remark 5.6. The bound on \( m \) in 5.5 is not optimal. Indeed, when \( L/K \) is unramified, \( W_N \) is determined by the \( G \)-action on \( W_m' \) with \( m = N \).

Proposition 5.7. Suppose \( E \) has semi-stable reduction over \( \mathcal{O}_K \). Then for any \( N \geq 0 \), \( W_N \) is determined by \( (W_m', G) \) if \( m \geq 2[v_K(\mathcal{D}_{L/K})] + N \). In particular, \( W_N \) is determined by \( (W_{m}', G) \) if \( L/K \) is tamely ramified.

Proof. Suppose we are given a system of isomorphisms \( (\text{Iso}_m) \). Then the special fiber of \( W_o \) is semi-stable because it is isomorphic to the special fiber of \( W \). Hence \( W_o \) is semi-stable. The minimal Weierstrass model \( W \) commutes with base changes by Dedekind domains when \( E \) has semi-stable reduction. Therefore \( W = W_{o,L} \), \( W' = (W_o)_{\mathcal{O}_L} \), and
\[
H^1(G, L(6\epsilon')) = H^1(G, L(6\epsilon) \otimes \mathcal{O}_L) = H^1(G, \mathcal{O}_L) \otimes L(6\epsilon)
\]
is killed by \( \pi^{2[v_K(\mathcal{D}_{L/K})]} \) (Proposition 3.6). Let \( m_1 = m - 2[v_K(\mathcal{D}_{L/K})] \geq N \). As in the proof Theorem 5.5, we have a commutative diagram
\[
\begin{array}{ccc}
L(3\epsilon'o)_{m_1} = (L(3\epsilon'o')^G)_{m_1} & \overset{\varphi_{m_1}}{\longrightarrow} & (L(3\epsilon'o')_{m_1})^G \\
\downarrow & & \downarrow \\
L(3\epsilon')_{m_1} = (L(3\epsilon')^G)_{m_1} & \overset{\varphi_{m_1}}{\longrightarrow} & (L(3\epsilon')^G)_{m_1}
\end{array}
\]
As in Lemma 5.2, the image of a Weierstrass basis \( \{1, x_o, y_o\} \) of \( W_o \) by \( \varphi_{m_1} \) lifts to a Weierstrass basis \( \{1, x, y\} \) of \( W \). Therefore \( W_{m_1} \simeq W_{o,m_1} \).

\[ \square \]

Theorem 5.8. Suppose \( K \) is henselian, \( \text{char}(K) = 0 \), and the residue field is algebraically closed of characteristic \( p > 0 \). Then there exist a positive integer \( l \), depending only on the absolute ramification index \( v_K(p) \), such that for any elliptic curve \( E \) over \( K \), and for \( L/K \) the minimal extension such that \( E_L \) has semi-stable reduction, \( W_N \) is determined by the \( G \)-action on \( W_{N+l}^G \) for any \( N \geq 0 \).

Proof. The extension \( L/K \) is Galois ([8], théorème 5.15), and it is well known that \( |G| = [L : K] \) divides 24. The corollary then follows from Remark 3.9 and Theorem 5.5.

\[ \square \]

6. From Weierstrass models to regular models

Recall that the residue field of \( K \) is perfect (to use Kodaira-Néron’s classification, and Ogg-Saito’s formula). The minimal regular model \( \mathcal{X} \) is the minimal desingularization of \( W \) ([16], Corollary 9.4.37). The models we consider are pointed with the Zariski closure of the neutral
element of $E$. In this section, we will prove that $W_{N+c}$ determines $X_Y$ for an explicit constant $c$ depending on the type of $E$ (Corollary 6.7). The next lemma results from an easy computation on the tangent spaces.

**Lemma 6.1.** Let $Y$ be a scheme flat and locally of finite type over $O_K$, of pure relative dimension $d$ and with regular generic fiber. Let $y_0$ be a closed point of the special fiber $Y_0$ of $Y$. Then $Y$ is regular at $y_0$ if and only if either $\dim T_{Y_0,y_0} = d$, or $\dim T_{Y_0,y_0} = d + 1$ and $\pi \in \mathfrak{m}^2_{Y,y_0}$. In particular, the singular locus of $Y$ is determined by $Y_1$.

**Notation** For any $O_K$-algebra $A$ and for any $n \geq 0$, we denote by

$$A[\pi^n] = \{ x \in A \mid \pi^n x = 0 \}, \quad A_{\text{tors}} = \bigcup_{n \geq 1} A[\pi^n].$$

By convention $\pi^0 = 1$ and $A[\pi^0] = 0$. We use similar notation when $A$ is replaced with a sheaf of $O_K$-algebras. The next lemma is easy.

**Lemma 6.2.** Let $U'$ be a noetherian $O_K$-scheme, and let $U = V(J)$ be the closed subscheme of $U'$ defined by a coherent sheaf of ideals $J$. Suppose $U$ is flat over $O_K$ and contains $U_K$. Then

1. $J = O_{U',\text{tors}} = O_{U'[\pi^c]}$ for some $c \geq 0$;
2. the composition of the closed immersions $U_N \to U'_N \to U'_{N+c}$ induces an isomorphism $U_N \cong V(O_{U'_{N+c}}[\pi^c])$.

Next we will give a bound for $c$ in a specific situation.

**Proposition 6.3.** Let $Z$ be a noetherian regular scheme, let $\rho : \tilde{Z} \to Z$ be the blowing-up of $Z$ along a closed point $q$, let $Y$ be an integral hypersurface in $Z$ (thus an effective Cartier divisor) passing through $q$ and let $\tilde{Y}$ be the strict transform of $Y$. Then the following properties hold:

1. $\tilde{Y}$ is an integral hypersurface in $\tilde{Z}$ and, if $\rho^*Y$ denotes the pullback of $Y$ as Cartier divisor, $\rho^*Y = \tilde{Y} + \mu_q(Y)E$

where $\mu_q(Y)$ is the multiplicity of $Y$ at $q$ and $E$ is the prime exceptional divisor $\rho^{-1}(q)$.

2. Let $\tilde{q} \in \tilde{Y}$ be a closed point lying over $q$. Then $\mu_{\tilde{q}}(\tilde{Y}) \leq \mu_q(Y)$.

3. Suppose further that $Z$ is an $O_K$-scheme, $Y$ is flat over $O_K$ and $q$ belongs to the closed fiber of $Y$. Let $r_E \geq 1$ be the multiplicity of $E$ in the special fiber of $Z$ (equal to the multiplicity at $q$ of the special fiber of $Z$) and let $c = [\mu_q(Y)/r_E]$ be the smallest integer bigger than or equal to $\mu_q(Y)/r_E$. Then $O_{\rho^*Y,\text{tors}} = O_{\rho^*Y[\pi^c]} \simeq O_{\rho^*Y[\pi^{c-1}]}$.

**Proof.** (1) is well known and (2) is a particular case of [2], Chap. I, Theorem 0.

(3) The decomposition $\rho^*Y = \tilde{Y} + \mu E$ gives an exact sequence $0 \to O_{\tilde{Z}}(-\tilde{Y})|_{\mu E} = O_{\tilde{Z}}(-\tilde{Y})/O_{\tilde{Z}}(-\tilde{Y} - \mu E) \to O_{\rho^*Y} \to O_{\tilde{Y}} \to 0.$

As $\tilde{Y}$ is flat over $O_K$, and $O_{\tilde{Z}}(-\tilde{Y})|_{\mu E}$ is of torsion (namely killed by $\pi^c$ because, by the definition of $c$, $\mu E$ is contained in $c$ times the closed fiber of $\tilde{Z}$), we have $O_{\rho^*Y,\text{tors}} = O_{\tilde{Z}}(-\tilde{Y})|_{\mu E} = O_{\rho^*Y[\pi^c]}.$
and $\pi^{-1}O_{\rho_Y,\text{tors}} \neq 0$.

**Theorem 6.4.** Let $Y$ be a flat $O_K$-scheme of finite type and let $q$ be a closed point of $Y$ contained in the closed fiber. Suppose further that $Y$ is locally at $q$ a hypersurface in a regular $O_K$-scheme of finite type. Let $Y \to \tilde{Y}$ be the blowing-up along $q$, and let $\ell \geq c$ (the constant defined as in 6.3 (3)). Then $Y_{N+\ell}$ determines $\tilde{Y}_N$ for all $N \geq 0$.

More precisely, suppose we have a discrete valuation ring $O_K$, and $(Y_0, q_0)$ over $O_K$ with similar properties and such that $q_0 \leq \ell$. If we have an isomorphism $\phi_{N+\ell}: O_{K,N+\ell} \to O_{K,N+\ell}$ and a compatible isomorphism

$$Y_{N+\ell} \simeq Y_0,N+\ell,$$

then we have an isomorphism

$$(\tilde{Y})_N \simeq (\tilde{Y}_0)_N$$

compatible with the isomorphism $O_{K,N} \simeq O_{K,N}$ induced by $\phi_{N+\ell}$.

**Proof.** We can suppose $Y$ is singular at $q$ (then $Y_0$ is also singular at $q_0$ by Lemma 6.1). Then $\dim k(q) T_{Y,q} = d + 1$ if $d = \dim O_{Y,q}$. Let $f_0, \ldots, f_d$ be a system of generators of $m_q O_{Y,q}$, let $Y'$ be the gluing of $Y \setminus \{q\}$ and of $\text{Proj} O_{Y,q}[T_0, \ldots, T_d]/(f_i T_j - f_j T_i)_{0 \leq i,j \leq d}$ along $\text{Spec}(O_{Y,q}) \setminus \{q\}$. Then $Y'$ is a flat closed subscheme of $Y'$ with generic fiber equal to that of $Y'$. Therefore $Y' = V(O_{Y',\text{tors}})$. Using a lifting in $O_{Y_0,q_0}$ of the images of the $f_i$'s in $O_{(Y_0,N+\ell),q_0}$, we define $Y'_0$ and clearly we have an isomorphism

$$Y'_{N+\ell} \simeq Y_0',N+\ell$$

extending the isomorphism $Y_{N+\ell} \simeq Y_0,N+\ell$. So, by Lemma 6.2, to show that $(\tilde{Y})_N \simeq (\tilde{Y}_0)_N$, it is enough to show that $\pi' O_{Y',\text{tors}} = 0$. This property is local on $Y$. As it trivially holds outside of $q$, it is enough to work with a small open neighborhood of $q$ in $Y$.

Write locally $Y = \text{Spec}(C/fC)$ with $(C, m_C)$ local and regular. Lift $f_0, \ldots, f_d$ to $t_0, \ldots, t_d \in C$. Since $m_C/m_C^2 \to m_q/m_q^2$ is surjective and both vector spaces have the same dimension over $k(q)$, this is an isomorphism and $t_0, \ldots, t_d$ is a system of coordinates of $C$. Consider

$$B = C[T_0, \ldots, T_d]/(f_i t_j - t_j T_i)_{i,j}.$$ 

Let $\rho: \tilde{Z} \to Z := \text{Spec } C$ be the blowing-up of $Z$ along $q$. Then $Y' = \text{Proj } B = \rho^* Y$ and Proposition 6.3 shows that $\pi' O_{Y',\text{tors}} = 0$ and we are done.

**Remark 6.5.** Note that one can not determine $(\tilde{Y})_N$ with a blowup of $Y_{N+\ell}$ because the latter process produces a scheme which is birational to $Y_{N+\ell}$, while $(\tilde{Y})_N$ has more irreducible components than $Y_{N+\ell}$.

**Remark 6.6.** Let $f: X \to W$ be the desingularization morphism of $W$. If $W$ is singular, the pre-image of the singular point of $W$ consists in $(-2)$-curves. It is well known that such singular points are rational singularities (one can apply [1], Theorem 3, because the fundamental cycle $Z$ satisfies $2p_a(Z) - 2 = Z^2 < 0$). Let $Z \to W$ be the blowing-up of the singular point of $W$. Then $Z$ is normal ([15], Proposition 8.1) and its singular points are rational singularities ([15], Proposition 1.2). Therefore the morphism $f: X \to W$ consists in successive blowing-ups

$$X = W^{(t)} \to W^{(t-1)} \to \cdots W^{(1)} \to W^{(0)} = W$$

along (reduced and discrete) singular loci.
Corollary 6.7. Let \( W \) be the minimal Weierstrass model of \( E \). Let \( X \) be the minimal regular model of \( E \) over \( O_k \) and let \( t \) be the number of blowing-ups defined as above. Then

1. \( W_{N+t} \) determines \( X_N \) (in the sense of Theorem 6.4) if \( t \geq 2t + 1 \).
2. \( t + 1 \) is bounded by the number of irreducible components of \( X_0 \). In particular, \( t \leq 8 \) if the Kodaira type of \( E \) is different from \( I_n \) and \( I_n^* \).
3. Let \( \Delta \) be the minimal discriminant of \( E \). Then \( t \leq v_K(\Delta) - 1 \), except when \( E \) has good reduction.

Proof. (1) If \( W \) is regular, then \( X = W \) and there is nothing to prove. So we suppose \( W \) is singular. The scheme \( W \) is embedded in \( Z = \mathbb{P}^2 \) as a cubic. Around the singular point \( q \), \( W \) is defined by a regular function \( y^2 + (a_1x + a_2)y - (x^3 + \cdots) \in M_y^2 \setminus M_y^3 \). So \( \mu_q(W) = 2 \). Let \( \ell \geq 1 \) be any positive integer. Applying Theorem 6.4, we see that \( W_{N+t} \) determines \( W_{N+t-2}^{(1)} \). As \( W^{(1)} \) is embedded (and has codimension 1) in \( Z \) which is regular, Proposition 6.3 and Theorem 6.4 imply that \( W_{N+t-2}^{(1)} \) determines \( W_{N+t-4}^{(2)} \). Repeating the same arguments we see that \( W_{N+t} \) determines \( W_{N+t-2t}^{(t)} \). This means that \( W_{N+t} \simeq W_{o,N+t} \) implies that \( W_{N+t-2t}^{(t)} \simeq W_{o,N+t-2t}^{(t)} \). Note that by Lemma 6.1, the isomorphism \( W_{N+t-2t}^{(t)} \simeq W_{o,N+t-2t}^{(t)} \) maps the singular locus of \( W^{(t)} \) to that of \( W_{o}^{(t)} \), so \( W_{o}^{(t+1)} \rightarrow W_{o}^{(t)} \) is the blowing-up of the singular locus of \( W_{o}^{(t)} \).

Now taking \( t = 2t \) might not be enough (when \( N = 0 \)) because we don’t know whether \( W_{o}^{(t)} \) is the minimal regular model of \( E_o \). We have to go one step further. Namely if \( W_{N+2t+1} \simeq W_{o,N+2t+1} \), then \( W_{N+1} \simeq W_{o,N+1}^{(t)} \). By Lemma 6.1, we know that \( W_{o}^{(t)} \) is regular and \( W_{o}^{(t-1)} \) is singular. Therefore, \( t = t_o \) and \( W_{o}^{(t)} = X_0 \).

(2) As each blowing-up \( W_{o}^{(t+1)} \rightarrow W_{o}^{(t)} \) introduces at least one irreducible component, we see that \( t + 1 \) is at most equal to the number of irreducible components of \( X_0 \).

(3) This is a direct consequence of (2) and Ogg-Saito’s formula.

Remark 6.8. Tate’s algorithm shows that \( W_0 \) determines whether \( E \) has type \( I_n \) (for some indeterminate \( n \geq 0 \)), \( II, III, IV, II^*, III^*, IV^* \) or \( I_n^* \) (for some indeterminate \( n \)). But \( W_0 \) can not determine the value of \( n \) in general in the case \( I_n \) or \( I_n^* \). Below is a table with more precise value of \( t \), the number of blowing-ups necessary to solve the singularities of \( W \).

<table>
<thead>
<tr>
<th>type of ( E )</th>
<th>( I_0, I_1, II )</th>
<th>III, IV</th>
<th>( II^* )</th>
<th>III^*</th>
<th>IV^*</th>
<th>( I_n )</th>
<th>( I_n^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of ( t )</td>
<td>0</td>
<td>1</td>
<td>( \leq 8 )</td>
<td>( \leq 7 )</td>
<td>4</td>
<td>([n/2])</td>
<td>( \leq n + 4 )</td>
</tr>
</tbody>
</table>

Example 6.9. Suppose \( \text{char}(k) \neq 2 \). Let \( n \geq 1 \) and consider the elliptic curves

\[
E : y^2 = (x^2 + \pi^{2n+1})(x+1), \quad E_o : y^2 = (x^2 + \pi^{2n+2})(x+1).
\]

We have \( W_{2n} \simeq W_{o,2n} \), but \( X_0 \not\simeq X_{o,0} \). Here \( t = n \) but \( t_o = n + 1 \).

7. Congruences of minimal regular models

In this section we prove the main theorem of this paper (Theorem 7.3). The idea is to show that \( X_{N+t_1+t_2} \) determines \( W_{N+t_1+t_2} \) which determines \( W_{N+t_1} \) for some \( t_2 \) and finally that \( W_{N+t_1} \) determines \( X_N \) for some \( t_1 \).
As $W$ is regular in a neighborhood of $\epsilon$, $X \to W$ is an isomorphism above a neighborhood of $\epsilon$. So denote again by $\epsilon$ the closure in $X$ of the neutral element of $E$. The effective Cartier divisor $\epsilon$ on $X$ is ample on the generic fiber and meets in the special fiber $X_0$ only in the strict transform $W_0$ of the irreducible component of $W_0$. Therefore $W$ is the contraction in $X$ of the components different from $W_0$. By construction, there is a canonical isomorphism
\[ W \simeq \text{Proj}(\oplus_{n \geq 0} H^0(X, \mathcal{O}_X(n \epsilon))) \] (see [3], Theorem 6.7/1).

**Lemma 7.1.** Let $N \geq 0$. Then
\[ W_N \simeq \text{Proj}(\oplus_{n \geq 0} H^0(X_N, \mathcal{O}_X(n \epsilon_N))) \]

**Proof.** We have to show that the canonical map
\[ H^0(X, \mathcal{O}_X(n \epsilon))_N \to H^0(X_N, \mathcal{O}_X(n \epsilon_N)) \]
is an isomorphism for all $n \geq 0$. By standard arguments ([16], Theorem 5.3.20), it is enough to show that $H^1(X, \mathcal{O}_X(n \epsilon))_0 \to H^1(X_0, \mathcal{O}_X(n \epsilon_0))$ is surjective. By duality, the latter is isomorphic to $H^0(X_0, \mathcal{O}_X(-n \epsilon_0)) = 0$ and we are done. \qed

**Proposition 7.2.** Let $W', X'$ be the minimal Weierstrass (resp. minimal regular) model of $E_L$ over $O_L$. Let $N \geq 0$. Then $(W'_N, G)$ is determined by $(X'_N, G)$.

**Proof.** It is enough to apply the isomorphism (7.1) and the previous lemma to the models $W'$ and $X'$ of $E_L$. Note that the isomorphism of 7.1 is compatible with the action of $G$ because $\epsilon$ is invariant by $G$. \qed

**Theorem 7.3.** Let $K$ be a discrete valuation field with perfect residue field. Let $E$ be an elliptic curve over $K$ of minimal discriminant $\Delta$. Let $L/K$ be a Galois extension of group $G$, of different $\mathcal{D}_{L/K}$. Then for any $N \geq 0$, the scheme $X_N$ is determined by $(X'_{N+\ell}, G)$, where
\[ \ell = 2v_K(\Delta) + 12[v_K(\mathcal{D}_{L/K})] + 18. \]

**Proof.** By Proposition 7.2, $(X'_{N+\ell}, G)$ determines $(W'_{N+\ell}, G)$. Theorem 5.5 says that the latter determines $W_{N+2v_K(\Delta)+1}$. Finally Corollary 6.7 implies that $X_N$ is determined by the previous data. \qed

**Corollary 7.4.** Suppose $K$ is strictly henselian of mixed characteristics $(0,p)$. Let $L/K$ be the minimal extension such that $E_L$ has semi-stable reduction. Then the special fiber $X_0$ is determined by the $G$-action on $X'_0$ for some $\ell_0 \geq 0$ depending only on the absolute ramification index $v_K(p)$ of $K$.

**Proof.** If $E$ has potentially multiplicative reduction, D. Lorenzini ([18], Theorem 2.8) showed that $[L : K] \leq 2$, $E$ has reduction type $I_{r+s}$, where $n = -v_K(j(E)) > 0$ and $s = v_L(\mathcal{D}_{L/K}) - 1 \geq 0$. The curve of type $I_{s}$ is unique up to isomorphism for each $r > 0$ ([20], Theorem 5.18). Hence, using Lemma 5.1, $X_0$ is determined by $X'_0$ with $\ell_0 = v_L(\mathcal{D}_{L/K}) \leq 4v_K(p)/(p-1)$ (Remark 3.9).
If $E$ has potentially good reduction, then $v_K(\Delta) \leq 12(v_K(\mathcal{O}_L/K) + 1)$ with $[L : K]$ dividing 24, hence $v_K(\mathcal{O}_L/K) \leq 24v_K(p)/(p-1)$. Then we conclude with Theorem 7.3.

Next we give some inverse results of Theorem 5.5 and Theorem 7.3.

**Proposition 7.5.** Let $N \geq v_K(\Delta)$. Then $W_N$ determines $(W'_N, G)$ for any finite Galois extension $L/K$.

**Proof.** Let $K_o, E_o$ and $L_o/K_o$ be as at the beginning of §5 and suppose we have isomorphisms

$$\mathcal{O}_L \simeq \mathcal{O}_{o,L}, \quad W_N \to W_{o,N}, \quad \theta_N : \mathcal{O}_{L,N} \simeq \mathcal{O}_{o,N},$$

the last one being $G$-equivariant. We have to find a $G$-equivariant isomorphism $W'_N \to W'_{o,N}$.

Let $\{x, y\}$ (resp. $\{x', y'\}$) be a Weierstrass basis of $W$ (resp. of $W'$). By Lemma 4.4, we have a change of coordinates of $E_L$:

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t, \quad u, r, s, t \in \mathcal{O}_L.$$ 

Let $\phi : L(6e) \simeq L(6e_o)$ be the isomorphism induced by $W_{o,N} \simeq W_N$. Let $\{x_o, y_o\}$ be a Weierstrass basis of $W_o$ lifting the image by $\phi_N$ of the class of $\{x, y\}$ in $L(6e)$. Let $u_o, r_o, s_o, t_o \in \mathcal{O}_{L_o}$ be liftings of the images by $\theta_N$ of the classes $u, r, s, t \in \mathcal{O}_{L,N}$. Let

$$x'_o = (x_o - r_o)/u'_o, \quad y'_o = (y_o - u_o^2s_ix'_o - t_o)/u'_o \in \mathcal{O}_{L'_o} \cap L_o.$$ 

We claim that $\{x'_o, y'_o\}$ is a Weierstrass basis of $W'_o$. First, the fact that $\{x, y\}$ defines a Weierstrass model

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

over $\mathcal{O}_L$ implies that

$$u \mid a_1 + 2s, \quad u^2 \mid a_2 - sa_1 + 3r - s^2, ..., \quad u^6 \mid a_6 + ra_4 + ra_0^3 - ta_3 - t^2 - rta_1$$

(see [7], page 57, (1.6)). As

$$6v_L(u) = (v_L(\Delta) - v_L(\Delta'))/2 \leq v_L(\Delta)/2 \leq v_L(\pi^N),$$

the above divisibility relations hold in $L(6e'_o)$. Therefore $\{x'_o, y'_o\}$ is a Weierstrass basis of some Weierstrass model over $\mathcal{O}_{L'_o}$. In particular

$$v_L(\Delta') = v_L(\Delta) - 12v_L(u) = v_L(\Delta_o) - 12v_{L_o}(u_0) \geq v_L(\Delta'_o).$$

But by symmetry, $v_L(\Delta'_o) \geq v_L(\Delta'_o)$, so the equality holds and the Weierstrass model associated to $\{x'_o, y'_o\}$ is minimal over $\mathcal{O}_{L'_o}$.

As the change of variables from $\{x, y\}$ to $\{x', y'\}$ and from $\{x_o, y_o\}$ to $\{x'_o, y'_o\}$ are given by the same relations modulo $p^{N+1}$ (up to $\theta_N$), and $\{x, y\}, \{x_o, y_o\}$ define the same equation up to $\mathcal{O}_{L,K} \simeq \mathcal{O}_{o,L,K}$, we have an isomorphism $W_N \to W'_{o,N}$ corresponding to $x' \to x'_o$ and $y' \to y'_o$.

**Proposition 7.6.** Let $\Delta$ be the minimal discriminant of $E$. Then for any $N \geq 0$, $(X'_N, G)$ is determined by $X_{N+2v_L(\Delta) - 1}$.

**Proof.** Let $\ell = 2v_K(\Delta) - 1$. First, by Proposition 7.2 (for $L = K$), $X_{N+\ell}$ determines $W_{N+\ell}$. Second, $W_{N+\ell}$ determines $(W'_{N+\ell}, G)$ by Proposition 7.5. Finally, the latter determines $(X'_N, G)$ by similar arguments than Corollary 6.7 (note that $v_L(\Delta)$ is bigger than or equal to the $v_L$ valuation of the minimal discriminant of $E_L$).
Proposition 7.7. Suppose char(K) = p > 0. Let N ≥ 0. Then there exists a discrete valuation field Kn of characteristic 0, with residue field equal to k, and an elliptic curve Em over Kn such that
\[ \mathcal{O}_{K_n,N} \cong \mathcal{O}_{K,N}, \quad W_{o,N} \cong W_N, \quad X_{o,N} \cong X_N. \]

Proof. Let \( n \geq \max\{5, N + 2v_K(\Delta) - 1\} \) so that \( X_N \) is determined by \( W_n \) (Corollary 6.7). Let \( \mathcal{O}_{K_n} = W(k)[t]/(t^{n+1} - p) \) with uniformizing element \( \pi_n = t \). Then \( \mathcal{O}_{K_n,n} \cong \mathcal{O}_{K,n} \). Lifting \( W_n \) to a Weierstrass equation \( \mathcal{O}_n \) over \( \mathcal{O}_{K_n} \), then we have \( v_{K_n}(\Delta_n) = v_K(\Delta) \) and \( W_n \) is minimal by Lemma 5.3(2). By construction, \( W_n \cong W_{o,n} \) (hence \( W_N \cong W_{o,N} \)). Again by Corollary 6.7, we have an isomorphism \( X_N \cong X_{o,N} \).

8. Lifting equivariant infinitesimal sections

In our settings, Weierstrass models come with a fixed section. But in Proposition 4.3, we saw that up to isomorphism, the choice of a section does not really matter. We can wonder whether in Theorem 5.5 we can dismiss the given section of \( W_m \). Note that, at least in our proof, we use the fact that this section of \( W_m \) is \( G \)-equivariant and even more, that it extends to a section of \( W \) over \( O_L \) induced by a rational point in \( E(K) \). Now suppose we are given \( (\text{Iso}_{o,m}) \) as at the beginning of §5 but without the condition that the isomorphism \( W_m \cong W_{o,m} \) maps \( \epsilon_m \) to \( \epsilon_{o,m} \). The image of \( \epsilon_m \) is a \( G \)-equivariant section of \( W_{o,m} \) contained in the smooth locus of \( W_{o,m} \). If for some \( m \geq 1 \), the image of \( \epsilon_m \) in \( W_{o,m} \) extends to a section \( Q \) of \( W_o \) induced by a rational point of \( q \in E_o(K_o) \) (equivalently, \( Q \) is a \( G \)-equivariant section of \( W_o \)), then by Proposition 4.3 we have a \( G \)-equivariant isomorphism \( W_m \cong W_{o,m} \) which maps \( \epsilon_m \) to \( \epsilon_{o,m} \), and we can apply Theorem 5.5 with \( m \) instead of \( m \). See Corollary 8.4 for some results on \( m \).

Let \( S \) be a scheme. Let \( f : X' \to S' \) be a morphism of \( S \)-schemes and let \( H \) be a group acting on the \( S \)-schemes \( X' \), \( S' \) compatibly with \( f \) (in other words, \( f \) is \( H \)-equivariant). Then \( H \) acts on the set of sections \( X'(S') \) in the following way: for any section \( \rho : S' \to X' \) and for any \( \sigma \in H \), we put \( \sigma \circ \rho = \sigma \circ \rho \circ \sigma^{-1} \in X'(S') \). A section \( \rho \) is said \( H \)-equivariant if \( \sigma \circ \rho = \rho \) for all \( \sigma \in H \). The set of \( H \)-equivariant sections will be denoted by \( X'(S')^H \). The above question is to study the image of the canonical map
\[ W'(O_L)^G \to W'(O_L/\pi^{m+1}O_L)^G. \]
Suppose from now on that \( S' \to S \) is finite and locally free and \( X' \to S' \) is quasi-projective. Then the Weil restriction \( R_{S'/S} X' \) exists ([3], Theorem 7.6/4) over \( S \) and is endowed with a canonical action of \( H \). Moreover, for any \( S \)-scheme \( T \), letting \( H \) act trivially on \( T \) and denoting \( Y = R_{S'/S} X' \), \( X'(S' \times_S T)^H \) is canonically isomorphic to \( Y(T)^H \). Suppose further that \( H \) is finite. Let \( Y^H(T) \) be the scheme of fixed points under \( H \) (see e.g. [10], §3). Then by definition \( Y(T)^H = Y^H(T) \).

Let \( S = \text{Spec} \mathcal{O}_K, S' = \text{Spec} \mathcal{O}_L, S_m = \text{Spec}(\mathcal{O}_K/\pi^{m+1}O_K) \) and \( S'_m = S' \times_S S_m \).

Proposition 8.1. Let \( Z' \) be a flat quasi-projective scheme over \( S' \) endowed with an equivariant action of \( G = \text{Gal}(L/K) \). Then the following properties hold.
(1) Let \( Z = Z'/G \). Then the canonical map \( Z(S) \to (Z')^G \) is bijective.
(2) Suppose \( S' \to S \) is étale. Then the canonical morphism \( Z' \to Z \times_S S' \) is an isomorphism and the canonical morphism \( Z \to (R_{S'/S} Z')^G \) is an isomorphism.
(3) Suppose that $K$ is henselian, $Z'$ is smooth over $S'$ and $L/K$ is tamely ramified. Then the canonical map

$$Z'(S')^G \to Z'(S_m')^G$$

is surjective for all $m \geq 0$.

(4) Suppose that $K$ is henselian and that $Z'_K$ is smooth over $L$. Then there exist $m_0, r_0 \geq 0$ such that for all $m \geq m_0$, and for any $t_m \in Z'(S_m')^G$, the image of $t_m$ in $Z'(S_{m-r_0})^G$ lifts to a section in $Z'(S')^G$.

Proof. (1) First notice that the quotient $Z'/G$ exists because $Z'$ is quasi-projective over $O_L$. The canonical morphism $Z'_K \to (Z_K)_L$ is an isomorphism by Lemma 3.2. The canonical map

$$Z'(O_L)^G \to Z(O_K) \subseteq Z_K(K)$$

is injective. Conversely, any section in $Z(O_K)$ induces a rational point in $Z_K(K) = Z'_L(L)^G \subseteq Z'_L(L)$. The valuative criterion of properness for $Z' \to Z$ implies that the point in $Z'_L(L)$ we obtain actually belongs to $Z'(O_L) \cap Z'_L(L)^G = Z'(O_L)^G$. Therefore $Z'(O_L)^G \to Z(O_K)$ is surjective.

(2) The canonical morphism $Z' \to Z \times_S S'$ is an isomorphism by Proposition 3.3. For any $O_K$-module $M$ with trivial action of $G$, the canonical map $M \to (M \otimes_{O_K} O_L)^G$ is an isomorphism (use a normal basis of $O_L/O_K$). For any $S$-scheme $T$, the canonical map

$$Z(T) \to R_{S'/S} Z'(T)^G = (Z'(T \times_S S'))^G = (Z(T \times_S S'))^G = Z(T)$$

is bijective. So $Z \to (R_{S'/S} Z')^G$ is an isomorphism.

(3) Let $Y = R_{S'/S} Z'$. We saw above that $Z'(S')^G \to Z'(S_m')^G$ can be identified with the canonical map

$$Y^G(S) \to Y^G(S_m).$$

Let $I \subset G$ be the inertia group, let $L_1 = L^I$, $H = G/I$ and let $S' = S'/I$. Denote by $Z' = R_{S'/S} Z'$. It is smooth over $S'$ ([3], Proposition 7.6/5) as well as $Z_1 := (Z')^f$ ([10], Proposition 3.4). Let $T$ be an $S$-scheme with trivial action of $G$. Then

$$Z'(T \times_S S')^G = (Z'(T \times_S S'))^H = (Z_1(T \times_S S'))^H.$$

Let $Z_2 = Z_1/H$. By (2), $Z_2$ is smooth over $S$ and $(Z_1(T \times_S S'))^H = Z_2(T)$. Thus $Z'(T \times_S S')^G = Z_2(T)$. As $O_K$ is henselian and $Z_2$ is smooth, $Z_2(S) \to Z_2(S_m)$ is surjective and (3) is proved.

(4) Applying (2) to Spec $L \to$ Spec $K$, we see that $Z_K$ is smooth over $K$ and $(Y^G)_K = (Y_K)^G = Z_K$. Our statement then results from Elkik’s approximation theorem ([11], Corollaire 1, page 567) and the identity $Y^G(T) = Z'(T \times_S S')^G$ for all $S$-schemes $T$.

Remark 8.2. Keep the notation of Proposition 8.1.

(i) If $K$ is henselian, $L/K$ is tamely ramified and $Z'$ is smooth, it is probably true that the canonical map

$$Z'(O_L)^G \to Z'(\text{Spec}(O_L/\pi_L^{m+1} O_L))^G$$

is surjective for all $m \geq 0$. Note that the right-hand side is not $Z'(S_m')^G$.

(ii) The constants $m_0, r_0$ in 8.1(4) depend on the scheme $Z'$. When the latter is smooth over $S'$, it is probably true that one can find bounds $m_0, r_0$ depending only on $v_K(D_{L/K})$.

Next we give an explicit bound on the constants $m_0, r_0$ of 8.1 (4) for abelian varieties.
Proposition 8.3. Suppose $K$ is complete with $\text{char}(K) = 0$ and residue characteristic $p \geq 0$. Let $m = 2[v_K(\mathcal{O}_L/K)]$ and let

$$m + 1 > h + v_K(p) - 1 + \frac{v_K(p)}{p - 1}.$$  

($m \geq 0$ if $p = 0$.) Let $A$ be an abelian variety over $K$ and let $A'$ be its Néron model over $\mathcal{O}_L$. Then for any $G$-equivariant section $t_m \in A'((\mathcal{O}_L/\pi^m+1\mathcal{O}_L)^G)$ of $A'$, there exists a $G$-equivariant section in $A'((\mathcal{O}_L)^G = A(K)$ whose image in $A'((\mathcal{O}_L/\pi^m+1\mathcal{O}_L)$ coincides with that of $t_m$.

Proof. Let $\bar{A}'$ be the formal group over $\mathcal{O}_L$ attached to $A'$. Let $r$ be the smallest integer $\geq h/v_K(p)$. For all integers $n > rv_K(p) \geq 0$, we have a canonical commutative diagram with exact horizontal lines

$$0 \longrightarrow \bar{A}'(\pi^n\mathcal{O}_L) \longrightarrow A'(\mathcal{O}_L) \longrightarrow A'(\mathcal{O}_L/\pi^n\mathcal{O}_L) \longrightarrow 0$$

Taking Galois cohomology, we get

$$\bar{A}'(\mathcal{O}_L)^G \longrightarrow A'(\mathcal{O}_L/\pi^n\mathcal{O}_L)^G \longrightarrow H^1(G, \bar{A}'(\pi^n\mathcal{O}_L))$$

$$0 \longrightarrow \bar{A}'(\pi^n\mathcal{O}_L) \longrightarrow A'(\mathcal{O}_L) \longrightarrow A'(\mathcal{O}_L/\pi^n\mathcal{O}_L) \longrightarrow 0.$$  

So it is enough to show that $f_{n,r} = 0$ when $n > rv_K(p) + v_K(p)/(p - 1)$. By general results on the formal groups of abelian varieties (see [19], §1, [13], Theorem 1, or [28], §2.4, p. 196), $\bar{A}'(\pi^n\mathcal{O}_L)$ is canonically isomorphic to $\pi^n\mathcal{O}_L$ for all $\ell \geq v_K(p)/(p - 1)$. So the canonical map $f_{n,r}$ can be identified with the multiplication-by-$p^r$ map on $H^1(G, \mathcal{O}_L)$. This is the zero map by Proposition 3.6(2), thus $f_{n,r} = 0$. As $rv_K(p) \leq h + v_K(p) - 1$, the proposition is proved.

Corollary 8.4. Let $K, K_o$ be henselian discrete valuation fields. Let $E, E_o, L, L_o, G$ and $W', W'_o$ be as at the beginning of §5.

(1) Let $\epsilon', \epsilon'_o$ be the unit sections of $W', W'_o$ respectively. Then there exists an integer $r$ such that for any $m \geq r$, if there are compatible $G$-equivariant isomorphisms

$$\theta_m : \mathcal{O}_{L,m} \simeq \mathcal{O}_{L_o,m}, \quad f_m : W'_m \simeq W'_o,$$

then there exists a $G$-equivariant isomorphism $W'_{m-r} \simeq W'_o$, compatible with the isomorphism $\mathcal{O}_{L,m-r} \simeq \mathcal{O}_{L_o,m-r}$ and which maps $\epsilon'_{m-r}$ to $\epsilon'_o$.

(2) Suppose that $K$ is complete of characteristic 0. Then one can take $r = 2[v_K(\mathcal{O}_L/K)]$ provided $m$ satisfies the inequality of Proposition 8.3.

Proof. The image of $\epsilon'_{o,m}$ by $f_m^{-1}$ is a $G$-equivariant section of $W'_o$. Let $r$ be the maximum of $m_0, r_0$ given in Proposition 8.1 (4). Then $f_m^{-1}(\epsilon'_{o,m-r}) \in W'_{m-r}((\mathcal{O}_{L,m-r})$ lifts to a $G$-invariant section $a \in W'(\mathcal{O}_L)$. Let $t : W' \to W'$ be the translation by $a$. This is a $G$-equivariant isomorphism, and the composition $f_{m-r} \circ t_{m-r} : W'_{m-r} \to W'_{o,m-r}$ is an isomorphism compatible with $\mathcal{O}_{L,m-r} \simeq \mathcal{O}_{L_o,m-r}$, and taking $\epsilon'_{m-r}$ to $\epsilon'_{o,m-r}$. This proves (1).

To prove (2), it is enough to notice that the smooth locus of $W'$ is the neutral component of $E'$. As $\epsilon'_{o,m}$ is contained in the smooth locus, we have $f_m^{-1}(\epsilon'_{o,m}) \subset E'_m((\mathcal{O}_L,m)$. By 8.3, $f_m^{-1}(\epsilon'_{o,m-r})$ lifts to a section in $E'(\mathcal{O}_L)^G \subset W'(\mathcal{O}_L)^G$. The proof is then achieved as in Part (1).
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