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INEQUALITY FOR CONDUCTOR AND DIFFERENTIALS OF A CURVE OVER A LOCAL FIELD

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Let $\mathcal{O}_K$ be a discrete valuation ring with perfect residue field. Consider a proper smooth and geometrically connected curve X_K over the field of fractions K of $\mathcal{O}_K$. Let X be a regular, proper flat model of X_K over $\mathcal{O}_K$. In this paper, we study and establish an inequality (Theorem 1) between the conductor of the curve X_K and the length of the cokernel of the canonical map

$$H^0(X, \Omega_{X/\mathcal{O}_K}) \to H^0(X, \omega_{X/\mathcal{O}_K}). \tag{1}$$

Here $\Omega = \Omega_{X/\mathcal{O}_K}$ denotes the sheaf of regular differentials and $\omega = \omega_{X/\mathcal{O}_K}$ denotes the relative canonical sheaf $\det \Omega_{X/\mathcal{O}_K}$ of X respectively. The map (1) is an isomorphism on the generic fiber, hence its cokernel has finite length over $\mathcal{O}_K$. Denote

$$\operatorname{Effcond}(X_K/K) := \operatorname{length}_{\mathcal{O}_K} \operatorname{Coker}(H^0(X, \Omega) \to H^0(X, \omega)).$$

In [PS], this integer is called *the efficient conductor* (conducteur efficace) of $X/\mathcal{O}_K$. It depends only on the generic fiber X_K (Lemma 4). The main result of this paper is the following :

Theorem 0. *Assume that X_k is not a multiple fiber. Then*

$$\operatorname{Effcond}(X_K/K) \leq \operatorname{Art}(X_K/K)$$

where the right hand side is the Artin conductor of X_K (see Theorem 1).

The main ingredients of the proof are the injectivity of the map (1) and a theorem of Bloch ([B], Theorem (2.3)) which computes $\operatorname{Art}(X_K/K)$ in terms of the characteristic of the complex $\Omega \to \omega$. When X_K is an elliptic curve, this inequality is proved by Pesenti and Szpiro ([PS], Théorème 2.2) in equal characteristic case using a different method. They also conjectured that this inequality is true in mixed characteristic case (op.cit., Conjecture 2.3). The next corollary is a generalization of an inequality of Szpiro ([Sz], page 8) for families of elliptic curves. In a talk at the Japan-US Mathematical Institute conference in 1997 where the second author attended, Szpiro presented Theorem 0 for elliptic curves (in equal characteristic) as a mean of proving this corollary. This is done in [PS], Théorème 0.2. The result was announced in [GS], Theorem 3.

Corollary 0. *Let C be a geometrically connected projective smooth curve over a perfect field k of characteristic $p \geq 0$. Let $f : E \to C$ be a non-isotrivial elliptic fibration (with a section). Let $\Delta_{E/C}$ and $\mathrm{Art}(E/C) \in \mathrm{Pic}(C)$ be respectively the minimal discriminant and conductor divisors of $E \to C$. Then*

$$\deg \Delta_{E/C} \leq 6p^e(2g(C) - 2 + \deg \mathrm{Art}(E/C))$$

where p^e is the modular inseparability degree of $E \to C$.

The proof of Corollary 0 (see Corollary 4) is exactly as in [Sz] if $e = 0$. When $e > 0$, our proof is different from (and in some sense orthogonal to) that of [PS]. Finally we end the paper by giving a lower bound of $\mathrm{Effcond}(X_K/K)$ (Propositions 2 and 3).

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1. Vanishing of $H^0(X,\ (\Omega_{X/\mathcal{O}_K})_{\mathrm{tors}})$.

Throughout this paper, π denotes a prime element of K and $k = \mathcal{O}_K/\pi\mathcal{O}_K$ is the residue field of K.

To simplify notations, and when no confusion is possible, we denote $\Omega = \Omega_{X/\mathcal{O}_K}$, $\omega = \omega_{X/\mathcal{O}_K}$. Assume that X is a curve over $\mathcal{O}_K$, then we have a canonical map $\Omega \to \omega$. Let Ω_{tors} be the submodule of Ω consisting in the $\mathcal{O}_X$-torsion elements. Since ω is invertible, Ω_{tors} is also the kernel of $\Omega \to \omega$. In this section, we prove the vanishing of the global sections of Ω_{tors} under a certain mild condition. Before stating the main result of this section, we collect some technical results on the sheaf $\Omega_{X/\mathcal{O}_K}$.

Lemma 1. *Let X be a connected regular n-dimensional scheme, flat and of finite type over $\mathcal{O}_K$, with smooth generic fiber X_K. Denote by $\Omega = \Omega_{X/\mathcal{O}_K}$, and by $\omega = \omega_{X/\mathcal{O}_K}$ the dualizing sheaf. Let $D = V(\mathcal{A}nn(\Omega_{\mathrm{tors}}))$. Then we have the following properties :*

(1) *Let $x \in X$. Then in some neighborhood of x, X is a divisor $V(f)$ in a smooth scheme P over $\mathcal{O}_K$.*
(2) *Consider the natural map from $\Omega^{n-1} := \Omega^{n-1}_{X/\mathcal{O}_K}$ to ω. Let $x \in X_k$ be a closed point and let $\pi, z_1, \ldots, z_n$ be a system of coordinates of P at x. Then the image of Ω^{n-1}_x in ω_x is $(a_1, \ldots, a_n)\omega_x$, where a_i is the image of $\partial f/\partial z_i$ in $\mathcal{O}_{X,x}$.*
(3) *The closed subscheme D is a divisor on X, and Ω_{tors} is an invertible $\mathcal{O}_D$-module. The image of Ω^{n-1} in ω is contained in $\omega(-D)$, and the cokernel $\mathrm{Coker}(\Omega^{n-1} \to \omega(-D))$ has support in codimension ≥ 2.*
(4) *Let C be an irreducible component of X_k of multiplicity r. Let $\nu_C(D)$ denote the coefficient of C in D. Then $\nu_C(D) \geq r - 1$, and the equality holds if and only if r is prime to $p := \mathrm{char}\, k$.*
(5) *Let $x \in X_k$. Let $C_1, \ldots, C_m$ be the irreducible components of X_k passing through x and let r_j denote the multiplicity of C_j in X_k.*
   (5.1) *The image of Ω^{n-1}_x in ω_x is contained in $\sum_j \omega_x(-X_k + C_j)$.*
   Assume that $X_{k,\mathrm{red}}$ is a simple normal crossings divisor at x.

(5.2) *If exactly one of the r_j's, say r_1, is prime to p, then $\nu_{C_1}(D) = r_1 - 1$, $\nu_{C_j}(D) = r_j$ for $j \neq 1$, and the image of Ω_x^{n-1} in ω_x is $\omega_x(-D)$.*

(5.3) *If $m = 2$, $n = 2$ and r_1, r_2 are prime to p, then the image of Ω_x in ω_x is $\mathfrak{m}_{X,x}\omega_x(-D)$.*

Proof. (1). One can embed locally X in a smooth scheme P over $\mathcal{O}_K$. Assume that $\dim_x P > \dim_x X + 1$, then $\dim_{k(x)} T_{P_k,x} = \dim_x P - 1 > \dim_x X \geq \dim_{k(x)} T_{X_k,x}$. It is easy to see that there exists $f \in \operatorname{Ker}(\mathcal{O}_{P,x} \to \mathcal{O}_{X,x}) \backslash (\pi\mathcal{O}_{P,x} + \mathfrak{m}_{P,x}^2)$, where $\mathfrak{m}_{P,x}$ is the maximal ideal of $\mathcal{O}_{P,x}$. Replacing P by $V(f)$, we get a local embedding of X in a smooth scheme of dimension $\dim_x P - 1$. Repeating this process if necessary, we get a local embedding of X at x in a smooth scheme P of dimension $\dim_x X + 1$, and statement (1) is proved

(2). For simplicity, we denote by dz_i the image of $dz_i \in \Omega_{P,x}$ in $\Omega_{X,x}$. Then Ω_x is generated by the dz_i's with one relation $df = 0$. For any $i \leq r$, put

$$\delta_i = dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n \in \Omega_x^{n-1}.$$

Then $(-1)^i a_i \delta_j = (-1)^j a_j \delta_i$ for all i, j. Denote by $\tilde{\delta}_i$ the canonical image of δ_i in $\Omega^{n-1}_{K(X)/K}$. Then ω_x, considered as a subgroup of $\Omega^{n-1}_{K(X)/K}$, is generated over $\mathcal{O}_{X,x}$ by the rational differential $a_i^{-1}\tilde{\delta}_i$ for any $a_i \neq 0$. On the other hand, if $a_j = 0$, then $\delta_j = 0$. Since Ω_x^{n-1} is generated by the δ_i's, the image of Ω_x^{n-1} in ω_x is $(a_1, \ldots, a_n)\omega_x$.

(3). The property is local on X, and it is enough to consider closed points $x \in X_k$. Then it is easy to check that $\mathcal{A}nn(\Omega_{\text{tors}})_x = \operatorname{Ann}(\Omega_{\text{tors},x})$ is generated by $c := \gcd(a_1, \ldots, a_n)$, and $\Omega_{\text{tors},x}$ is generated by $\sum_i (c^{-1}a_i)dz_i$. This implies that Ω_{tors} is an invertible $\mathcal{O}_D$-module. Further, the image of Ω_x^{n-1} in ω_x is $(a_1, \ldots, a_n)\omega_x \subseteq c \cdot \omega_x = \omega_x(-D)$. Since the elements $c^{-1}a_i$ have no g.c.d in $\mathcal{O}_{X,x}$, they generated an ideal of height at least 2. This proves the assertion on $\operatorname{Coker}(\Omega^{n-1} \to \omega(-D))$.

(4). Let x be a closed point of C smooth in $(X_k)_{\text{red}}$. It is enough to compute $\operatorname{Ann}(\Omega_{\text{tors},x})$. If $r = 1$, then $\nu_C(D) = 0 = r - 1$. Otherwise, up to a multiplicative factor in $\mathcal{O}^*_{P,x}$, we can write f as $f = uh^r - \pi$, with $u \in \mathcal{O}^*_{P,x}$ and $h = z_1$ is an element of a system of coordinates of P at x. Now the statement follows immediately from the calculation of $\partial f/\partial z_i$ and the description of $\operatorname{Ann}(\Omega_{\text{tors},x})$ as above.

(5). We can assume that x is closed and $\sum_j r_j \geq 2$. Let $h_j \in \mathcal{O}_{P,x}$ whose image in $\mathcal{O}_{X,x}$ is a local equation of C_j at x. Then up to a multiplicative factor in $\mathcal{O}^*_{P,x}$, f has the form $f = u\prod_j h_j^{r_j} - \pi$ with $u \in \mathcal{O}^*_{P,x}$. Moreover, if $(X_k)_{\text{red}}$ has strict normal crossings at x, we can find a system of coordinates $\pi, z_1, \ldots, z_n$ of P at x such that $z_j = h_j$ for all $1 \leq j \leq m$. Now the statements follow easily from the calculation of $\partial f/\partial z_i$.

Proposition 1. *Let X be a proper regular flat curve over $\mathcal{O}_K$ with smooth and geometrically connected generic fiber X_K. We write $X_k = \sum_i r_i C_i$ as Cartier divisors, where the C_i's are the irreducible components of X_k. Assume that the greatest common divisor d of the multiplicities r_i is equal to 1. Then we have*

$$H^0(X,\ \Omega_{\text{tors}}) = 0 \tag{1}$$

$$H^1(X,\ \operatorname{Coker}(\Omega \to \omega)) = 0. \tag{2}$$

Proof. (1). Let D be the closed subscheme of X defined by the annihilator ideal $\mathcal{A}nn(\Omega_{\text{tors}})$. It is a divisor on X (Lemma 1 (2)). The vanishing will be deduced from the following two facts.

(A) Ω_{tors} is "numerically isomorphic" to $\mathcal{O}_D(D)$.
(B) D does not contain an effective divisor proportional to the closed fiber X_k.

For a divisor D supported on the closed fiber, invertible $\mathcal{O}_D$-modules L and M are said to be *numerically isomorphic* if $\deg(L|_C) = \deg(M|_C)$ for all irreducible components C of D. A divisor D satisfies the condition (B) if and only if $D \not\geq \frac{1}{d}X_k$ where d denotes the greatest common divisor of the multiplicities of the closed fiber X_k. The following lemma shows that the conditions (A) and (B) imply the first vanishing in Proposition 1.

Lemma 2. *Let X be a regular proper and flat curve over $\mathcal{O}_K$, let D be an effective divisor of X supported on the closed fiber X_k. Suppose moreover that D does not contain an effective divisor proportional to X_k. If L is an invertible $\mathcal{O}_D$-module numerically isomorphic to $\mathcal{O}_D(D)$, then we have*

$$H^0(D, L) = 0.$$

Proof. Write $D = \sum_i m_i C_i$ with C_i irreducible. We start by proving the existence of an irreducible component C of D such that $\deg L|_C < 0$. It is enough to find a component C satisfying $D \cdot C < 0$. Actually, we have $\deg L|_C = \deg \mathcal{O}_D(D)|_C = \deg \mathcal{O}_X(D)|_C = D \cdot C$. Since D is not proportional to X_k, we have $\sum_i m_i D \cdot C_i = D^2 < 0$ (SGA7 II, exposé XII, corollaire 1.8). This implies the existence of a component C of D with $D \cdot C < 0$.

Now we prove the vanishing by induction on $m = \sum_i m_i > 0$. If $m = 1$, then D is integral and $\deg L = D^2 < 0$, thus $H^0(D, L) = 0$. Assume $m > 1$. Let C be as above. We put $D' = D - C$ and define an invertible $\mathcal{O}_{D'}$-module L' by the exact sequence $0 \to L' \to L \to L|_C \to 0$. Then since $\deg L|_C < 0$, we have $H^0(C, L|_C) = 0$ and $H^0(D, L) = H^0(D', L')$. The pair D' and L' satisfies the assumption of Lemma 2 since $L' \simeq L|_{D'} \otimes \mathcal{O}_X(-C)$ is numerically isomorphic to $\mathcal{O}_{D'}(D - C) = \mathcal{O}_{D'}(D')$. We have $D' < D$, so by the assumption of induction, we get $H^0(D, L) = H^0(D, L') = H^0(D', L') = 0$.

Continuing proof of Proposition 1. Now to prove the first part of Proposition 1, it is enough to check that the conditions (A) and (B) are satisfied. Firstly, (A) is essentially [B] Lemma (3.1) (cf. [Sa1] Proposition (3.1)). Let $i : D \to X$ denote the closed immersion. The local resolution $\Omega \simeq \mathrm{Coker}(\mathcal{O}_X \to \mathcal{O}_X dz_1 \oplus \mathcal{O}_X dz_2 : 1 \mapsto df)$ gives an isomorphism $\Omega_{\text{tors}} \simeq L^1 i^* \Omega(D)$ of invertible $\mathcal{O}_D$-modules. In fact, writing the resolution as $L \to E \to \Omega$, we get $\Omega_{\text{tors}} = L(D)|_D$ and $L^1 i^* \Omega = L|_D$. It is easily checked that this gives a globally well-defined isomorphism. The invertible $\mathcal{O}_D$-module $L^1 i^* \Omega$ is $\mathcal{L}^* \otimes_{\mathcal{O}_Z} \mathcal{O}_L$ for $\mathcal{L}$ in [B] loc.cit. and (A) is nothing other than Lemma (3.1) loc.cit.

Finally let us check the condition (B). More precisely, let d denote the g.c.d of the multiplicities of the closed fiber X_k, we will show that the condition $d = 1$ is equivalent to $D = \sum_i \ell_i C_i \not\geq \frac{1}{d}X_k$ (which is the condition (B)). If $d = 1$, then there is a component C_i with $p \nmid r_i$. Hence $\ell_i = r_i - 1$ by Lemma 1 (4) and $D \not\geq X_k = \frac{1}{d}X_k$. If $d > 1$, then again by Lemma 1 (4), we have $D \geq X_k - \frac{1}{d}X_k \geq \frac{1}{d}X_k$.

Proof of Proposition 1 (2). Using the exact sequence $0 \to \omega(-D) \to \omega \to \omega|_D \to 0$ and Lemma 1 (3), we get a new exact sequence

$$0 \to \mathrm{Coker}(\Omega \to \omega(-D)) \to \mathrm{Coker}(\Omega \to \omega) \to \omega|_D \to 0.$$

Since the support of $\mathrm{Coker}(\Omega \to \omega(-D))$ consists of finite number of closed points, we have $H^1(X, \mathrm{Coker}(\Omega \to \omega(-D))) = 0$ and thus an isomorphism

$$H^1(X, \mathrm{Coker}(\Omega \to \omega)) \simeq H^1(D, \omega|_D).$$

By Grothendieck duality and the adjunction formula $\omega_D = \omega(D)|_D$, the cohomology $H^1(D, \omega|_D)$ is the dual of $H^0(D, \mathcal{O}_D(D))$ and the latter is 0 by Lemma 2.

We easily deduce the following from Proposition 1.

Corollary 1. *Let X be as in Proposition 1. Then $H^0(X, \Omega) \to H^0(X, \omega)$ is injective and $H^1(X, \Omega) \to H^1(X, \omega)$ is surjective.*

The following elementary fact will be used in the sequel.

Fact 1. *Let X be a proper curve over $\mathcal{O}_K$. Let $\mathcal{F}$ be a coherent sheaf over X, flat over $\mathcal{O}_K$. Then*

(1) $H^1(X, \mathcal{F}) \otimes k \simeq H^1(X_k, \mathcal{F}|_{X_k})$.

(2) $H^1(X, \mathcal{F})$ *is torsion free if and only if* $H^0(X, \mathcal{F}) \otimes k \simeq H^0(X_k, \mathcal{F}|_{X_k})$.

For the kernel of the surjection $H^1(X, \Omega) \to H^1(X, \omega)$, we know the following fact due to Raynaud.

Lemma 3 (cf. [R] Théorème (8.2.1)). *Keep the assumptions of Proposition 1. Then $\mathrm{Ker}(H^1(X, \Omega) \to H^1(X, \omega))$ is equal to the torsion part $H^1(X, \Omega)_{\mathrm{tors}}$.*

Proof. Since $H^1(X, \Omega) \to H^1(X, \omega)$ is surjective and is an isomorphism over K, the lemma is equivalent to say that $H^1(X, \omega)$ is torsion free. Using Fact 1 and Grothendieck duality, the latter is equivalent to $H^0(X_k, \mathcal{O}_{X_k}) = k$. Here we give a proof of this equality using Lemma 2 (see also [A-W], Lemma 2.6). Let C be an irreducible component of X_k. It is enough to show that $H^0(X_k, \mathcal{O}_{X_k}) \to H^0(C, \mathcal{O}_C)$ is injective. Actually, this will imply that $H^0(X_k, \mathcal{O}_{X_k})$ is a finite separable extension of k. Since X_k is geometrically connected, we will get $H^0(X_k, \mathcal{O}_{X_k}) = k$.

Let $D = X_k - C$ as divisor. We can assume that $D \neq 0$. Then we have the following exact sequence

$$0 \to \mathcal{O}_D(-C) \to \mathcal{O}_{X_k} \to \mathcal{O}_C \to 0$$

By assumption, the g.c.d of the multiplicities is 1, therefore D does not contain an effective divisor proportional to X_k. Applying Lemma 2, and since $\mathcal{O}_D(-C) \simeq \mathcal{O}_D(D)$, we get $H^0(D, \mathcal{O}_D(-C)) = 0$, so $H^0(X_k, \mathcal{O}_{X_k}) \to H^0(C, \mathcal{O}_C)$ is injective.

Next we give an application of Proposition 1 to the computation of the Artin conductor

$$\mathrm{Art}(X/\mathcal{O}_K) = -\chi(X_{\bar{K}}) + \chi(X_{\bar{k}}) + \mathrm{Sw}\, H^1(X_{\bar{K}}, \mathbb{Q}_\ell).$$

Here cohomology denotes the ℓ-adic étale cohomology for a prime $\ell \neq p$ and Sw denotes the Swan conductor of the quasi-unipotent ℓ-adic representation $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ of an inertia group I of K (See [B], page 297). Here we switch the usual sign of $\mathrm{Art}(X/\mathcal{O}_K)$ to get a positive integer. This number is known to be independent of the choice of the prime ℓ (SGA7 I, exposé IX, corollaire 4.6).

Corollary 2. *Let X be as in Proposition 1. Then*

$$\mathrm{Art}(X/\mathcal{O}_K) = \mathrm{Effcond}(X_K/K) + \mathrm{length}\, H^1(X,\Omega)_{\mathrm{tors}}.$$

Proof. According to [B] Theorem (2.3), $\mathrm{Art}(X/\mathcal{O}_K)$ can be computed using the two term complex $\Omega \to \omega$:

$$\mathrm{Art}(X/\mathcal{O}_K) = \chi(H^1(\Omega \to \omega)) - \chi(H^0(\Omega \to \omega))$$

For simplicity we denote by $H^q(\mathcal{F})$ the group $H^q(X,\mathcal{F})$ for any sheaf $\mathcal{F}$ over X. Then we have

$$\begin{aligned}\mathrm{Art}(X/\mathcal{O}_K) =&\, \mathrm{length}\,\mathrm{Coker}(H^0(\Omega) \to H^0(\omega)) - \mathrm{length}\,\mathrm{Ker}(H^0(\Omega) \to H^0(\omega)) \\ &- \mathrm{length}\,\mathrm{Coker}(H^1(\Omega) \to H^1(\omega)) + \mathrm{length}\,\mathrm{Ker}(H^1(\Omega) \to H^1(\omega)).\end{aligned}$$

And the result follows from Corollary 1 and Lemma 3.

2. Inequality for conductor.

We keep the notation that X_K is a proper smooth and geometrically connected curve over K. We put

$$\mathrm{Art}(X_K/K) = \dim H^1(X_{\bar{K}}, \mathbb{Q}_\ell) - \dim H^1(X_{\bar{K}}, \mathbb{Q}_\ell)^I + \mathrm{Sw}\, H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$$

and call it the (Artin) *conductor* of X_K. Here the superscript ^{I} denotes the fixed part by an inertia group I of K. Let J_K be the Jacobian of the curve X_K. Then $\mathrm{Art}(X_K/K)$ is also the conductor of J_K. Our main result in this paper is the following inequality for the conductor $\mathrm{Art}(X_K/K)$.

Theorem 1. *Let $\mathcal{O}_K$ be a discrete valuation ring with perfect residue field k. Let X be a proper regular flat curve over $\mathcal{O}_K$ with smooth and geometrically connected generic fiber X_K. Assume that the g.c.d of the multiplicities of the irreducible components of X_k is 1. Then we have an inequality*

$$\mathrm{Effcond}(X_K/K) \le \mathrm{Art}(X_K/K).$$

Furthermore, if X is semi-stable, we have an equality.

Proof. Since everything here commutes with étale base change, we may and do assume that the residue field k is algebraically closed. Let $C_1, \ldots, C_n$ be the irreducible components of the closed fiber X_k. By [B] Lemma (1.2) (i) (or [L], Proposition 1 for a proof using Néron models), we have $\mathrm{Art}(X_K/K) = \mathrm{Art}(X/\mathcal{O}_K) - (n-1)$. Hence by Corollary 2 of Proposition 1 it is enough to prove $\mathrm{length}\, H^1(X,\Omega)_{\mathrm{tors}} \ge n-1$. Since

$$\mathrm{length}\, H^1(X,\Omega)_{\mathrm{tors}} \ge \dim(H^1(X,\Omega)_{\mathrm{tors}} \otimes k) = \dim(H^1(X,\Omega) \otimes k) - 1$$

(the second equality comes from the fact $H^1(X,\Omega) \otimes K \simeq K$), it is enough to show that the canonical map $H^1(X,\Omega) \to \bigoplus_i H^1(\bar{C}_i, \Omega^1_{\bar{C}_i})$ is surjective. Here $\bar{C}_i$ denotes the normalization of C_i. Note that $H^1(\bar{C}_i, \Omega^1_{\bar{C}_i})$ is a k-vector space of dimension 1. Since the cokernel of $\Omega \to \bigoplus_i \Omega^1_{\bar{C}_i}$ is supported on finitely many closed points, the surjectivity follows from the fact that H^1 is right exact and annihilates a skyscraper sheaf. Thus the inequality is proved.

In the course of the proof of the inequality above, we saw that the equality holds if and only if the following conditions (C) and (D) are satisfied.

$$\text{(C)} \qquad H^1(X,\Omega)_{\mathrm{tors}} \text{ is annihilated by } \pi.$$

$$\text{(D)} \qquad H^1(X,\Omega)\otimes k \to \bigoplus_i H^1(\bar{C}_i, \Omega^1_{\bar{C}_i}) \text{ is an isomorphism.}$$

Now let us prove that these conditions hold for semi-stable X. Let π be a prime element of K. By an elementary local computation, we see that the sequence

$$0 \to \omega \xrightarrow{\times\pi} \Omega \to \bigoplus_i \Omega^1_{\bar{C}_i} \to 0$$

is exact. It induces an exact sequence

$$H^1(X,\omega) \xrightarrow{\times\pi} H^1(X,\Omega) \to \bigoplus_i H^1(\bar{C}_i, \Omega^1_{\bar{C}_i}) \to 0.$$

Since $H^1(X,\omega)$ is torsion free (Lemma 3), $\pi H^1(X,\omega)\cap H^1(X,\Omega)_{\mathrm{tors}} = \{0\}$ as subgroups of $H^1(X,\Omega)$. Hence the above exact sequence implies (C). Further, since the canonical map $H^1(X,\Omega)\to H^1(X,\omega)$ is surjective, we have $\pi H^1(X,\omega) = \pi H^1(X,\Omega)$ and (D) follows.

Theorem 1 has an immediate consequence over global fields.

Corollary 3. *Let $\mathcal{O}_K$ be the ring of integers of a number field K. Let X be a regular, proper flat curve over $\mathcal{O}_K$ such that X_K is smooth and has a rational point over K. Let $\mathfrak{f}\subseteq \mathcal{O}_K$ be the ideal conductor of the Jacobian J_K of X_K. Then*

$$\mathrm{card}(H^0(X,\omega_{X/\mathcal{O}_K})/H^0(X,\Omega_{X/\mathcal{O}_K})) \le \mathrm{Norm}_{\mathcal{O}_K/\mathbb{Z}}(\mathfrak{f}).$$

Furthermore, if $X\to \operatorname{Spec}\mathcal{O}_K$ is semi-stable then the equality holds.

Proof. Because $X_K(K)\neq\emptyset$, X_K is geometrically connected and, for any maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$, at least one irreducible component of the fiber $X_{\mathfrak{p}}$ has multiplicity 1. So we can apply Theorem 1 to $X_{\mathcal{O}_{K,\mathfrak{p}}}\to \operatorname{Spec}\mathcal{O}_{K,\mathfrak{p}}$.

Next we will give an application to Szpiro's inequality between discriminant and conductor of elliptic curves over a function field. Let $f: E\to C$ be an *elliptic fibration* (i.e. f is a proper flat morphism of connected proper smooth varieties over k, $\dim C = 1$ and the generic fiber of f is an elliptic curve over $K := k(C)$). Consider the morphism $j: C\to \mathbb{P}^1_k$ induced by the j-invariant of smooth fibers of $E\to C$. Then $k(\mathbb{P}^1_k)$ is generated by $j(E_K)$ over k. Assume in the sequel that $E\to C$ is not isotrivial. Then j is finite and decomposes into a purely inseparable morphism $C\to C'$ followed by a finite separable morphism $C'\to\mathbb{P}^1_k$. The degree p^e of $C\to C'$ is called the *modular inseparability degree of* $E\to C$ ([Sz2], Déf. 2). By convention $p^e := 1$ if $p = 0$. Note that since k is perfect, $k(C') = k(C)^{p^e}$. In particular, $j(E_K)\in k(C)^{p^e}$.

Corollary 4. *Let C be a geometrically connected projective smooth curve over a perfect field k of characteristic $p \geq 0$. Let $f : E \to C$ be a non-isotrivial elliptic fibration. Let $\Delta_{E/C}$ and $\mathrm{Art}(E/C) \in \mathrm{Pic}(C)$ be respectively the minimal discriminant and conductor divisors of $E \to C$. Then*

$$\deg \Delta_{E/C} \leq 6p^e(2g(C) - 2 + \deg \mathrm{Art}(E/C))$$

where p^e is the modular inseparability degree of $E \to C$.

Proof. Denote $K = k(C)$. By assumption, E_K has a rational point over K, hence $E \to C$ has no multiple fiber. In the case $e = 0$, the proof of the corollary goes exactly as in [Sz], page 8. One has just to remark that following Theorem 1, $f_*\Omega^1_{E/C} = f_*\omega_{E/C}(-S)$ where S is an effective divisor over C such that $S \leq \mathrm{Art}(E/C)$.

Now assume that $e > 0$. Then $j(E_K) \in K$ is a p^e-th power in K. Since $E \to C$ is non-isotrivial, $j(E_K) \notin k$. In particular $j(E_K) \neq 0, 1728$. Using explicit Weierstrass equations of E_K (as in [Sil], Appendix A), we see that there exists an elliptic curve E'_K such that E_K is the pull-back of E'_K by Frob^e_K, where Frob_K is the absolute Frobenius of $\mathrm{Spec}\, K$. Moreover, extending E'_K to an elliptic fibration $E' \to C$, it is clear that

$$\deg \Delta_{E/C} \leq p^e \deg \Delta_{E'/C}.$$

On the other hand, the e-th power of the relative Frobenius of E over K is an isogeny $E_K \to E'_K$, thus $E \to C$ and $E' \to C$ have the same conductor:

$$\mathrm{Art}(E/C) = \mathrm{Art}(E'/C).$$

(This elegant argument is due to Minhyong Kim. Our original proof consists in the following facts : (1) if K/F is a finite purely inseparable extension of a discrete valuation field, then for any finite separable extension L/F, the tensor product $L \otimes_F K$ is a separable extension of K, the map $\mathrm{Gal}(L/F) \to \mathrm{Gal}(L \otimes_F K/K)$ is an isomorphism and induces isomorphisms on their respective ramification subgroups. (2) Let $F = K^{p^e}$. Then $E_K = E'_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, K$. One has $T_l(E'_F) = T_l(E_K)$ and this equality is compatible with the isomorphism $\mathrm{Gal}(F^s/F) \simeq \mathrm{Gal}(K^s/K)$. Hence $\mathrm{Art}(E'_F/F) = \mathrm{Art}(E_K/K)$.) Therefore the corollary comes from the inequality for the fibration $E' \to C$ which has modular inseparable degree $p^e = 1$.

Remark 1. After Szpiro's work, Frey ([F], Proposition 2.3), Hindry and Silverman ([HS], Theorem 5.1) have proved Corollary 4 when k has characteristic zero by applying Hurwitz formula to $j : C \to \mathbb{P}^1_k$. Their inequality is slightly better in the sense that $\mathrm{Art}(E/C)$ can be replaced by $\mathrm{Art}(E/C) - \deg S$, where S is the reduced divisor on C supported by the closed points $x \in C$ such that E_K has additive and potentially good reduction at $x \in C$.

Remark 2. There is an example where we have a strict inequality in Theorem 1. See Remark 4.

Next we will show that $\mathrm{Effcond}(X_K/K)$ is independent of the choice of a regular model X and hence is an invariant of the generic fiber X_K. More precisely, we have the following :

Lemma 4. *Let $\phi : X' \to X$ be a birational morphism between regular proper models over $\mathcal{O}_K$ of a proper smooth curve X_K over K. Then the canonical morphisms*

$$H^0(X, \Omega^1_{X/\mathcal{O}_K}) \to H^0(X', \Omega^1_{X'/\mathcal{O}_K}), \quad H^0(X, \omega_{X/\mathcal{O}_K}) \to H^0(X', \omega_{X'/\mathcal{O}_K})$$

are isomorphisms.

Proof. It is enough to prove the lemma assuming X' is the blowing up of X at a closed point x. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Consider the canonical map $\alpha : \mathcal{F} \to \phi_*\phi^*\mathcal{F}$. In a small affine neighborhood U of x, $\phi^{-1}(U)$ is covered by two affine open subsets V_1, V_2 of X'. Since $R^1\phi_*\mathcal{O}_{X'} = 0$, we have an exact sequence

$$0 \to \mathcal{O}_{X'}(\phi^{-1}(U)) \to \mathcal{O}_{X'}(V_1) \oplus \mathcal{O}_{X'}(V_2) \to \mathcal{O}_{X'}(V_{12}) \to 0.$$

Tensoring by $\mathcal{F}(U)$ over $\mathcal{O}_X(U)$ and using the fact that $\phi_*\mathcal{O}_{X'} = \mathcal{O}_X$, we see that α is surjective with kernel supported in $\{x\}$. Thus $H^0(X, \mathcal{F}) \to H^0(X', \phi^*\mathcal{F})$ is surjective. If $\mathcal{F} = \omega_{X/\mathcal{O}_K}$, then $\mathrm{Ker}(\alpha) = 0$ since ω has no torsion. If $\mathcal{F} = \Omega^1_{X/\mathcal{O}_K}$, then $\mathrm{Ker}(\alpha)$ is contained in the torsion part of $\Omega^1_{X/\mathcal{O}_K}$ which is an invertible sheaf on a divisor of X (Lemma 1), thus $\mathrm{Ker}(\alpha)$ is again 0. So in both cases $H^0(X, \mathcal{F}) = H^0(X', \phi^*\mathcal{F})$.

Let E be the exceptional divisor. First we prove the assertion for Ω. By the exact sequence $0 \to \phi^*\Omega^1_{X/\mathcal{O}_K} \to \Omega^1_{X'/\mathcal{O}_K} \to \Omega^1_{E/k(x)} \to 0$ (because X is regular) and $H^0(E, \Omega^1_{E/k(x)}) = 0$, the canonical map $H^0(X', \phi^*\Omega^1_{X/\mathcal{O}_K}) \to H^0(X', \Omega^1_{X'/\mathcal{O}_K})$ is an isomorphism. The proof runs similarly for ω, by using the exact sequence

$$0 \to \phi^*\omega_{X/\mathcal{O}_K} \to \omega_{X'/\mathcal{O}_K} \to \omega_{X'/\mathcal{O}_K}|_E \to 0$$

and the vanishing $H^0(E, \omega_{X'/\mathcal{O}_K}|_E) \simeq H^1(E, \mathcal{O}_E(-1)) = 0$.

Remark 3. In the situation of Lemma 4, it is easy to see that for any quasi-coherent sheaf $\mathcal{F}$ on X, one has $R^1\phi_*(\phi^*\mathcal{F}) = 0$, and then

$$H^1(X, \mathcal{F}) \simeq H^1(X', \phi^*\mathcal{F}).$$

Using the fact that $H^q(X', \omega_{X'/\mathcal{O}_K}|_E) = 0$ for $q = 0, 1$, one obtains an isomorphism $H^1(X, \omega_{X/\mathcal{O}_K}) \simeq H^1(X', \omega_{X'/\mathcal{O}_K})$. Similarly we see that $H^1(X, \Omega_{X/\mathcal{O}_K}) \to H^1(X', \Omega_{X'/\mathcal{O}_K})$ is injective, but is not surjective unless $\phi : X' \to X$ is an isomorphism.

Let J_K be the Jacobian of the proper smooth curve X_K, let J be the Néron model of J_K over $\mathcal{O}_K$. Denote by a, t and u the dimension of the abelian part, the toric part and of the unipotent part of the connected component of J_k respectively. Then $\mathrm{Art}(X_K/K) = t + 2u + \delta$, where $\delta \geq 0$ is the Swan conductor. Here we will say that X_K has *tame reduction* if the action of the wild ramification subgroup $P \subset I$ is trivial on $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ (see also [Ser], §2.1). If X_K has genus $g \geq 2$, this condition is equivalent to X_K having semi-stable reduction over a tamely ramified extension of K. Thus we have an inequality $\mathrm{Art}(X_K/K) \geq t + 2u$ and the equality holds if and only if X_K has tame reduction. In the next proposition, we prove a similar inequality for $\mathrm{Effcond}(X_K/K)$.

Proposition 2. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ with smooth generic fiber. Assume that the g.c.d of the multiplicities of the components of X_k is equal to* 1. *Then we have*

$$\mathrm{Effcond}(X_K/K) \geq t + u.$$

Moreover if $X_{k,\mathrm{red}}$ is a simple normal crossings divisor, then the equality holds if and only if $H^0(X, \Omega) = H^0(X, \mathcal{E})$, where $\mathcal{E}$ is the subsheaf $\sum_i \omega(-X_k + C_i)$ of ω.

Proof. Using embedded resolution of singularities and Lemma 4, we may assume that the irreducible components of X_k are regular. Following Lemma 1 (5.1), the canonical map $\Omega \to \omega$ has image in $\mathcal{E}$. Hence by Lemma 5 below we have

$$\mathrm{Effcond}(X_K/K) \geq \mathrm{length}(H^0(X, \omega)/H^0(X, \mathcal{E})) = g - a'.$$

Now the assumption on the multiplicities implies that $\mathrm{Pic}^0_{X_k/k}$ is isomorphic to the connected component of J_k ([BLR], Theorem 9.5.4), hence $g - a' = (a+t+u) - a = t + u$. The last assertion is then trivial.

Lemma 5. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ as in Proposition 1. Then we have the equality*

$$\mathrm{length}(H^0(X, \omega)/H^0(X, \mathcal{E})) = g - a',$$

where $g = g(X_K)$ and $a' = \sum_i \dim_k H^1(C_i, \mathcal{O}_{C_i})$.

Proof. We have the inclusions $\pi\omega \subseteq \mathcal{E} \subseteq \omega$. Therefore

$$\mathrm{length}(H^0(X, \omega)/H^0(X, \mathcal{E})) = \dim_k H^0(X, \omega) \otimes k - \dim_k H^0(X, \mathcal{E})/\pi H^0(X, \omega).$$

The dimension $\dim_k H^0(X, \omega) \otimes k$ is equal to g. Now it remains to compute the dimension of $H^0(X, \mathcal{E})/\pi H^0(X, \omega)$. We have

$$\text{(E)} \qquad \mathcal{E}/\pi\omega \simeq \bigoplus_i (\omega(-X_k + C_i)/\pi\omega) \simeq \bigoplus_i \omega_{C_i}.$$

Since $H^1(X, \omega)$ is torsion free (see the beginning of the proof of Lemma 3), we then have an exact sequence

$$\text{(F)} \qquad 0 \to H^0(X, \omega) \xrightarrow{\times\pi} H^0(X, \mathcal{E}) \to \bigoplus_i H^0(C_i, \omega_{C_i}) \to 0.$$

Thus $\dim_k H^0(X, \mathcal{E})/\pi H^0(X, \omega) = a'$, and the lemma is proved.

One may suspect that we have an equality in Proposition 2 when the action of the wild ramification subgroup $P \subset I$ is trivial on $H^1(X_{\bar{K}}, \mathbb{Q}_\ell)$ similarly as for the conductor $\mathrm{Art}(X_K/K)$. Recall that under this tameness assumption, $\mathrm{Art}(X_K/K) = t + 2u$. The authors do not know a counterexample. For a positive direction, we give the following proposition.

Proposition 3. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ with smooth generic fiber such that $X_{k,\mathrm{red}}$ is a simple normal crossings divisor. Assume that X_K has tame reduction and the g.c.d of the multiplicities of the irreducible components of X_k is equal to 1. Then we have*

$$\mathrm{Effcond}(X_K/K) \geq \mathrm{length}(H^0(X,\omega)/H^0(X,\Omega/\Omega_{\mathrm{tors}})) = t+u.$$

Proof. The first inequality is trivial. Let us prove the second equality. We exclude the trivial case $g(X_K)=0$ since then $H^0(X,\omega)=0$. Denote $\Omega_d=\Omega/\Omega_{\mathrm{tors}}$. As in Proposition 2, we have $\mathrm{length}(H^0(X,\omega)/H^0(X,\Omega_d)) \geq t+u$. Similarly as Lemma 4 we see that $H^0(X,\Omega_d)$ increases by blowing up, so $\mathrm{length}(H^0(X,\omega)/H^0(X,\Omega_d))$ decreases by the same operation. Thus it is enough to prove the proposition when X is the minimal regular model such that $X_{k,\mathrm{red}}$ is a simple normal crossings divisor. Again as in Proposition 2, the assertion is equivalent to the equality $H^0(X,\Omega_d)=H^0(X,\mathcal{E})$. The latter will be a consequence of the exact sequence

$$0 \to \Omega_d \to \mathcal{E} \to \bigoplus_C \omega_C \to 0$$

where $\bigoplus_C \omega_C$ runs the rational irreducible components of X_k with multiplicity divisible by p. Next we prove exactness of the sequence.

Since both members of the equality are invariant under étale base change, we can assume that k is algebraically closed. Let $x \in X_k$. If the components of X_k containing x have multiplicities prime to p, then using Lemma 1 (4)-(5) we see that $\Omega \to \mathcal{E}$ is surjective at x. Now assume that x is contained in a component C of multiplicity r divisible by p. By [Sa2], Theorem 3.11, C is a smooth rational curve, it intersects only components with multiplicity prime to p and there are exactly two intersection points x_1, x_2 in C. Following Lemma 1 (5.2), this implies that $\nu_C(D)=r$. Using the isomorphism (E) of the proof of Lemma 5, it is enough to show that $\Omega_x \to \omega_x(-D)$ is surjective for all $x \in C$. This is already true for the intersection points x_1, x_2 (Lemma 1 (5.2)).

Let Γ be the union of the (reduced) irreducible components of X_k different from C. Let $\{U_i\}_i$ be a covering of X such that $\mathcal{O}_X(-C)|_{U_i}$ is generated by a single function h_i for all i. Let $u_i=\pi h_i^{-r} \in \mathcal{O}_X(U_i) \cap \mathcal{O}_X(U_i \setminus \Gamma)^*$. Then $d\log u_i$ is a section of $\Omega_{X/\mathcal{O}_K}(\log \Gamma)|_{U_i}$. Since $\Gamma|_C = x_1+x_2$, we get local sections $(d\log u_i)|_{U_i \cap C} \in H^0(C\cap U_i, \Omega_C(x_1+x_2))$. Using the fact that p divides r we see that these sections glue together and give rise to a section $s_C \in H^0(C,\Omega_C(x_1+x_2))$ (see also [Sa3], Def. 2.4). On the other hand, one checks easily that s has poles of exact order 1 at x_1 and x_2. Since $\Omega_C(x_1+x_2) \simeq \mathcal{O}_C$, this implies that s is a generator of $\Omega_C(x_1+x_2)$. Let $x \in C \setminus \{x_1,x_2\}$, let U_{i_0} be an open containing x, let $h=h_{i_0}$, $u=u_{i_0}$. Then $s_C=(d\log u)|_C$ is a generator of $\Omega_{C,x}$, and so is $du|_C$. Let $z \in \mathcal{O}_{X,x}$ be a lifting of a local coordinate of C at x, then $(\partial u/\partial z)|_C \in \mathcal{O}^*_{C,x}$, hence $\partial u/\partial z \in \mathcal{O}^*_{X,x}$. Using the relation $uh^r-\pi=0$ and Lemma 1 (2), we see that $\Omega_x \to \omega_x(-D)$ is surjective.

By construction, if $\mathrm{Art}(X_K/K)=t+2u$, then X_K has tame reduction. We can ask similarly whether the equality in Proposition 2 implies tameness. The next lemma is a partial result in this direction. (See also the Remark 4).

Lemma 6. *Let X be a proper flat and regular curve over $\mathcal{O}_K$ with smooth generic fiber. Assume that the g.c.d of the multiplicities of the components of X_k is equal to 1 and we have an equality*

$$\operatorname{length}(H^0(X,\omega)/H^0(X,\Omega/\Omega_{\mathrm{tors}})) = t+u.$$

Then any irreducible component of X_k with multiplicity divisible by $p = \operatorname{char} k$ is a smooth rational curve over k.

Proof. Let C be a component with multiplicity r divisible by p, then $\nu_C(D) \geq r$ (Lemma 1 (4)). Thus $\omega(-D) \subseteq \pi\omega$ in an open dense subset of C. This implies that the composition $\Omega_d \to \mathcal{E} \to \omega_C$, where $\Omega_d := \Omega/\Omega_{\mathrm{tors}}$, is the 0-map. By Proposition 2 and the exact sequence (F), the map $H^0(X,\Omega_d) \to H^0(C,\omega_C)$ must be surjective. Hence the assertion is proved.

Example. Let X_K be an elliptic curve with reduction of type II. This means that the special fiber X_k of the minimal regular model X has exactly one singular point e (necessarily rational over k). We have $a = t = 0$ and $u = 1$. Denote for simplicity

$$F := \operatorname{Effcond}(X_K/K), \quad f := \operatorname{Art}(X_K/K).$$

In this example, we will show that

$$\begin{cases} F = [(f+2)/3], & \text{if } p \neq 2 \\ [(f+2)/4] \leq F \leq [f/2], & \text{if } p = 2. \end{cases}$$

Note that when K varies, there is no upper bound for f if $p = 2, 3$.

We have $\Omega_{\mathrm{tors}} = 0$ and ω/Ω is only supported at e. We then deduce an exact sequence

$$0 \to H^0(X,\Omega) \to H^0(X,\omega) \to \omega_e/\Omega_e \to H^1(X,\Omega)_{\mathrm{tors}} \to 0.$$

This implies that $\operatorname{length}(\omega_e/\Omega_e) = \operatorname{Art}(X/\mathcal{O}_K) = f$ (Corollary 2), the second equality holds because X_k is geometrically irreducible. In a neighborhood of e, X is a divisor $V(y^2 - x^3 + \pi u)$ in a smooth scheme P over $\mathcal{O}_K$ such that x, y, π is a system of coordinates of P at e and u is invertible. Dividing $\omega \simeq \mathcal{O}_X$ by a generator, the injection $H^0(X,\omega)/H^0(X,\Omega) \hookrightarrow \omega_e/\Omega_e$ becomes

$$\text{(G)} \qquad \mathcal{O}_K/(\pi^F) \hookrightarrow A := \mathcal{O}_{P,e}/(y^2 - x^3 + \pi u, 2y + \pi u'_y, 3x^2 - \pi u'_x),$$

where $u'_y = \partial u/\partial y$ and $u'_x = \partial u/\partial x$. Let $\mathfrak{m}$ be the maximal ideal of A, let N be the smallest integer such that $\mathfrak{m}^N = 0$. Then

$$f = \operatorname{length} A = 1 + \sum_{1 \leq i \leq N-1} \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

For any element $a \in \mathcal{O}_{P,e}$, denote by $\tilde{a}$ its image in A. By the injection (G), we see that F is the smallest integer such that $\tilde{\pi}^F = 0$. From the relation $\tilde{y}^2 - \tilde{x}^3 + \tilde{\pi}\tilde{u} = 0$ we deduce that $\tilde{\pi} \in \mathfrak{m}^2$.

Assume that $p \neq 2$. From the relation $2\tilde{y} + \tilde{\pi}\widetilde{u'_x} = 0$ we get $\tilde{y} \in \tilde{\pi}A \subseteq \mathfrak{m}^2$ and $\tilde{\pi}A = \tilde{x}^3 A$. Thus $\mathfrak{m} = \tilde{x}A$, and $\mathfrak{m}^i = \tilde{x}^i A$ for all $i \geq 1$. So $\operatorname{length} A = N$ and F is the smallest integer such that $\tilde{x}^{3F} = 0$. Hence $F = [(N+2)/3] = [(f+2)/3]$.

Now assume that $p = 2$. Similarly as above, it is easy to see that $\tilde{x}^2 \in \tilde{\pi}A$ and $\tilde{\pi}A = \tilde{y}^2 A$. Thus $\mathfrak{m}^i = (\tilde{x}\tilde{y}^{i-1}, \tilde{y}^i)$. So $1 \leq \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1} \leq 2$ and $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. This implies that $N + 1 \leq f \leq 2N - 1$, and $(N-1)/2 \leq F \leq (N+1)/2$. These two inequalities imply that $[(f+2)/4] \leq F \leq [f/2]$.

Remark 4. In this example, when $p = 2, 3$, X_K does not have tame reduction ([Sa2], loc. cit.), hence $f \geq 3$. The equality in Proposition 2 holds if and only if $f = 3$. For any p, the inequality in Theorem 1 is strict.

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