DESINGULARIZATION OF DOUBLE COVERS OF REGULAR SURFACES

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Abstract. Let Z be a noetherian integral excellent regular scheme of dimension 2. Let Y be an integral normal scheme endowed with a finite flat morphism $Y \to Z$ of degree 2. We give a description of Lipman's desingularization of Y by explicit equations, leading to a desingularization algorithm for Y.

Contents

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1. Introduction

The existence of the desingularization of excellent *surfaces* (reduced noetherian excellent schemes of dimension 2) is well known from the theoretical point of view. However, in practice, the implementation of an algorithm of desingularization in full generality is rather complicated, see [3] and [8] for an overview of various approaches. In view of applications in arithmetic geometry, a lot of work also has been done for smooth projective curves C over a discrete valuation field K. The aim is then to find a projective regular scheme over the valuation ring $\mathcal{O}_K$ of K, with generic fiber isomorphic to C, see [9], [16] and the references therein, when C is a cyclic cover of $\mathbb{P}^1_K$, mostly assuming the residue characteristic of K to be prime to the order of the cover. For hypersurfaces in the affine plane $\mathbb{A}^2$ over $\mathcal{O}_K$, see also [6] with some powerful applications in number theory.

In this work we deal with a special type of surfaces for which Lipman's method works very well and is completely explicit.

Theorem 1.1 (Lipman, [19])**.** *Let Y be a noetherian integral normal excellent surface. Consider a sequence of birational morphisms*

$$... \to Y_n \to \cdots \to Y_1 \to Y_0 := Y \tag{1.1}$$

where each $Y_{i+1} \to Y_i$ is the normalization of the blowing-up of Y_i along some singular closed (reduced) point. Then the sequence is necessarily finite.

The surfaces we are interested in are double covers of regular surfaces. For simplicity we only work with noetherian excellent schemes (*e.g.*, algebraic varieties over a field, schemes of finite type over Dedekind domains of characteristic 0 or complete discrete valuation rings, see [12], §8.2.3).

Definition 1.2 Let Z be an integral noetherian regular scheme. We define a *double cover of* Z as a reduced scheme Y endowed with a finite flat morphism of degree 2

$$\psi : Y \to Z.$$

We call Y a *normal double cover of* Z if moreover Y is integral and normal.

Example 1.3 Let C be a hyperelliptic curve over $\mathbb{Q}$, with the canonical degree 2 morphism $C \to \mathbb{P}^1_{\mathbb{Q}}$. Let $Z = \mathbb{P}^1_{\mathbb{Z}}$ and let W be the normalization of Z in the function field $K(C)$ of C. Then Z is a regular excellent surface, and $W \to Z$ is a normal double cover of Z. The surface W is a so called Weierstrass model of C. It is singular in general. A resolution of singularities on W gives a projective regular model of C over $\mathbb{Z}$.

Example 1.4 More generally, let Z be an integral noetherian excellent surface. Let L be a (possibly inseparable) quadratic extension of the function field $K(Z)$ of Z and let Y be the normalization of Z in L. Then $Y \to Z$ is a normal double cover. Indeed, $Y \to Z$ is finite because Z is excellent. The scheme Y is a noetherian integral normal surface, hence Cohen-Macaulay, hence flat over Z because the latter is regular.

Now come back to the general situation. The key point in our situation is that if we start with a normal double cover $Y \to Z$ of a regular surface, then in any sequence like (1.1), all the Y_i's are again normal double covers of regular surfaces.

More precisely (Theorem 3.4), we have a canonical commutative diagram

$$\begin{tikzcd}
Y_n \arrow[r] \arrow[d] & Y_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_1 \arrow[r] \arrow[d] & Y_0 = Y \arrow[d] \\
Z_n \arrow[r, "\mathrm{bl}_{q_{n-1}}"] & Z_{n-1} \arrow[r, "\mathrm{bl}_{q_{n-2}}"] & \cdots \arrow[r, "\mathrm{bl}_{q_1}"] & Z_1 \arrow[r, "\mathrm{bl}_{q_0}"] & Z_0 := Z
\end{tikzcd}$$

where the vertical arrows are normal double covers of regular surfaces, and the lower horizontal arrows are blowing-ups of (reduced) closed points $q_i \in Z_i$. Each morphism $Y_{i+1} \to Y_i$ is a *normalized blowing-up*, meaning that we blow-up the unique singular point (with the reduced structure) of Y_i lying over q_i, see Proposition 2.8(3), then we take the normalization.

Moreover, we can give explicit equations for the Y_i's as follows. First one can cover Z_0 by small affine open subsets V's over which Y_0 is defined by a monic polynomial of degree 2 (Lemma 2.2):

$$y^2 + ay + b = 0, \quad a, b \in \mathcal{O}_Z(V). \tag{1.2}$$

If $Z_1 \to Z_0$ is the blowing-up of a point $q_0 \in Z_0$, and if $\lambda_{q_0}(Y_0)$ is the multiplicity of Y_0 at any point lying over q_0 (Definitions 2.3 and 2.13), we set $r = [\lambda_{q_0}(Y_0)/2]$, the integral part of $\lambda_{q_0}(Y_0)/2$. Let s, t be a system of coordinates of Z_0 at q_0. Then Y_1 is covered by the affine schemes defined respectively by

$$y_1^2 + (a/s^r)y_1 + (b/s^{2r}) = 0, \quad y_1 = y/s^r$$

and

$$z_1^2 + (a/t^r)z_1 + (b/t^{2r}) = 0, \quad z_1 = y/t^r$$

(Proposition 3.3). And we continue in the same way for the Y_i's.

The computation of the multiplicity $\lambda_q(Y) \in \mathbb{N}$ is straightforward when $2 \in \mathcal{O}_{Z,q}^*$. Indeed we can reduce to $a = 0$. Then $\lambda_q(Y)$ is just the order of b in the regular local ring $\mathcal{O}_{Z,q}$ (Proposition 2.6(2)). If 2 is not invertible, there is no canonical choice of the equation (1.2). Lemma 2.10 gives a simple algorithm to determine $\lambda_q(Y)$ in this case.

As a byproduct of Theorem 3.4 we prove the existence of a simultaneous resolution of singularities for covers of degree 2 of excellent normal surfaces (Corollary 3.12).

When 2 is invertible in $\mathcal{O}_Z$, the sequence $(Z_i)_{i\geq 0}$ is classical and consists in making the branch locus of $Y \to Z$ regular. In the case $Z = \mathbb{P}^1_R$ over a discrete valuation ring R, this is essentially equivalent to locate the branch locus of the generic fiber of $Y \to Z$. The language of clusters ([7], §1.1, [16]) is then more explicit and convenient. The novelty here is that our method works in any (mixed or equal) characteristic. It can also be applied to the case when Z is a regular model over R of a smooth curve Z_K over $K := \mathrm{Frac}(R)$ and $Y_K \to Z_K$ is a double cover of smooth curves over K.

Our initial motivation is to give an easily implementable algorithm to finding a desingularization of the surface W in Example 1.3, in the spirit of Tate's algorithm for elliptic curves [22]. Indeed, a regular model of C over $\mathbb{Z}$ allows to compute some important arithmetic invariants of C as its Artin conductor, the Tamagawa number of its Jacobian $J(C)$ as well as the (finite part of) L-function of $J(C)$. Applications will be given in [14]. Implementation in PARI/gp [20] by B. Allombert and in MAGMA are ongoing.

Now let us describe the content of this paper. In § 2, we introduce our multiplicity $\lambda_p(Y)$ for any double cover $Y \to Z$ of a regular scheme and for any point $p \in Y$. This invariant already appeared in various works, when Z is a smooth surface over a perfect field, or in the context of Hironaka's characteristic polyhedra (see *e.g.* [4], §5). In our setting it is particularly simple to define.

Normalized blowing-ups are defined in § 3 as blowing-ups followed by a normalization. These are the morphisms appearing in the top line of Sequence (1.1). We prove the main theorem 3.4 about the description of a sequence of normalized blowing-ups. The equation of each Y_i is given in Proposition 3.3.

In § 4, we give a bound on the sum of the multiplicities running through the closed points of the exceptional locus of the normalized blowing-up $Y_1 \to Y$ (Theorem 4.6). Let S be a regular scheme. The relative dualizing sheaves $\omega_{Y_i/S}$ are determined in § 5 when Z is flat of finite type over an integral noetherian regular scheme S of dimension ≤ 1. This will be used later in [14] to determine a canonical volume form on the Néron model of the Jacobian $J(C)$ of a hyperelliptic curve C. Finally in § 6 we apply the above results to describe a desingularization algorithm for the double covers of regular excellent surfaces.

The general setting in this work is the following: Z is an *integral noetherian excellent regular scheme*, $Y \to Z$ is a *double cover*. No assumption are made on the residue fields of Z. Except in § 2, Z will be of dimension 2.

2. The multiplicity $\lambda_p(Y)$

The multiplicity of Y at a closed point p is a local invariant depending only on $\mathcal{O}_{Y,y}$. Let A be a noetherian regular domain. Let $B = A[y]$ be an A-algebra with $y^2 + ay + b = 0$ for some $a, b \in A$. We will define a multiplicity $\lambda_{\mathfrak{m}}(B) \in \mathbb{N}$ for any maximal ideal $\mathfrak{m}$ of A. It is a measure of singularity of B at the maximal ideals lying over $\mathfrak{m}$.

2.1. Definition and basic properties for local rings.

Let us first recall some elementary facts on regular rings. Let $\mathfrak{m}$ be a maximal ideal of A, let $d = \dim A_{\mathfrak{m}}$ and $k = A/\mathfrak{m}$. Consider

$$\mathrm{Gr}_{\mathfrak{m}}(A) = \oplus_{\ell \geq 0} \mathfrak{m}^{\ell}/\mathfrak{m}^{\ell+1}.$$

It is canonically a homogeneous k-algebra. As A is regular, it is isomorphic to $\mathrm{Sym}_k(\mathfrak{m}/\mathfrak{m}^2)$, the polynomial ring in d variables over k.

For any non-zero element $a \in A$, $\mathrm{ord}_{\mathfrak{m}}(a)$, or simply $\mathrm{ord}(a)$ if $\mathfrak{m}$ is implicitly given, is the supremum of the integers n such that $a \in \mathfrak{m}^n$. It depends only on the ideal aA. By convention $\mathrm{ord}(0) = +\infty$. We have $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$ for $a, b \in A \setminus \{0\}$. As $\mathrm{Gr}_{\mathfrak{m}}(A)$ is a domain, ord extends to a valuation on $\mathrm{Frac}(A)$.

Definition 2.1 Keep the above notation. If $\mathrm{ord}_{\mathfrak{m}}(a) = \ell \neq +\infty$, the *initial form of* a is the image of a in $\mathfrak{m}^{\ell}/\mathfrak{m}^{\ell+1}$. The latter can be viewed as the k-vector space of homogeneous polynomials in d variables and of degree ℓ.

Lemma 2.2. *Let R be a noetherian ring. Let D be an R-algebra, free of rank 2 as R-module. Then there exists $y \in D$ such that $D = R \oplus Ry$, and we have for some $a, b \in R$*

$$y^2 + ay + b = 0.$$

Moreover, any other z such that $D = R[z]$ satisfies $z = uy + v$ with $u \in R^*$ and $v \in R$.

Proof. Locally on $\operatorname{Spec} R$, R is a direct summand in D ([2], II, §3, Proposition 6 with $M = R$ and $N = D$), then apply *op. cit.*, Cor. 1 to Proposition 12 with $M = D$ and $N = R$, which implies that R is a direct summand in D. Thus there exists $y \in D$ such that $\{1, y\}$ is a basis of D over R. The statement on z is obvious. □

Definition 2.3 Let A be a noetherian regular domain. Let B be a reduced A-algebra, free of rank 2 as A-module. Write $B = A[y]$ with

$$y^2 + ay + b = 0$$

for some $a, b \in A$.

Fix a maximal ideal $\mathfrak{m} \subset A$. For each generator y as above, denote by

$$\lambda_{\mathfrak{m}}(y) = \min\{2 \operatorname{ord}_{\mathfrak{m}}(a), \operatorname{ord}_{\mathfrak{m}}(b)\}.$$

Define

$$\lambda_{\mathfrak{m}}(B) = \sup_y \{\lambda_{\mathfrak{m}}(y)\} \in \mathbb{N} \cup \{+\infty\},$$

where the supremum runs through all generators y as above. Note that when an y is given, we only need to compute $\lambda_{\mathfrak{m}}(y + c)$ for all $c \in A$ to determine $\lambda_{\mathfrak{m}}(B)$. This follows from the description of the generators in Lemma 2.2, and the fact that multiplying y by a unit of A does not change $\lambda_{\mathfrak{m}}(y)$.

We define the *discriminant ideal of B over A* as

$$\Delta_{B/A} := (a^2 - 4b)A$$

using a generator y. Again by Lemma 2.2, it is independent on the choice of y. Furthermore it commutes with base changes on A with regular domains.

Remark 2.4 The above multiplicity is very likely twice the invariant $\delta(B) \in \mathbb{Q}_{\geq 0}$ considered by [4], §5, in a much more general context involving Hironaka's characteristic polyhedra. We use the notation $\lambda_{\mathfrak{m}}(B)$ to keep coherent with our previous work [11], Définition 10 (with $r = 1$). Note that as $\operatorname{Spec} B$ is defined by an equation of degree 2 in the regular scheme $\operatorname{Spec}(A[y])$, its Hilbert-Samuel multiplicity is 2 at every singular point.

The multiplicity $\lambda_{\mathfrak{m}}(B)$ is local, in the sense that it depends only on $A_{\mathfrak{m}} \to B_{\mathfrak{m}} := A_{\mathfrak{m}} \otimes_A B$ as shown in the lemma below.

Lemma 2.5. *Keep the notation of Definition 2.3.*

(1) *We have* $\operatorname{ord}_{\mathfrak{m}} = \operatorname{ord}_{\mathfrak{m}A_{\mathfrak{m}}}$ *on* $\operatorname{Frac}(A)$.

(2) *Consider the $A_{\mathfrak{m}}$-algebra $B_{\mathfrak{m}}$, then*

$$\lambda_{\mathfrak{m}}(B) = \lambda_{\mathfrak{m}A_{\mathfrak{m}}}(B_{\mathfrak{m}}).$$

Proof. (1) This is because $\mathfrak{m}^n A_{\mathfrak{m}} \cap A = \mathfrak{m}^n$ for all $n \geq 0$.

(2) Clearly $\lambda_{\mathfrak{m}}(B) \leq \lambda_{\mathfrak{m}A_{\mathfrak{m}}}(B_{\mathfrak{m}})$. Let z be a generator of $B_{\mathfrak{m}}$ as $A_{\mathfrak{m}}$-algebra. Let $B = A[y]$. Then there exists $u \in A_{\mathfrak{m}}^*$, $v \in A_{\mathfrak{m}}$ such that $y = uz - v$. Approximate v by a sequence $v_n \in A$: $v = v_n + \alpha_n$ with $\alpha_n \in \mathfrak{m}^n A_{\mathfrak{m}}$. Then $B = A[y + v_n]$ and we have, if n is big enough,

$$\lambda_{\mathfrak{m}A_{\mathfrak{m}}}(z) = \lambda_{\mathfrak{m}A_{\mathfrak{m}}}(uz) = \lambda_{\mathfrak{m}A_{\mathfrak{m}}}(y + v_n) = \lambda_{\mathfrak{m}}(y + v_n) \leq \lambda_{\mathfrak{m}}(B).$$

This proves the inverse inequality. □

Proposition 2.6. *Let $A, \mathfrak{m}$ and B be as in Definition 2.3, let $k = A/\mathfrak{m}$.*

(1) *We have $\mathrm{ord}_{\mathfrak{m}}(\Delta_{B/A}) = 0$ if and only if $A \to B$ is étale above $\mathfrak{m}$.*
(2) *We have $\lambda_{\mathfrak{m}}(B) \le \mathrm{ord}_{\mathfrak{m}}(\Delta_{B/A})$, with equality when $\mathrm{char}(k) \neq 2$.*
(3) *Consider the following properties:*
   (a) $\lambda_{\mathfrak{m}}(B) = +\infty$;
   (b) $\mathrm{char}(A) = 2$, *and for any generator y we have $a = 0$ and b is a square in the $\mathfrak{m}$-adic completion $\hat{A}_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$;*
   (c) $B \otimes_A \hat{A}_{\mathfrak{m}}$ *is non-reduced;*
   (d) $\Delta_{B/A} = \{0\}$;
   (e) $B \otimes_A \mathrm{Frac}(A)$ *is an inseparable quadratic extension of* $\mathrm{Frac}(A)$.

   Then
$$(a) \iff (b) \iff (c) \implies (d) \iff (e).$$
   In particular, if A is excellent, then $\lambda_{\mathfrak{m}}(B)$ is finite.

Proof. (1) is clear.

(2) We have
$$\mathrm{ord}_{\mathfrak{m}}(\Delta_{B/A}) = \mathrm{ord}_{\mathfrak{m}}(a^2 - 4b) \ge \min\{\mathrm{ord}_{\mathfrak{m}}(a^2), \mathrm{ord}_{\mathfrak{m}}(b)\} = \lambda_{\mathfrak{m}}(y).$$
Thus $\lambda_{\mathfrak{m}}(B) \le \mathrm{ord}_{\mathfrak{m}}(\Delta_{B/A})$. When $\mathrm{char}(k) \neq 2$, by Lemma 2.5 we can suppose $2 \in A^*$. Then $B = A[y + a/2]$ and $\mathrm{ord}_{\mathfrak{m}}(\Delta_{B/A}) = \lambda_{\mathfrak{m}}(y + a/2) \le \lambda_{\mathfrak{m}}(B)$ which implies the equality.

(3) We can suppose A is local.

(a) $\implies$ (b) Suppose that $\lambda_{\mathfrak{m}}(B) = +\infty$. Then there exists a sequence $(c_n)_n$ in A such that $\lambda_{\mathfrak{m}}(y + c_n) \ge 2n$. Thus
$$a - 2c_n \in \mathfrak{m}^n, \quad b - ac_n + c_n^2 \in \mathfrak{m}^{2n}.$$
If $\mathrm{char}(A) \neq 2$, then c_n converges to some $\hat{c} \in \hat{A}_{\mathfrak{m}}$ and we have $a = 2\hat{c}$. As $A \to \hat{A}_{\mathfrak{m}}$ is faithfully flat, $a \in A \cap 2\hat{A}_{\mathfrak{m}} = 2A$. Thus $\hat{c} = c \in A$. Then $b = c^2$ and $(y - c)^2 = 0$, contradiction because B is reduced by hypothesis. Therefore $\mathrm{char}(A) = 2$. Then $a \in \cap_n \mathfrak{m}^n = \{0\}$ and $b + c_n^2 \in \mathfrak{m}^{2n}$. This implies that c_n converges to some $\hat{c} \in \hat{A}_{\mathfrak{m}}$ with $\hat{c}^2 = b$.

(b) $\implies$ (c) is clear. Let us prove (c) $\implies$ (a). By Lemma 2.7(1) below, there exists $\hat{\alpha} \in \hat{A}_{\mathfrak{m}}$ such that $(y + \hat{\alpha})^2 = 0$. If $(\alpha_n)_n$ is a sequence in A convergent to $\hat{\alpha}$, then $\lambda_{\mathfrak{m}}(y + \alpha_n)$ tends to $+\infty$.

We have $(a) \implies (d)$ by (2), and $(d) \iff (e)$ by Lemma 2.7(2) below.

Finally, if A is excellent then so is B, hence $B \otimes_A \hat{A}_{\mathfrak{m}}$ is reduced. □

Lemma 2.7. *Let R be a normal domain, let $D = R[T]/(T^2 + aT + b)$.*

(1) *The ring D is non-reduced if and only if $T^2 + aT + b = (T + c)^2$ for some $c \in R$.*
(2) *Suppose that D is reduced. Then $a^2 - 4b = 0$ if and only if D is a domain and the extension $\mathrm{Frac}(D)/\mathrm{Frac}(R)$ is inseparable.*

Proof. Let $K = \mathrm{Frac}(R)$. As D is flat over R, $D \to D \otimes_R K$ is injective.

(1) The "if" part is obvious. Let us prove the converse. As D is non-reduced, then so is $D \otimes_R K$, thus there exists $c \in K$ such that $T^2 + aT + b = (T + c)^2$. Then c is integral over R, thus belongs to R.

(2) As D is reduced, the K-algebra $D \otimes_R K$ is either a quadratic field extension or is isomorphic to the product of two copies of K. If D is a domain, then $D \otimes_R K$ is a field, hence equal to $\mathrm{Frac}(D)$. The latter is inseparable over K if and only if

$a^2-4b=0$. This proves the "only if" part. Conversely, if this vanishing condition holds, then $D\otimes_R K$ is not separable over K, hence is a field and then D is a domain. □

Proposition 2.8. *Let A be a noetherian regular domain, let B be a reduced A-algebra, free of rank 2 as A-module. Let $\mathfrak{m}$ be a maximal ideal of A.*

(1) *The Artinian ring $B/\mathfrak{m}B$ is reduced if and only if $\lambda_{\mathfrak{m}}(B)=0$. If $k:=A/\mathfrak{m}$ is perfect or of $\operatorname{char}(k)\neq 2$, then this is equivalent to $A\to B$ being étale above $\mathfrak{m}$.*
(2) *The ring B is regular at all maximal ideals lying over $\mathfrak{m}$ if and only if $\lambda_{\mathfrak{m}}(B)\leq 1$.*
(3) *If $\lambda_{\mathfrak{m}}(B)\geq 1$ (e.g. when B is singular at some maximal ideal lying over $\mathfrak{m}$), then there is only one maximal ideal $\mathfrak{n}\subset B$ lying over $\mathfrak{m}$ and $A/\mathfrak{m}\to B/\mathfrak{n}$ is an isomorphism.*

Proof. By Lemma 2.5 we can suppose A is local. Suppose $\operatorname{char}(k)=2$ (otherwise the proof is easier). Let $\lambda_{\mathfrak{m}}(y)=\lambda_{\mathfrak{m}}(B)$.

If $\lambda_{\mathfrak{m}}(B)=0$ and $\operatorname{ord}_{\mathfrak{m}}(a)=0$, then $A\to B$ is étale and B is regular. Suppose $\lambda_{\mathfrak{m}}(B)=0$ and $\operatorname{ord}_{\mathfrak{m}}(a)>0$. Then $B/\mathfrak{m}B\simeq k[T]/(T^2+\bar{b})$. If $T^2+\bar{b}\in k[T]$ is reducible, there exists $\alpha\in A^*$ such that $b-\alpha^2\in\mathfrak{m}$. This implies that $\lambda_{\mathfrak{m}}(B)\geq\lambda_{\mathfrak{m}}(y+\alpha)>0$. Contradiction. So $B/\mathfrak{m}B$ is a radicial quadratic extension of k and B is regular.

If $\lambda_{\mathfrak{m}}(B)\geq 1$, then $a,b\in\mathfrak{m}$ and $B/\mathfrak{m}B\simeq k[T]/(T^2)$. This implies that B is local with maximal ideal $\mathfrak{n}$, not étale over A, and we have $k\simeq B/\mathfrak{n}$. Note that if B is singular at some maximal ideal lying over $\mathfrak{m}$, then $\lambda_{\mathfrak{m}}(B)\geq 2$ by (2).

Finally when $\lambda_{\mathfrak{m}}(B)\geq 1$, we have $y\in\mathfrak{n}$ and the relation $y^2+ay+b=0$ implies that B is regular if and only if $b\notin\mathfrak{m}^2$, equivalently, $\lambda_{\mathfrak{m}}(B)=1$. □

2.2. Computation of the multiplicity $\lambda_p(Y)$.

We now discuss how to compute $\lambda_{\mathfrak{m}}(B)$ and find a generator y of B reaching $\lambda_{\mathfrak{m}}(B)$.

Lemma 2.9. *Let A be a noetherian regular domain, let B be a reduced A-algebra, free of rank 2 over A, let $y\in B$ be a generator of B with the relation $y^2+ay+b=0$ over A and let $\mathfrak{m}$ be a maximal ideal of A. Suppose that $\operatorname{char}(A/\mathfrak{m})\neq 2$. Let $c\in A$ be such that $2\operatorname{ord}_{\mathfrak{m}}(a-2c)>\operatorname{ord}_{\mathfrak{m}}(a^2-4b)$. Then $\lambda_{\mathfrak{m}}(y+c)=\lambda_{\mathfrak{m}}(B)$.*

Proof. First, c exists because 2 is invertible in $A/\mathfrak{m}^n$ for any $n\geq 1$. Let $z=y+c$. Then
$$z^2+(a-2c)z+(b-ac+c^2)=0 \tag{2.1}$$
with
$$\operatorname{ord}_{\mathfrak{m}}(b-ac+c^2)=\operatorname{ord}_{\mathfrak{m}}(4b-4ac+4c^2)=\operatorname{ord}_{\mathfrak{m}}(4b-a^2+(a-2c)^2)=\operatorname{ord}_{\mathfrak{m}}(4b-a^2).$$
Therefore $\lambda_{\mathfrak{m}}(z)=\operatorname{ord}_{\mathfrak{m}}(4b-a^2)=\lambda_{\mathfrak{m}}(B)$ by Proposition 2.6(2). □

The lemma below is a generalization of [13], Lemma 3.3(1-2). An algorithm to finding a generator y of B with $\lambda_{\mathfrak{m}}(B)=\lambda_{\mathfrak{m}}(y)$ can be easily derived from it in a similar way to [13], Algorithm 6.2.

Lemma 2.10. *Let A be a noetherian regular domain, let B be a reduced A-algebra, free of rank 2 over A, let $y\in B$ be a generator of B with the relation $y^2+ay+b=0$ over A and let $\mathfrak{m}$ be a maximal ideal of A with $\operatorname{char}(A/\mathfrak{m})=2$ and such that $\lambda_{\mathfrak{m}}(B)$ is finite (see Proposition 2.6(3)). We omit $\mathfrak{m}$ from the subscripts of ord and λ.*

(1) *If $2\operatorname{ord}(a)\leq\operatorname{ord}(b)$, then $\lambda(B)=\lambda(y)$.*

(2) *Suppose* $2\operatorname{ord}(a) > \operatorname{ord}(b)$. *Let* $\tilde{b} \in \mathrm{Gr}_{\mathfrak{m}}(A)$ *be the initial form of* b *(see Definition 2.1).*
  (a) *If* $\tilde{b}$ *is not a square in* $\mathrm{Gr}_{\mathfrak{m}}(A)$*, then* $\lambda(B) = \lambda(y)$.
  (b) *Suppose* $\tilde{b}$ *is a square in* $\mathrm{Gr}_{\mathfrak{m}}(A)$*: there exists* $c \in A$ *such that* $b - c^2 \in \mathfrak{m}^{\ell+1}$ *where* $\ell = \operatorname{ord}(b)$. *Then*

$$\lambda(y+c) > \lambda(y).$$

Proof. We know that $\lambda(B) = \lambda(y+c)$ for some $c \in A$ (see the discussions in Definition 2.3). The generator $z = y + c$ satisfies Equation (2.1). Using the fact that ord is a valuation on $\mathrm{Frac}(A)$, the proof of (1) is similar to that of Lemma 3.3(1) in [13].

(2.a) If $\lambda(B) > \lambda(y)$, there exists $c \in A$ such that

$$\operatorname{ord}(b - ac + c^2) > \lambda(y) = \operatorname{ord}(b).$$

As $2\operatorname{ord}(a) > \operatorname{ord}(b)$, this implies that $\operatorname{ord}(b) = 2\operatorname{ord}(c)$ and $\operatorname{ord}(b + c^2) > \operatorname{ord}(b)$. This is equivalent to the initial form of b being a square in $\mathrm{Gr}_{\mathfrak{m}}(A)$.

(2.b) We have $\operatorname{ord}(b) = 2\operatorname{ord}(c)$. Moreover

$$2\operatorname{ord}(a - 2c) > \operatorname{ord}(b), \quad \operatorname{ord}(b - ac + c^2) = \operatorname{ord}(b - c^2 - (a-2c)c) > \operatorname{ord}(b).$$

Hence $\lambda(y + c) > \operatorname{ord}(b) = \lambda(y)$. □

Example 2.11 Let $A = \mathbb{Z}[x]$. Let

$$B = \mathbb{Z}[x,y]/(y^2 + x^5 - 1).$$

and let $\mathfrak{m} \subset A$ be a maximal ideal.

(1) As $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is regular, we have $\lambda_{\mathfrak{m}}(B) \leq 1$ if $\mathfrak{m} \cap \mathbb{Z} = \{0\}$.
(2) Suppose now that $\mathfrak{m} \cap \mathbb{Z} = \ell\mathbb{Z}$ for some prime number ℓ. There exists a monic $f(x) \in \mathbb{Z}[x]$ such that $f(x)$ is irreducible in $\mathbb{F}_\ell[x]$ and $\mathfrak{m} = (\ell, f(x))$.
  (a) If $\ell \neq 2, 5$, then $\lambda_{\mathfrak{m}}(B) \leq 1$, with equality if and only if $f(x) \mid x^5 - 1$ in $\mathbb{F}_\ell[x]$.
  (b) If $\ell = 5$, then $\lambda_{\mathfrak{m}}(B) = 0$ if $f(x)$ is prime to $x - 1$ in $\mathbb{F}_5[x]$. Let $\mathfrak{m}_5 := (5, x-1)$. Then $\lambda_{\mathfrak{m}_5}(B) = \operatorname{ord}_{\mathfrak{m}_5}(x^5 - 1) = 2$.
  (c) Let $\ell = 2$. Let $z = y - 1 \in B$. Then $z^2 + 2z + x^5 = 0$. Consider $\mathfrak{m}_2 = (2, x)$. By Lemma 2.10(1), we have $\lambda_{\mathfrak{m}_2}(B) = 2\operatorname{ord}_{\mathfrak{m}_2}(2) = 2$.
  (d) Let $\ell = 2$ and $\mathfrak{m} \neq \mathfrak{m}_2$, then $\lambda_{\mathfrak{m}}(B) = 1$. Indeed, the curve $z^2 + x^5$ over $\mathbb{F}_2$ is smooth away from the point $x = z = 0$, hence $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}$ is smooth at $\mathfrak{m}$ and $B_{\mathfrak{m}}$ is regular. Thus $\lambda_{\mathfrak{m}}(B) = 1$ (it is non-zero by Proposition 2.8(2)).

Proposition 2.12. *Let* A *be a noetherian regular domain, let* B *be a reduced* A*-algebra, free of rank* 2 *as* A*-module. Let* $\mathfrak{m}$ *be a maximal ideal of* A*. Let* $(A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *be an étale extension of regular local rings. Then*

$$\lambda_{\mathfrak{m}}(B) = \lambda_{\mathfrak{m}'}(B \otimes_A A').$$

Proof. Let k, k' be the respective residue fields of A, A'. As $\mathrm{Gr}_{\mathfrak{m}}(A) \to \mathrm{Gr}_{\mathfrak{m}'}(A') \simeq \mathrm{Gr}_{\mathfrak{m}}(A) \otimes_k k'$ is injective, $\operatorname{ord}_{\mathfrak{m}}$ coincides with $\operatorname{ord}_{\mathfrak{m}'}$ on A.

Let $B = A[y]$. Then $B' := B \otimes_A A' = A'[y]$ and $\lambda_{\mathfrak{m}}(y) = \lambda_{\mathfrak{m}'}(y) \leq \lambda_{\mathfrak{m}'}(B')$. This implies that $\lambda_{\mathfrak{m}}(B) \leq \lambda_{\mathfrak{m}'}(B')$. So we can suppose $\lambda_{\mathfrak{m}}(B)$ finite. Assume that $\lambda_{\mathfrak{m}}(y) = \lambda_{\mathfrak{m}}(B)$. If $\mathrm{char}(k) \neq 2$, then $\lambda_{\mathfrak{m}'}(B') = \operatorname{ord}_{\mathfrak{m}'}(\Delta_{B/A}) = \operatorname{ord}_{\mathfrak{m}}(\Delta_{B/A})$. Suppose $\mathrm{char}(k) = 2$. By Lemma 2.10, $\lambda_{\mathfrak{m}}(y) = \lambda_{\mathfrak{m}'}(y) = \lambda_{\mathfrak{m}'}(B')$ (in the case

2.10(2) use the fact that if an element of $\mathrm{Gr}_{\mathfrak{m}}(A)$ is a square in $\mathrm{Gr}_{\mathfrak{m}}(A) \otimes_k k'$, then it is already a square in $\mathrm{Gr}_{\mathfrak{m}}(A)$ because k' is separable over k.) $\square$

2.3. **Multiplicity in the global case.**

Definition 2.13 Let Z be an integral noetherian regular scheme. Let $\psi : Y \to Z$ be a double cover (Definition 1.2). Let $p \in Y$ and let $q \in Z$ be its image. Define

$$\begin{array}{rcl} \lambda_q(Y) & = & \lambda_{\mathfrak{m}_q}(\psi_* \mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,q}), \\ \lambda_p(Y) & = & \lambda_q(Y), \end{array}$$

where $\mathfrak{m}_q$ denotes the maximal ideal of the noetherian regular local ring $\mathcal{O}_{Z,q}$. Note that it follows from the definition of a double cover that $\psi_* \mathcal{O}_Y \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,q}$ is reduced and free of rank 2 over $\mathcal{O}_{Z,q}$.

By Proposition 2.8(2)-(3), Y is singular at p if and only if $\lambda_p(Y) \geq 2$, and this implies that $\psi^{-1}(q) = \{p\}$.

Definition 2.14 Let $Y \to Z$ be as in Definition 2.13 above. As the discriminant ideal in the affine case is compatible with localization, we obtain by glueing a unique sheaf of ideals in $\mathcal{O}_Z$. We call it the *discriminant ideal of* $Y \to Z$ and denote it by $\Delta_{Y/Z}$.

3. Normalized blowing-up

For any scheme Y and closed subscheme $Y_0 \subset Y$, we call *the normalized blowing-up of* Y *along* Y_0 the normalization of the blowing-up of Y along Y_0. In this section we study the normalized blowing-up of Y at a singular point when Y is a *normal double cover* $\psi : Y \to Z$ of an integral noetherian excellent regular surface Z.

3.1. **Blowing-up regular schemes along a point.** Let A be a ring and let $\mathfrak{m} \subset A$ be a maximal ideal. We recall below the explicit description of the blowing-up

$$\mathrm{bl}_{\mathfrak{m}} : Z' \to Z = \operatorname{Spec} A$$

of Z along $\mathfrak{m}$. Denote by $\widetilde{A}$ the Rees algebra $\oplus_{n \geq 0} \mathfrak{m}^n$. For any $s \in \mathfrak{m} \setminus \{0\}$, denote by $\tilde{s}$ the element s viewed as a homogeneous element of degree 1 in $\widetilde{A}$ and by $\widetilde{A}_{(\tilde{s})}$ the homogeneous localization of $\widetilde{A}$ with respect to the positive powers of $\tilde{s}$. Denote by $D_+(\tilde{s}) \subseteq Z'$ the principal open subset defined by $\tilde{s}$.

Proposition 3.1. *Let A be a noetherian regular domain, let $\mathfrak{m} \subset A$ be a maximal ideal with $\dim A_{\mathfrak{m}} = 2$. Suppose that $\mathfrak{m}$ can be generated by two elements s, t (e.g., if A is local). Denote $k = A/\mathfrak{m}$.*

(1) *Let $Z' \to Z = \operatorname{Spec} A$ be the blowing-up along $\mathfrak{m}$.*
   (a) *We have a natural isomorphism*
   $$Z' \to \operatorname{Proj} A[S,T]/(tS - sT)$$
   induced by the isomorphism of homogeneous A-algebras
   $$A[S,T]/(tS - sT) \longrightarrow \widetilde{A}, \quad S \mapsto \tilde{s},\ T \mapsto \tilde{t}.$$
   (b) *The homomorphism*
   $$A[T_1]/(sT_1 - t) \longrightarrow \widetilde{A}_{(\tilde{s})} = \mathcal{O}_{Z'}(D_+(\tilde{s})), \quad T_1 \mapsto \tilde{t}/\tilde{s}$$
   of A-algebras is an isomorphism, it takes $\mathfrak{m}^r \subseteq A$ into $s^r \widetilde{A}_{(\tilde{s})}$ for all $r \geq 0$.
   (c) *The scheme Z' is regular of dimension 2, covered by the principal open subsets $D_+(\tilde{s})$ and $D_+(\tilde{t})$.*

(2) *Let* $E = V(\mathfrak{m}\mathcal{O}_{Z'}) \subset Z'$ *be the exceptional divisor. Then*

$$E \cap D_+(\tilde{s}) = V(s\widetilde{A}_{(\tilde{s})}) \simeq \operatorname{Spec} k[\bar{t}_1]$$

where $\bar{t}_1$ *is the images of* $t_1 := \tilde{t}/\tilde{s}$ *modulo* s. *Any maximal ideal of* $\widetilde{A}_{(\tilde{s})}$ *containing* s *is generated by* s *and one other element.*

(3) *Let* ν_E *be the normalized discrete valuation on* $\operatorname{Frac}(A)$ *associated to the generic point of* E. *Then* $\nu_E(\alpha) = \operatorname{ord}_{\mathfrak{m}}(\alpha)$ *for all* $\alpha \in \operatorname{Frac}(A)$.

(4) *If* A *is factorial then so is* $\widetilde{A}_{(\tilde{s})}$.

(5) *Let* $f \in A$ *with* $\operatorname{ord}_{\mathfrak{m}}(f) = d \geq 1$. *Let* $\tilde{f} \in \widetilde{A}$ *be the corresponding homogeneous element of degree* d. *Then* $f/s^d = \tilde{f}/\tilde{s}^d \in \widetilde{A}_{(\tilde{s})}$.

   (a) *Let* $V(f)'$ *be the strict transform of* $V(f)$ *in* Z'. *Then we have* $V(f)' \cap D_+(\tilde{s}) = V(f/s^d)$.

   (b) *The image of* $f/s^d \in \widetilde{A}_{(\tilde{s})}$ *in* $\widetilde{A}_{(\tilde{s})}/(s) \simeq k[\bar{t}_1]$ *has degree at most equal to* n.

Proof. (1), (2) are standard, see *e.g.*, the proof of [12], Theorem 8.1.19.

(3) Let ξ_E be the generic point of E. By (2), $\xi_E \in D_+(\tilde{s})$ and $\mathcal{O}_{Z',\xi_E}(-E)$ is generated by s. Therefore $\nu_E(s) = 1$. If $\alpha \in \mathfrak{m}^\ell \setminus \mathfrak{m}^{\ell+1}$, then $\alpha/s^\ell \in \widetilde{A}_{(\tilde{s})}$, hence $\nu_E(\alpha) \geq \ell = \operatorname{ord}_{\mathfrak{m}}(\alpha)$.

Conversely, suppose $\nu_E(\alpha) = \ell$. We have

$$\operatorname{div}(\alpha s^{-\ell}) = \operatorname{div}(\alpha) - \ell E \geq 0$$

as Weil divisors on $D_+(\tilde{s})$. Thus $\alpha \in s^\ell \widetilde{A}_{(\tilde{s})}$. So there exists $N \geq \ell$ such that $\alpha s^{N-\ell} \in \mathfrak{m}^N$. Because the class of s in $\mathfrak{m}/\mathfrak{m}^2 \subset \operatorname{Gr}_{\mathfrak{m}}(A)$ is a regular element, $\alpha \in \mathfrak{m}^\ell$, hence $\operatorname{ord}_{\mathfrak{m}}(\alpha) \geq \nu_E(\alpha)$.

(4) Notice that $s \in \widetilde{A}_{(\tilde{s})}$ is a prime element by (2), and $(\widetilde{A}_{(\tilde{s})})_s = A_s$ is factorial. Hence $\widetilde{A}_{(\tilde{s})}$ is factorial by Nagata's criterion.

(5.a) By definition, the strict transform of $V(f)$ is the closed subscheme of Z' defined by the homogeneous ideal $\oplus_{n\geq 0}(fA \cap \mathfrak{m}^n) \subseteq \widetilde{A}$. As $\operatorname{Gr}_{\mathfrak{m}}(A)$ is a domain, this ideal is equal to $\tilde{f}\widetilde{A}$ and $V(f)' = V_+(\tilde{f})$. Therefore $V(f)' \cap D_+(\tilde{s}) = V(f/s^d)$.

(5.b) Write

$$f = \sum_{0\leq i\leq d} \alpha_i t^i s^{d-i} + \epsilon_{d+1}$$

with $\alpha_i \in A$ and $\epsilon_{d+1} \in \mathfrak{m}^{d+1}$. This implies that

$$f/s^d = \sum_{0\in i\leq d} \alpha_i t_1^i + sc, \quad c \in \widetilde{A}_{(\tilde{s})}.$$

Its class mod s has degree $\leq d$. □

3.2. **Normalization after blowing-up.** To determine the integral closure of a domain we will use Serre's (S_2) + (R_1) criterion ([12], Theorem 8.2.23).

Lemma 3.2. *Let* (R, ν) *be a discrete valuation ring with residue field* κ, *let*

$$D = R[T]/(T^2 + \alpha T + \beta)$$

be an R*-algebra with* $\alpha^2 - 4\beta \neq 0$.

(1) *If* $\nu(\alpha^2 - 4\beta) = 0$, *then* D *is an étale* R*-algebra of rank* 2*;*

(2) *If* $\nu(\alpha^2 - 4\beta) = 1$, *then* D *is a discrete valuation ring totally ramified over* R *with ramification index* 2*;*

(3) *The conclusion of* (2) *holds if* $\mathrm{char}(\kappa) = 2$, $\nu(\alpha) > 0$, *and* $\nu(\beta) = 1$*;*
(4) *If* $\mathrm{char}(\kappa) = 2$, $\nu(\alpha) > 0$, *and* β *is not a square in* κ, *then* D *is a discrete valuation ring with ramification index* 1 *over* R *and inseparable quadratic residue extension.*

Proof. (1) is a special case of Proposition 2.6(1).

(2)-(3) The condition in (2) implies that $\mathrm{char}(\kappa) \neq 2$ and we can reduce to $\alpha = 0$. Then in both cases $T^2 + \alpha T + \beta$ is an Eisenstein polynomial and the conclusion is clear.

(4) Let π be a uniformizing element of R. Then $D/\pi D = \kappa[T]/(T^2 + \bar{\beta})$ is a field. This implies that π generates the unique maximal ideal of D, and the latter is a discrete valuation ring with ramification index 1 over R. The residue field $D/\pi D$ of D is inseparable over κ because $T^2 + \bar{\beta} \in \kappa[T]$ is inseparable. $\square$

Proposition 3.3. *Let* A *be an excellent noetherian regular domain and let* $\mathfrak{m}$ *be a maximal ideal of* A *with* $\dim A_{\mathfrak{m}} = 2$. *Let* B *be a normal domain, free of rank* 2 *over* A *such that* $\lambda_{\mathfrak{m}}(B) \geq 2$. *Fix a generator* y *of* B, *with the relation* $y^2 + ay + b = 0$, *such that* $\lambda_{\mathfrak{m}}(y) = \lambda_{\mathfrak{m}}(B)$ *and set* $r = [\lambda_{\mathfrak{m}}(B)/2] \geq 1$.

(1) *Let* $Z' \to Z := \operatorname{Spec} A$ *be the blowing-up along* $\mathfrak{m}$, *and let* $Y' \to Y := \operatorname{Spec} B$ *be the blowing-up along the maximal ideal of* B *lying over* $\mathfrak{m}$.
   (a) *Let* $s \in \mathfrak{m} \setminus \mathfrak{m}^2$. *Put* $y_0 = y/s$, $a_0 = a/s$, $b_0 = b/s^2 \in A_s$ *and let* $B_0 = \widetilde{A}_{(\tilde{s})}[y_0] \subseteq \mathrm{Frac}(B)$ *(see Proposition 3.1 for the notation). Then* $a_0 \in \mathfrak{m}^{r-1}\widetilde{A}_{(\tilde{s})}$, $b_0 \in \mathfrak{m}^{2r-2}\widetilde{A}_{(\tilde{s})}$, *and*
$$y_0^2 + a_0 y_0 + b_0 = 0 \tag{3.1}$$
   is the equation defining the $\widetilde{A}_{(\tilde{s})}$*-algebra* B_0.
   (b) *We have a canonical commutative diagram*
$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "\phi"'] & Y \arrow[d] \\ Z' \arrow[r] & Z \end{tikzcd}$$
   with ϕ *finite.*
(2) *Let* $s \in \mathfrak{m} \setminus \mathfrak{m}^2$. *Put*
$$y_1 = y/s^r,\ a_1 = a/s^r,\ b_1 = b/s^{2r} \in A_s.$$
   and let $B_1 := \widetilde{A}_{(\tilde{s})}[y_1] \subset \mathrm{Frac}(B)$. *Then* $a_1, b_1 \in \widetilde{A}_{(\tilde{s})}$,
$$y_1^2 + a_1 y_1 + b_1 = 0 \tag{3.2}$$
   and B_1 *is the integral closure of* $\widetilde{A}_{(\tilde{s})}$ *in* $\mathrm{Frac}(B)$. *In particular* B_1 *is the integral closure of* $\widetilde{A}_{(\tilde{s})}[y_0]$.
(3) $\Delta_{B_1/\widetilde{A}_{(\tilde{s})}} = s^{-2r}\Delta_{B/A}\widetilde{A}_{(\tilde{s})}$.

Proof. The hypothesis A excellent insures that $\lambda_{\mathfrak{m}}(B)$ is finite (Proposition 2.6). Part (3) is an immediate consequence of (2).

(1.a) Let $t \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that $\mathfrak{m} = (s, t)$. The maximal ideal of B lying over $\mathfrak{m}$ is generated by s, t and y. As $y \in \sqrt{sB + tB}$, the scheme Y' is covered by the affine charts $\mathrm{Spec}(A[y/s, t/s])$ and $\mathrm{Spec}(A[y/t, s/t])$. We have $a \in \mathfrak{m}^{r-1}\mathfrak{m} \subseteq s\mathfrak{m}^{r-1}\widetilde{A}_{(\tilde{s})}$ by Proposition 3.1(1.b). Hence $a_0 \in \mathfrak{m}^{r-1}\widetilde{A}_{(\tilde{s})}$. Similarly $b_0 \in \mathfrak{m}^{2r-2}\widetilde{A}_{(\tilde{s})}$. The relation (3.1) is deduced from the equation of y. As $A[y/s, t/s] = \widetilde{A}_{(\tilde{s})}[y_0]$, (a) is proved.

(1.b) As one can cover Z' by $D_+(\tilde{s})$ and $D_+(\tilde{t})$, the assertion follows from (1.a) applied to s and to t.

(2) The assertion $a_1, b_1 \in \widetilde{A}_{(\tilde{s})}$ follows from Proposition 3.1(1.b). As B_1 contains $\widetilde{A}_{(\tilde{s})}$ and is finite over $\widetilde{A}_{(\tilde{s})}$, it is enough to show that B_1 is integrally closed. As $\widetilde{A}_{(\tilde{s})}$ is regular and

$$B_1 \simeq \widetilde{A}_{(\tilde{s})}[T]/(T^2 + a_1 T + b_1),$$

B_1 is a local complete intersection, hence satisfies (S_n) for all $n \geq 1$ ([12], Corollary 8.2.18 and Example 8.2.20). By Serre's criterion it is enough to show that B_1 is integrally closed at points of codimension 1. Moreover, as $\operatorname{Spec} B_1 \to \operatorname{Spec} B$ is an isomorphism away from $V(s)$, it is enough to check the property for codimension 1 points belonging to $V(s)$, so lying over the generic point ξ_E of E (exceptional divisor of $Z' \to Z$).

Let $R = \mathcal{O}_{Z', \xi_E}$. Then

$$B_1 \otimes_{\widetilde{A}_{(\tilde{s})}} R \simeq R[y_1]/(y_1^2 + a_1 y_1 + b_1).$$

We will apply Lemma 3.2. We have $\min\{2\nu_E(a_1), \nu_E(b_1)\} \leq 1$. So we only have to check that when $\operatorname{char}(k(\xi_E)) = 2$, $\nu_E(a_1) > 0$ and $\nu_E(b_1) = 0$, then b_1 is not a square in $k(\xi_E)$. Under these hypothesis, we have $2\operatorname{ord}_{\mathfrak{m}}(a) = 2\nu_E(a) > 2r = \nu_E(b) = \operatorname{ord}_{\mathfrak{m}}(b)$ (Proposition 3.1(3)).

Suppose that b_1 is a square in $k(\xi_E) = \operatorname{Frac}(\widetilde{A}_{(\tilde{s})}/(s))$. As $\widetilde{A}_{(\tilde{s})}/(s)$ is integrally closed (Proposition 3.1(2)), b_1 is a square in $\widetilde{A}_{(\tilde{s})}/(s)$: there exist $c_1, c_2 \in \widetilde{A}_{(\tilde{s})}$ such that

$$b_1 = c_1^2 + s c_2.$$

There exist $N \geq r$, $\alpha_N \in \mathfrak{m}^N$ and $\beta_{2N} \in \mathfrak{m}^{2N}$ such that $c_1 = \alpha_N / s^N$ and $c_2 = \beta_{2N}/s^{2N}$. Then

$$s^{2N-2r} b = \alpha_N^2 + s\beta_{2N}.$$

So the initial form of $s^{2N-2r} b \in \mathfrak{m}^{2N}$ is a square in $\operatorname{Gr}_{\mathfrak{m}}(A)$, thus the same is true for that of b. This contradicts Lemma 2.10(2.b). Finally, B_1 is the integral closure of $\widetilde{A}_{(\tilde{s})}[y_0]$ because the latter is a sub-$\widetilde{A}_{(\tilde{s})}$-algebra of B_1. This proves (2).

(3) Using Equation (3.2), we get

$$\Delta_{B_1/\widetilde{A}_{(\tilde{s})}} = (a_1^2 + 4b_1)\widetilde{A}_{(\tilde{s})} = s^{-2r}(a^2 + 4b)\widetilde{A}_{(\tilde{s})} = s^{-2r}\Delta_{B/A}\widetilde{A}_{(\tilde{s})}.$$

□

3.3. **Lipman's resolution of singularities.**

Theorem 3.4. *Let Z be an integral noetherian excellent regular surface, let $\psi : Y \to Z$ be a normal double cover. Let $p_0 \in Y$ be a singular point, let $q_0 = \psi(p_0)$.*

(1) *Let Z' (resp. Y') be the blowing-up of Z (resp. of Y) along the (reduced) closed point q_0 (resp. p_0) and let $\tilde{Y} \to Y'$ be the normalization. Then we have a canonical commutative diagram*

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[dr, "\psi_1"'] & Y' \arrow[r, "\mathrm{bl}_{p_0}"] \arrow[d] & Y \arrow[d, "\psi"] \\ & Z' \arrow[r, "\mathrm{bl}_{q_0}"] & Z \end{tikzcd}$$

and $\psi_1 \colon \tilde{Y} \to Z'$ is a normal double cover with discriminant ideal

$$\Delta_{\tilde{Y}/Z'} = \mathcal{O}_{Z'}(2rE) \otimes_{\mathcal{O}_Z} \Delta_{Y/Z},$$

where $r = [\lambda_{q_0}(Y)/2]$ and E is the exceptional divisor of $Z' \to Z$.

(2) *If*

$$Y_n \to Y_{n-1} \to ... \to Y_1 \to Y_0 = Y$$

is a sequence of normalized blowing-ups of (reduced) singular points $p_i \in Y_i$, then we have a commutative diagram

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d, "\psi_n"] & Y_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}"] & \cdots \arrow[r] & Y_1 \arrow[r] \arrow[d, "\psi_1"] & Y_0 \arrow[d, "\psi_0=\psi"] \\ Z_n \arrow[r] & Z_{n-1} \arrow[r] & \cdots \arrow[r] & Z_1 \arrow[r] & Z_0 := Z \end{tikzcd}$$

where $Z_{i+1} \to Z_i$ is the blowing-up of the reduced closed point $q_i \in Z_i$, image of p_i, and each $\psi_i : Y_i \to Z_i$ is a normal double cover.

(3) *Suppose moreover that $Y \to Z$ is generically étale. Then the Galois group $G := \mathrm{Gal}(K(Y)/K(Z))$ acts on each Y_i and ψ_i induces an isomorphism $Y_i/G \simeq Z_i$.*

Proof. (1) This follows from the local description of Y' and $\tilde{Y}$ by Proposition 3.3.

(2) Use repeatedly (1).

(3) As Y_i is the normalization of Z_i in $K(Y)$, G acts on Y_i. The morphism $\psi_i : Y_i \to Z_i$ factors through $Y_i/G \to Z_i$ which is finite and birational, hence an isomorphism because Z_i is normal. □

Remark 3.5 Keep the settings of Theorem 3.4(1) and suppose that $Z = \operatorname{Spec} A$ is affine and $Y = \operatorname{Spec} B$ is free over A. Consider the graded A-algebra $\tilde{A}[W]$ with $\deg W = r$. Then $\tilde{Y}$ is isomorphic to the closed subscheme of $\mathrm{Proj}(\tilde{A}[W])$ defined by the equation

$$W^2 + \tilde{a}W + \tilde{b} = 0,$$

where $\tilde{a}, \tilde{b}$ are respectively a and b considered as homogeneous elements in $\tilde{A}$ of respective degrees $r, 2r$.

Indeed, the natural homomorphism of graded A-algebras

$$\widetilde{A} \to \widetilde{B} := \widetilde{A}[W]/(W^2 + \tilde{a}W + \tilde{b})$$

is finite, thus induces a finite morphism of Z-schemes $\phi : \operatorname{Proj} \widetilde{B} \to \operatorname{Proj} \widetilde{A} = Z'$. Let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to q_0. For any $s \in \mathfrak{m} \backslash \mathfrak{m}^2$, Proposition 3.3 implies that $\psi_1^{-1}(D_+(\tilde{s}))$ is the normalization of $D_+(\tilde{s})$ in $\mathrm{Frac}(B)$. Covering Z by the $D_+(\tilde{s})$'s, we see that ϕ is the normalization of Z in $K(Y)$, hence isomorphic to $\widetilde{Y}$ by Theorem 3.4(1).

3.4. **The exceptional locus of Lipman's resolution of singularities.** Let Z be an integral noetherian excellent regular surface, let $\psi : Y \to Z$ be a normal double cover (Definition 1.2).

Definition 3.6 We call a sequence

$$Y_N \to Y_{N-1} \to \cdots \to Y_1 \to Y_0 := Y$$

where each $Y_{i+1} \to Y_i$ is the normalized blowing-up of a singular point of Y_i, and where Y_N is regular (see Theorem 1.1), a *Lipman's desingularization sequence of Y*.

Let $p_0 \in Y$ be a singular point. In this subsection we study the *exceptional locus* of $Y_N \to Y$ above p_0, that is, the scheme $Y_N \times_Y \operatorname{Spec} k(p_0)$. We fix a prime divisor $E_0 \ni q_0$ of Z, regular at q_0. Typically, for our applications in mind, Z is a regular model over a discrete valuation ring $\mathcal{O}_K$ of a smooth curve over $\operatorname{Frac}(\mathcal{O}_K)$ and E_0 is an irreducible component of the closed fiber of Z, regular at q_0. Let

$$\mathcal{E} := Y_N \times_Y \psi^* E_0 = Y_N \times_Z E_0 \subset Y_N,$$

where $\psi^* E_0 = Y \times_Z E_0$. This is the union of the strict transform of $\psi^* E_0$ in Y_N and the exceptional locus of $Y_N \to Y$ above p_0. In the above example, $\mathcal{E}$ is the closed fiber of Y_N. We would like to know the structure of each irreducible component of $\mathcal{E}$, their multiplicities in $\mathcal{E}$, and how they intersect each other.

Remark 3.7 The intermediate Y_i's are not necessarily unique as there may be more than one singular point in Y_{i-1}, and Y_i depends on the choice of a singular point in Y_{i-1}. But the final Y_N is also obtained by repeatedly taking normalized blowing-up of the singular locus, rather than just take one singular point; the integer N is the total number of singular points in the intermediate Y_i's. Therefore N and Y_N are unique for a given normal double cover $Y \to Z$.

Note that in general, $Y_N \to Y$ is not the minimal desingularization, see the example of § 7.

The double cover $\psi : Y \to Z$ induces a finite flat morphism $\mathcal{E} \to Z_N \times_Z E_0$ of degree 2. We start by describing $Z_N \times_Z E_0$. By Theorem 3.4, there exists a sequence of blowing-ups of closed points

$$Z_N \to Z_{N-1} \to \cdots \to Z_1 \to Z_0 := Z$$

such that Y_i is the normalization of Z_i in $K(C)$. For all $1 \le i \le N$, denote by $E_i \subset Z_i$ the exceptional divisor of $Z_i \to Z_{i-1}$ and by E_i' (including $i = 0$) its strict transform in Z_N.

Lemma 3.8. *Write*

$$Z_N \times_Z E_0 = \sum_{0 \le i \le N} m_i E_i'$$

as Cartier divisors of Z_N.

(1) *For $1 \le i \le N$, $E_i = \mathbb{P}^1_{k(q_{i-1})}$. For all $0 \le i \le N$, the canonical morphism $E_i' \to E_i$ is an isomorphism.*
(2) *The union $\cup_{0 \le i \le N} E_i' \subset Z_N$ is a tree of irreducible components with transverse intersections.*
(3) *The multiplicities m_i can be determined inductively as follows: $m_0 = m_1 = 1$. Let $i \ge 1$. If q_i is the intersection point of E_i with the strict transform of E_{i-1} in Z_i, then $m_{i+1} = m_i + m_{i-1}$. Otherwise, $m_{i+1} = m_i$.*

Proof. (1) By Castelnuovo's criterion, $E_i \simeq \mathbb{P}^1_{k(q_{i-1})}$ if $i \ge 1$. For all $i \ge 1$, as $E_i' \to E_i$ is finite birational and E_i is normal, $E_i' \to E_i$ is an isomorphism. For $i = 1$, $E_0' \to E_0$ is an isomorphism above $E_0 \setminus \{q_0\}$ and E_0 is regular at q_0. This also implies that $E_0' \to E_0$ is an isomorphism.

(2)-(3) Apply inductively Proposition 3.1 to the Z_i's. One can take for s a generator of $\mathcal{O}_Z(-E_0)_{q_0}$. □

Proposition 3.9. *Keep the notation of Lemma 3.8. For $1 \le i \le N$, set*

$$\Theta_i' = \psi_N^{-1}(E_i')$$

with the reduced structure.

(1) *The scheme Θ'_i is irreducible if and only if $\psi_i^{-1}(E_i)$ is irreducible. When it is reducible, it has two irreducible components, each of them is isomorphic to E'_i by ψ_N, and has multiplicity m_i in $\mathcal{E}$.*
(2) *Suppose that $\lambda_{q_{i-1}}(Y_{i-1})$ is odd, then $\Theta'_i \to E'_i$ is an isomorphism, and the multiplicity of Θ'_i in $\mathcal{E}$ is $2m_i$.*
(3) *Suppose that $\lambda_{q_{i-1}}(Y_{i-1})$ is even and Θ'_i is irreducible. Then the latter has multiplicity m_i in $\mathcal{E}$. Moreover, if $\mathrm{char}(k(q_0)) \neq 2$, then $\Theta'_i \to E'_i$ is generically étale.*

Proof. Note that any finite birational morphism onto E'_i is an isomorphism because the latter is normal. So it is enough to study Θ'_i at its generic points. Note that the multiplicity in $\mathcal{E}$ of an irreducible component of Θ'_i is equal to m_i multiplied by the ramification index over E'_i.

As $Y_N \to Y_i$ is an isomorphism over an open neighborhood of the generic point of E_i, it is enough to show the statement for $\Theta_i := \psi_i^{-1}(E_i)$. Moreover without loss of generalities we can assume that $i = 1$. Then all the statements of the proposition follow from direct observations on an affine equation (3.2) of Y_1 □

Remark 3.10 Let us see how to check the conditions of Proposition 3.9 above in the concrete situation given by Proposition 3.3 of which we keep the notation. As usual we let $Z = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$.

(1) The scheme $\psi_1^{-1}(E_1)$ is irreducible if and only if the polynomial
$$T^2 + \bar{a}_1 T + \bar{b}_1 \in k[\bar{t}_1][T]$$
(where $\bar{*}$ means the class modulo $s\widetilde{A}_{(\tilde{s})}$) is irreducible. The same statement applies to $\psi_i^{-1}(E_i)$ if a similar equation of Y_i is known.
(2) Suppose $\mathrm{char}(k(q_0)) = 2$ and Θ_1 is irreducible. The $\Theta'_1 \to E'_1$ is generically étale if and only if $2\,\mathrm{ord}_{\mathfrak{m}}(a) = \lambda_{q_0}(Y)$.
(3) The structure of Θ'_1, in Case (3) of Proposition 3.9 is more complicate to describe. We have to determine inductively the strict transform of Θ_1 after each blowing-up.

Proposition 3.11. *Keep the notation of Lemma 3.8 and Proposition 3.9. Let $0 \le i \neq j \le N$ be such that $E'_i \cap E'_j = \{q\}$.*

(1) *If Θ'_i has two irreducible components $\Theta'_{i,1}, \Theta'_{i,2}$ and similarly for Θ'_j, then up to renumbering, $\Theta'_{i,1}$ and $\Theta'_{j,1}$ (resp. $\Theta'_{i,2}$ and $\Theta'_{j,2}$) intersect transversely at a single $k(q)$-rational point, and $\Theta'_{i,1} \cap \Theta'_{j,2} = \Theta'_{i,2} \cap \Theta'_{j,1} = \emptyset$.*
(2) *If Θ'_i has two irreducible components and Θ'_j is irreducible, then Θ'_j intersects transversely each of these components at a single $k(q)$-rational point. These intersection points can be the same.*
(3) *Suppose that Θ'_i and Θ'_j are both irreducible. Denote by e_i, e_j the respective ramification indexes of ψ_N at the generic points of Θ'_i, Θ'_j. Then we have*
$$e_i[k(\Theta'_i) : k(E'_i)] = e_j[k(\Theta'_j) : k(E'_j)] = 2$$
and
$$\sum_{p \in \psi_N^{-1}(q)} i_p(\Theta'_i, \Theta'_j) e_i e_j [k(p) : k(q)] = 2$$
where $i_p(\Theta'_i, \Theta'_j)$ denotes the intersection multiplicity of Θ'_i and Θ'_j at p.

Proof. Let $\{\Theta'_{i,\alpha}\}_\alpha, \{\Theta'_{j,\beta}\}_\beta$ be the respective irreducible components of Θ'_i and Θ'_j. Let $e_{i,\alpha}$ be the ramification index of ψ_N at the generic point of $\Theta'_{i,\alpha}$, similarly for $e_{j,\beta}$. We have

$$\sum_\alpha e_{i,\alpha}[k(\Theta'_{i,\alpha}) : k(E'_i)] = [K(Y) : K(Z)] = 2.$$

Similarly for Θ'_j. By general intersection theory, we have

$$\sum_{p\in\psi_N^{-1}(q)} i_p(\psi_N^* E'_i, \psi_N^* E'_j)[k(p) : k(q)] = [K(Y) : K(Z)] i_q(E'_i, E'_j) = 2,$$

and by definition $\psi_N^* E'_i = \sum_\alpha e_{i,\alpha}\Theta'_{i,\alpha}$ (similarly for $\psi_N^* E'_j$). Therefore

$$\sum_{p\in\psi_N^{-1}(q)} \sum_{\alpha,\beta} i_p(\Theta'_{i,\alpha}, \Theta'_{j,\beta}) e_{i,\alpha} e_{j,\beta}[k(p) : k(q)] = 2.$$

All coefficients in the above equality are positive except the intersection multiplicities i_p which can be zero. This leaves very few possibilities for the combinations of the various coefficients.

We also notice that through any point of $\psi_N^{-1}(q)\cap\Theta'_i$ passes a point of $\psi_N^{-1}(q)\cap\Theta'_j$, by the going down property of $Y_N \to Z_N$ (see [21], tag 00HU). The proposition is then proved by straightforward case-by-case analysis. □

3.5. **Simultaneous resolution of singularities.** Let $f : X \to Y$ be a finite morphism of integral noetherian excellent surfaces. S. Abhyankar [1] asked for the existence of resolutions of singularities $\tilde{X} \to X$ and $\tilde{Y} \to Y$ such that f extends to a finite morphism $\tilde{X} \to \tilde{Y}$. He proved that the answer is negative in general if $\deg f > 3$, and is positive when f is a Galois cover of degree ≤ 3 of algebraic surfaces over a field of characteristic different from $\deg f$. For normal surfaces fibered over a discrete valuation ring, similar results are obtained in [15], §7 without assumption on the residue characteristics.

Corollary 3.12. *Let $f : X \to Y$ be a finite morphism of degree 2 of integral noetherian excellent surfaces. Then there are resolutions of singularities $\tilde{X} \to X$ and $\tilde{Y} \to Y$ such that f extends to a finite morphism $\tilde{X} \to \tilde{Y}$.*

Proof. Replacing Y with a resolution of singularities of Y and then X by the normalization of Y in $K(X)$, we can assume that Y is regular. Then $X \to Y$ is a normal double cover (Example 1.4). Applying Theorem 3.4 to $X \to Y$ then yields a simultaneous resolution of singularities. □

4. Variations of the multiplicities after normalized blowing-ups

Let $\psi : Y \to Z$ be a normal double cover (Definition 1.2) of an integral excellent noetherian regular surface Z. Let $Z' \to Z$ be the blowing-up of a closed point q_0, with exceptional divisor E, and let $\tilde{Y} \to Z'$ be the normalization of Z' in $K(Y)$. In this section we will give a bound on the multiplicities of the points of $\tilde{Y}$ lying over E in terms of $\lambda_{q_0}(Y)$ (Theorem 4.6). It says roughly that the sum of the new multiplicities is bounded by $\lambda_{q_0}(Y)$. This is done by comparing them to the multiplicities of the double cover $\tilde{Y} \times_{Z'} E \to E$ when $\tilde{Y} \times_{Z'} E$ is reduced, or of a "twisted" version of λ_q otherwise.

4.1. **Double covers of $\mathbb{P}^1_k$.** We first give some properties of the multiplicities λ_q on double covers of regular curves.

Proposition 4.1. *Let E be an integral regular curve over a field k. Let $\Theta \to E$ be a double cover (Θ may be reducible) and let $\pi : \tilde{\Theta} \to \Theta$ be the normalization map.*

(1) *Let $q \in E$ be a closed point. We have*

$$\dim_k((\pi_*\mathcal{O}_{\tilde{\Theta}})_q/\mathcal{O}_{\Theta,q}) = [\lambda_q(\Theta)/2][k(q):k].$$

(2) *Let $p \in \Theta$ be a point lying over q. If $\lambda_q(\Theta)$ is odd, then there is only one point $\tilde{p} \in \tilde{\Theta}_p$, and we have $k(p) = k(\tilde{p})$.*

(3) *If E is proper over k, then*

$$p_a(\Theta) = p_a(\tilde{\Theta}) + \sum_{q\in E}[\lambda_q(\Theta)/2][k(q):k],$$

where $p_a(C) = 1 - \chi(\mathcal{O}_C)$ denotes the arithmetic genus of any proper variety C over k.

Proof. (1)-(2) Similar to [11], Lemme 6. The proof of (3) is similar to [12], Proposition 7.5.4. □

Next we consider the case $E = \mathbb{P}^1_k$. Fix an integer $r \geq 1$ and two homogeneous polynomials $a(s,t), b(s,t) \in k[s,t]$ such that

$$\max\{2\deg a, \deg b\} = 2r.$$

Consider the curve Θ in the weighted projective space $\mathbb{P}(1,1,r)$ over k (see [5], §1) defined by the equation

$$z^2 + a(s,t)z + b(s,t) = 0.$$

Lemma 4.2. *Let Θ be as above and suppose that Θ is reduced. Denote by $\tilde{\Theta} \to \Theta$ the normalization map.*

(1) *We have a finite flat morphism $\Theta \to \mathbb{P}^1_k$ of degree 2. Moreover $H^0(\Theta, \mathcal{O}_\Theta) = k$ and $p_a(\Theta) = r - 1$.*

(2) *If $\Theta \to \mathbb{P}^1_k$ is generically étale then*

$$\sum_{q\in\mathbb{P}^1_k}\lambda_q(\Theta)[k(q):k] \leq 2r.$$

(3) *Suppose that $\Theta \to \mathbb{P}^1_k$ is purely inseparable.*
   (a) *If $H^0(\tilde{\Theta}, \mathcal{O}_{\tilde{\Theta}}) = k$, then*

$$\sum_{q\in\mathbb{P}^1_k} 2[\lambda_q(\Theta)/2][k(q):k] \leq 2r - 2.$$

   (b) *Otherwise, we have*

$$\sum_{q\in\mathbb{P}^1_k} 2[\lambda_q(\Theta)/2][k(q):k] = 2r,$$

   and $\lambda_q(\Theta)$ is even if $k(q) = k$.

(4) *In all cases, $\lambda_q(\Theta) \leq 2r$.*

Proof. (1) The first part is clear and the second part can be proved by deforming the curve to a hyperelliptic curve of genus $r-1$, or by computing the Čech cohomology of $\mathcal{O}_\Theta$ with the affine covering $\{D_+(s), D_+(t)\}$.

(2) Denote by $\Delta(t) = \Delta_{\Theta/\mathbb{P}^1_k} \in k[t]$. For any closed point $q \in \operatorname{Spec} k[t]$, $\lambda_q(\Theta) \leq \operatorname{ord}_q(\Delta)$ by Proposition 2.6(2). We have

$$\sum_{q \in \operatorname{Spec} k[t]} \lambda_p(\Theta)[k(q):k] \leq \deg \Delta(t) \leq 2r.$$

We are done if $t = \infty$ corresponds to a regular point. Otherwise we have $\lambda_\infty(\Theta) \leq \deg t^{2r}\Delta(1/t) = 2r - \deg \Delta(t)$.

(3.a) We have $p_a(\tilde{\Theta}) = \dim H^1(\tilde{\Theta}, \mathcal{O}_{\tilde{\Theta}}) \geq 0$. The desired inequalities results from Proposition 4.1(3) and (1).

(3.b) Let $\tilde{k} = H^0(\tilde{\Theta}, \mathcal{O}_{\tilde{\Theta}})$ with $k \subsetneq \tilde{k}$. Then $\tilde{\Theta}$ is a $\tilde{k}$-scheme. The morphism $\tilde{\Theta} \to \Theta \to \mathbb{P}^1_k$ induces a morphism $\tilde{\Theta} \to \mathbb{P}^1_{\tilde{k}}$:

$$\begin{tikzcd} \tilde{\Theta} \arrow[r] \arrow[d] & \mathbb{P}^1_{\tilde{k}} \arrow[d] \\ \Theta \arrow[r] & \mathbb{P}^1_k. \end{tikzcd}$$

We see that $\tilde{\Theta} \to \mathbb{P}^1_{\tilde{k}}$ has degree $2/[\tilde{k}:k]$, thus $[\tilde{k}:k] = 2$, and $\tilde{\Theta} \to \mathbb{P}^1_{\tilde{k}}$ is birational hence an isomorphism. Therefore $p_a(\tilde{\Theta}) = -1$ and

$$\sum_{q \in \mathbb{P}^1_k} [\lambda_q(\Theta)/2][k(q):k] = r$$

by Proposition 4.1 and (1). Note that $\tilde{k}/k$ is inseparable because $k(\Theta) = k(\tilde{\Theta}) = \tilde{k}(t)$ is inseparable over $k(t)$.

Let $k(q) = k$ and suppose that $\lambda_q(\Theta)$ is odd. Let $p \in \Theta_q$ and let $\tilde{p} \in \tilde{\Theta}_p$. Then $k(p) = k(q) = k$ because p is singular, and $k(\tilde{p}) = k(p) = k$ by Proposition 4.1(2). But $\tilde{k} = H^0(\tilde{\Theta}, \mathcal{O}_{\tilde{\Theta}}) \subseteq k(\tilde{p})$, contradiction.

(4) is an immediate consequence of (2) and (3). □

4.2. **Multiplicity along a regular closed subscheme.** Let $\psi : Y \to Z$ be a normal double cover of an integral noetherian regular excellent scheme. Let Γ be an integral regular closed subscheme of Z. We define in this subsection a multiplicity $\lambda_{\Gamma,q}(Y)$, for any closed point $q \in Z$, which is just $\lambda_q(Y \times_Z \Gamma)$ if $Y \times_Z \Gamma \to \Gamma$ is a double cover (equivalently if $Y \times_Z \Gamma$ is reduced). If this condition is not satisfied, we have to "twist" $Y \times_Z \Gamma$ to get a double cover.

Lemma 4.3. *Let A be a noetherian excellent regular domain. Let B be a normal and free A-algebra of rank 2. Suppose that $A \to B$ has ramification index 2 above some prime ideal $\mathfrak{q} \subset A$ of height 1, and that $A/\mathfrak{q}$ is normal.*

(1) *There exists a generator y of B, with $y^2 + ay + b = 0$, such that $a \in \mathfrak{q}$, $b \in \mathfrak{q} \setminus \mathfrak{q}^2$.*
(2) *Fix an y given by (1). If $\mathfrak{m} \in V(\mathfrak{q})$ is a maximal ideal of A such that $A_\mathfrak{m}/\mathfrak{q}A_\mathfrak{m}$ is regular, then $\lambda_\mathfrak{m}(B) = \lambda_\mathfrak{m}(y - c)$ for some $c \in \mathfrak{q}$ (depending on $\mathfrak{m}$).*

Proof. (1) Let y be a generator of B. As $A \to B$ has ramification index 2 above $\mathfrak{q}$, $B/\mathfrak{q}B \simeq (A/\mathfrak{q})[T]/(T^2 + \bar{a}T + \bar{b})$ is non-reduced. So $T^2 + \bar{a}T + \bar{b}$ is a square in $\operatorname{Frac}(A/\mathfrak{q})[T]$. But $A/\mathfrak{q}$ is integrally closed, so $T^2 + \bar{a}T + \bar{b}$ is in fact a square in $(A/\mathfrak{q})[T]$. Therefore translating y by a suitable element of A we have $a, b \in \mathfrak{q}$. Moreover, $b \notin \mathfrak{q}^2$ because $B \otimes_A A_{\mathfrak{q}}$ is normal.

(2) Suppose that for some $c_1 \in A$ we have

$$\lambda_{\mathfrak{m}}(y - c_1) = \min\{2 \operatorname{ord}_{\mathfrak{m}}(a + 2c_1), \operatorname{ord}_{\mathfrak{m}}(b + (a + 2c_1)c_1 - c_1^2)\} > \ell := \lambda_{\mathfrak{m}}(y).$$

This implies that $\operatorname{ord}_{\mathfrak{m}}(c_1) \geq \ell/2$ and $\operatorname{ord}_{\mathfrak{m}}(b - c_1^2) > \ell$. Therefore

$$c_1^2 \in b + \mathfrak{m}^{\ell+1} \subset \mathfrak{q} + \mathfrak{m}^{\ell+1}.$$

In $A/\mathfrak{p}$, we have $\bar{c}_1^2 \in \bar{\mathfrak{m}}^{\ell+1}$. As $A/\mathfrak{p}$ is regular at $\bar{\mathfrak{m}}$, this implies that $\operatorname{ord}_{\bar{\mathfrak{m}}}(\bar{c}_1) \geq r := \lceil (\ell+1)/2 \rceil$. In other words $c_1 = c + s$ with $c \in \mathfrak{q}$ and $s \in \mathfrak{m}^r$. As $r > \ell/2$, we have $\lambda_{\mathfrak{m}}(y - c) = \lambda_{\mathfrak{m}}((y - c_1) + s) > \ell$. Repeating this operation if necessary we will get a generator $y - c$ as desired, because $\lambda_{\mathfrak{m}}(B)$ is finite. $\square$

Definition 4.4 Let $\psi : Y \to Z$ be as in the beginning of this subsection. Let $\Gamma \subset Z$ be an integral regular closed subscheme of codimension 1 such that ψ has ramification index 2 above the generic point of Γ. Then $\Theta := Y \times_Z \Gamma \to \Gamma$ is finite flat of rank 2, but Θ is not reduced. So we can't define the multiplicity $\lambda_q(\Theta)$ as in Definition 2.3. Instead we will define a "twisted" version for Θ.

We construct an effective divisor D on Γ as follows. Let $U \subseteq Z$ be an affine open subset such that $\mathcal{O}_Z(-\Gamma)|_U = s\mathcal{O}_Z(U)$ is free, and that $\psi^{-1}(U) \to U$ is free, defined by some equation

$$y^2 + ay + b = 0$$

such that $a, b \in s\mathcal{O}_Z(U)$ and $b \notin s^2\mathcal{O}_Z(U)$ (Lemma 4.3(1)). Note that any another similar equation of $\psi^{-1}(U)$ will give another $b' = ub + s^2c$ with u a unit and $c \in \mathcal{O}_Z(U)$. Then define $D|_{U\cap\Gamma}$ as the zero divisor of $(s^{-1}b)|_{U\cap\Gamma}$. It does not depend on the choice of the equation. This allows to define an effective divisor D on Γ.

Now for any $q \in \Gamma$ we put

$$\lambda_{\Gamma,q}(Y) = \operatorname{ord}_q(\mathcal{O}_\Gamma(-D)_q) < +\infty$$

(order at q of the ideal $\mathcal{O}_\Gamma(-D)_q \subseteq \mathcal{O}_{\Gamma,q}$). It will be crucial for the proof of Theorem 4.6(1) that we can compute $\lambda_{\Gamma,q}(Y)$ with *the same equation* for all points in a suitable open subset of Γ.

Lemma 4.5. *Let $Y \to Z$ and Γ be as above. Let ξ_Γ be the generic point of Γ, let $q \in \Gamma$ be a closed point and let $\Theta = Y \times_Z \Gamma$.*

(1) *If $Y \to Z$ has ramification index 2 over ξ_Γ, then*

$$\lambda_{\Gamma,q}(Y) \geq \lambda_q(Y) - 1 \geq 0.$$

(2) *Suppose that $Y \to Z$ has ramification index 1 above ξ_Γ. Then $\Theta \to \Gamma$ is a double cover and $\lambda_q(\Theta) \geq \lambda_q(Y)$.*

Proof. We can suppose that $Z = \operatorname{Spec} A$ with $(A, \mathfrak{m})$ local. Observe that for any $\alpha \in A$,

$$\operatorname{ord}_{\mathfrak{m}}(\alpha) \leq \operatorname{ord}_{\mathfrak{m}}(\alpha|_\Gamma)$$

where $\alpha|_\Gamma$ denotes the image of α in $\mathcal{O}(\Gamma)$.

(1) We can suppose that $Z = \operatorname{Spec} A$ is local. Let $s \in \mathfrak{m}$ be a generator of the ideal $\mathcal{O}_Z(-\Gamma)$. Let $y \in B$ be such that $\lambda_{\mathfrak{m}}(B) = \lambda_{\mathfrak{m}}(y)$ and $a, b \in sA$ (Lemma 4.3(2)). By definition

$$\lambda_q(Y) \leq \operatorname{ord}_q(b) = \operatorname{ord}_q(b/s) + 1 \leq \operatorname{ord}_q((b/s)|_\Gamma) + 1 = \lambda_{\Gamma,q}(Y) + 1$$

(the last equality comes from the definition of $\lambda_{\Gamma,q}(Y)$). On the other hands, as $a, b \in \mathfrak{m}$, we have $\lambda_q(Y) \geq 1$.

(2) Let $B = A[y]$ with $\lambda_q(Y) = \lambda_{\mathfrak{m}}(y)$. The projection $\Theta \to \Gamma$ is finite flat of rank 2. The scheme Θ is reduced because it is a local complete intersection (Lemma 2.2) and reduced at the generic points. Hence $\Theta \to \Gamma$ is a double cover. An equation of Θ is given by

$$\bar{y}^2 + \bar{a}\bar{y} + \bar{b} = 0 \mod s$$

Then

$$\lambda_q(\Theta) \geq \min\{2\operatorname{ord}_q(\bar{a}), \operatorname{ord}_q(\bar{b})\} \geq \min\{2\operatorname{ord}_q(a), \operatorname{ord}_q(b)\} = \lambda_q(Y)$$

and the lemma is proved. □

4.3. **Upper bound on the multiplicities.**

Theorem 4.6. *Let Z be an integral noetherian excellent regular surface. Let $\psi : Y \to Z$ be a normal double cover (Definition 1.2). Let $p_0 \in Y$ be a singular point and $q_0 = \psi(p_0)$.*

Let $Z' \to Z$ be the blowing-up along q_0, with exceptional divisor $E \subset Z'$ and let $\phi : \tilde{Y} \to Z'$ be the normalization of Z' in $K(Y)$ (see Theorem 3.4(1)). Consider the scheme $\Theta := \tilde{Y} \times_{Z'} E$.

(1) *If $\lambda_{q_0}(Y)$ is odd, then $\lambda_q(\tilde{Y}) \geq 1$ for all $q \in E$, and*

$$\sum_{q\in E}(\lambda_q(\tilde{Y}) - 1)[k(q) : k(q_0)] \leq \sum_{q\in E} \lambda_{E,q}(\tilde{Y})[k(q) : k(q_0)] \leq \lambda_{q_0}(Y).$$

(See § 4.2 for the definition of $\lambda_{E,q}(\tilde{Y})$.)

(2) *Suppose that $\lambda_{q_0}(Y)$ is even. Then $\Theta \to E$ is a double cover.*

(a) *If $\lambda_{q_0}(Y) = \operatorname{ord}_{q_0}(\Delta_{Y/Z})$, then*

$$\sum_{q\in E} \lambda_q(\tilde{Y})[k(q) : k(q_0)] \leq \sum_{q\in E} \lambda_q(\Theta)[k(q) : k(q_0)] \leq \lambda_{q_0}(Y).$$

(b) *Otherwise,*

$$2\sum_{q\in E}[\lambda_q(\tilde{Y})/2][k(q) : k(q_0)] \leq 2\sum_{q\in E}[\lambda_q(\Theta)/2][k(q) : k(q_0)] \leq \lambda_{q_0}(Y).$$

(c) *In both cases we have*

$$\lambda_q(\tilde{Y})[k(q) : k(q_0)] \leq \lambda_q(\Theta)[k(q) : k(q_0)] \leq \lambda_{q_0}(Y)$$

for all $q \in E$.

Proof. We can suppose that $Z = \operatorname{Spec} A$ is local. Let $\mathfrak{m}$ be the maximal ideal of A, $k = A/\mathfrak{m}$, let $\lambda_{\mathfrak{m}}(B) = \lambda_{\mathfrak{m}}(y)$ and $r = [\lambda_{\mathfrak{m}}(B)/2]$.

(1) The cover $\phi : \tilde{Y} \to Z'$ has ramification index 2 at the generic point of Θ (Proposition 3.9(2)). Let s, t be a system of generators of $\mathfrak{m}$. We use the notation of Proposition 3.1. By Lemma 4.3(1), we can suppose that $a, b \in sA$. The equation of $\tilde{Y} \times_{Z'} D_+(\tilde{s})$ is

$$y_1^2 + a_1 y_1 + b_1 = 0, \quad a_1 = a/s^r, \ b_1 = b/s^{2r}$$

(Proposition 3.3). As $\operatorname{ord}_{\mathfrak{m}}(a) \geq r + 1$, we have $a_1 \in s\widetilde{A}_{(\tilde{s})}$. Write

$$b = \sum_{0\leq i\leq 2r+1} \beta_i t^i s^{2r+1-i} + \epsilon_{2r+2}, \quad \beta_i \in A, \ \epsilon_{2r+2} \in \mathfrak{m}^{2r+2}.$$

Then, if $t_1 = t/s$,

$$b_1/s = \sum_{0\le i\le 2r+1} \beta_i t_1^i + sc, \quad c \in \widetilde{A}_{(\tilde{s})}$$

Let $f(\bar{t}_1) = \sum_{0\le i\le 2r+1} \bar{\beta}_i \bar{t}_1^i \in k[\bar{t}_1]$ be the class of b_1/s mod s. By Lemma 4.5(1), for all $q \in E \cap D_+(\tilde{s})$, we have

$$1 \le \lambda_q(\tilde{Y}) \le \lambda_{E,q}(\tilde{Y}) = \mathrm{ord}_q(f(\bar{t}_1)).$$

Therefore

$$\sum_{q\in E\cap D_+(\tilde{s})} (\lambda_q(\tilde{Y}) - 1)[k(q) : k] \le \deg f(\bar{t}_1).$$

The only point missed is the rational point $\{s_1 = 0\}$ in $D_+(\tilde{t})$, where $s_1 = s/t$. The same reasoning as above shows that

$$\lambda_{s_1=0}(\tilde{Y}) \le \mathrm{ord}_{s_1=0}(\sum_{0\le i\le 2r+1} \bar{\beta}_i \bar{s}_1^{2r+1-i}) = 2r + 1 - \deg f(\bar{t}_1).$$

This implies the desired inequality on the sum running through E.

(2) The fact that $\Theta \to E$ is a double cover is Lemma 4.5(2). Let $\tilde{A} = \oplus_{n\ge 0}\mathfrak{m}^n$. Then

$$\tilde{A} \otimes_A k \simeq \mathrm{Gr}_{\mathfrak{m}}(A) \simeq k[s, t]$$

because A is regular. For any $f \in \mathfrak{m}^n$, if we consider f as a homogeneous element $\tilde{f} \in \tilde{A}_n$, then the image $\tilde{f}(q_0)$ of $\tilde{f}$ in $\tilde{A} \otimes_A k$ is 0 if $\mathrm{ord}_{\mathfrak{m}}(f) > n$, and is a homogeneous polynomial of degree n, equal to the initial form of f otherwise. By Remark 3.5, Θ is the closed subscheme of $\mathbb{P}(1, 1, r)$ defined by the equation

$$w^2 + \tilde{a}(q_0)w + \tilde{b}(q_0) = 0$$

where a, b are respectively considered as elements of $\mathfrak{m}^r$ and $\mathfrak{m}^{2r}$. This is a double cover of E of arithmetic genus $r - 1$. So (2) is an immediate consequence of Lemmas 4.2 and 4.5. □

Corollary 4.7. *Keep the assumption of Theorem 3.4. For any $i \ge 1$, denote by $E_i \subseteq Z_i$ the exceptional divisor of $Z_i \to Z_{i-1}$. Then for all $q \in E_i$, we have*

$$\lceil \lambda_q(Y_i)/2\rceil \le \lceil \lambda_{q_0}(Y)/2\rceil.$$

Proof. By induction on i and by using Theorem 4.6 (1) and (2.c). □

We refer to [18], §1, for the definition of rational singularities.

Corollary 4.8. *Keep the assumption of Theorem 4.6.*

(1) *If $\lambda_{p_0}(Y) = 2$, then p_0 is a rational singularity.*
(2) *If p_0 is a rational singularity, then $\lambda_{p_0}(Y) \le 3$.*

Proof. (1) By Proposition 3.3, the blowing-up $Y' \to Y$ at p_0 is normal, hence coincides with the normalized blowing-up $Y_1 \to Y$ at p_0. By Corollary 4.7, all singular points of Y_1 lying over p_0 also have multiplicity $\lambda = 2$. Repeating this reasoning, we see that any Lipman's desingularization sequence only consists in blowing-up singular points. Then [18], Proposition 23.5 implies that p_0 is a rational singularity.

(2) If $\lambda_{p_0}(Y) \ge 4$, then Y' is not normal by Proposition 3.3. By [18], Proposition 8.1, p_0 is not a rational singularity. □

Example 4.9 We give an example where the multiplicities $\lambda_{q_i}(Y_i)$ may increase in the sequence $(Y_i)_i$ of Theorem 3.4(2). It also shows that the inequality of Corollary 4.7 is sharp. Let $Z = \operatorname{Spec} k[s,t]$ over a field k. Let $n \geq 3$. Consider $Y \subset \operatorname{Spec} k[s,t,y]$ defined by

$$y^2 + s^n + t^3 = 0.$$

Let $q_0 = (0,0) \in Z$. Then $\lambda_{q_0}(Y) = 3$. We get a sequence of normalized blowing-ups (for $i+1 < n/3$)

$$\to Y_{i+1} \to Y_i \to \cdots \to Y_1 \to Y_0 = Y$$

where Y_i has an affine open subset defined by

$$y_i^2 + s^{n-3i+1} + st_i^3 = 0, \quad t_i = t/s^i, y_i = y/s^{3m+1},$$

if $i = 2m+1$, and by

$$y_i^2 + s^{n-3i} + t_i^3 = 0, \quad t_i = t/s^i, y_i = y/s^{3m},$$

if $i = 2m$. The center of $Y_{i+1} \to Y_i$ is $p_i = \{y_i = s = t_i = 0\}$ which has $\lambda_{p_i}(Y_i) = 4$ or 3 depending on whether i is odd or even. There is no other singular points on Y_i.

4.4. **The case when $\lambda_p(Y) = 2$.**

Corollary 4.10. *Keep the hypothesis and notation of Theorem 4.6. Suppose further that*

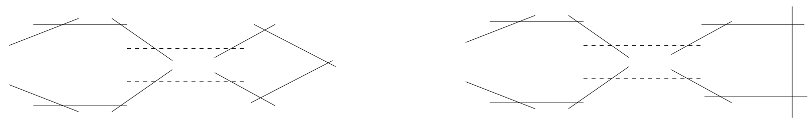
$$\lambda_{q_0}(Y) = 2.$$

Denote $k = k(q_0)$. Recall that $\Theta = \tilde{Y} \times_{Z'} E$.

(1) *The scheme Θ is a reduced projective conic over k. There are two cases:*
   (a) *either Θ is regular (hence integral). Then all points of Θ are regular in $\tilde{Y}$, and the singularity p_0 is solved;*
   (b) *or Θ has a unique singular point p_1. Then p_1 is rational over k. There are two subcases:*
      (i) *Θ is union of two copies of $\mathbb{P}^1_k$ intersecting transversely at p_1;*
      (ii) *Θ is integral with normalization isomorphic to $\mathbb{P}^1_{k'}$ for some quadratic extension k'/k.*

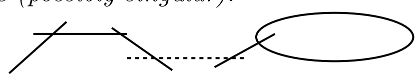
(2) *Suppose we are in Case (1.b) and that p_1 is singular in $\tilde{Y}$ (hence $\lambda_{p_1}(\tilde{Y}) = 2$). Let $Y_2 \to Y_1 := \tilde{Y}$ be the normalized blowing-up along p_1. Define similarly $\Theta_2 \subset Y_2$.*
   (a) *The strict transform Θ' of Θ in Y_2 is contained in the regular locus of Y_2. It is isomorphic to the disjoint union of two copies of $\mathbb{P}^1_k$ in Case (1.b.i), and to $\mathbb{P}^1_{k'}$ in Case (1.b.ii).*
   (b) *In Case (1.b.i), Θ' intersects Θ_2 transversely at two distinct k-rational points. If Θ_2 is irreducible, then it is isomorphic to $\mathbb{P}^1_k$.*
   (c) *In Case (1.b.ii), Θ_2 is integral and intersects transversely Θ' at one k'-rational point. The normalization of Θ_2 is isomorphic to $\mathbb{P}^1_{k'}$ if it is singular.*

(3) *The final resolution of singularity of p_0 has exceptional locus as follows:*
   (a) *In Case (1.b.i), all components are isomorphic to $\mathbb{P}^1_k$ and all intersection points are transverse and k-rational.*



(Read the resolution sequences from left to right.)

(b) *In Case (1.b.ii), all components but the last one are isomorphic to $\mathbb{P}^1_{k'}$, all intersection points are transverse and k'-rational. The last component is an irreducible conic (possibly singular).*



Proof. (1) In the proof of Theorem 4.6(2), we saw that one can apply Lemma 4.2 to $\Theta \to E$ with $r = [\lambda(p_0)/2] = 1$. Hence Θ is a reduced (Theorem 4.6(2)) conic over k. As E is a local complete intersection in Z' (they are both regular), Θ is a local complete intersection in $\tilde{Y}$. In particular, if Θ is regular, then $\tilde{Y}$ is regular at all points of Θ. This proves (1.a). The remaining part follows from the classification of reduced projective conics over k.

(2.a) Let $Y_1' \to Y_1$ be the blowing-up along p_1. By definition, the strict transform $\Sigma \subset Y_1'$ of Θ is isomorphic to the blowing-up of Θ along p_1. In Case (1.b.i), Σ is the disjoint union of two copies of $\mathbb{P}^1_k$, while in Case (1.b.ii), $\Sigma \to \Theta$ is the normalization map. In both cases Σ is normal.

Next, the normalization map $Y_2 \to Y_1'$ induces a finite birational morphism $\Theta' \to \Sigma$. Hence the latter is an isomorphism and Θ' is regular. Let us prove that Y_2 is regular at the points of Θ'. Let $Z_1 \to Z$ be the blowing-up along q_0 and let $Z_2 \to Z_1$ be the blowing-up along q_1, the image of p_1 in Z_1. Let $E' \subset Z_2$ be the strict transform of $E \subset Z_1$ by $Z_2 \to Z_1$. The closed immersion $\Theta' \to Y_2$ factors through $\Theta' \to Y_2 \times_{Z_2} E' \to Y_2$. So $\Theta' \to Y_2 \times_{Z_2} E'$ is birational and a closed immersion, hence an isomorphism. Then similarly to (1), we see that Y_2 is regular at the points of Θ'.

(2.b) As Θ' is contained in the regular locus of Y_2, we can apply Proposition 3.11(1) which shows the first part of the assertion. Applying (1) to Θ_2, we see that the latter is an integral conic over k, having a smooth rational point. So it is isomorphic to $\mathbb{P}^1_k$.

(2.c) As in (2.b), we can apply Proposition 3.11(2) which shows that Θ_2 is integral. Proposition 3.11(3) implies that Θ' and Θ_2 intersect each other transversely at a k'-rational point. Finally, if Θ_2 is singular, its normalization is $\mathbb{P}^1_{k''}$ over a quadratic extension k''/k. We have $k'' = k'$ because Θ_2 has a regular k'-point.

(3) Apply inductively (2). □

Remark 4.11 When $\lambda_{p_0}(Y) = 2$ as in Corollary 4.10, one can check that the invariant τ defined in [18], p. 259, is equal to 2 or 3. For such a singular point, Lipman proved that the exceptional locus of the desingularization is of type A_n or B_n in the notation of Dynkin diagrams, see [18], **Case I**, pages 259-260, and **Cases II a, II b**, page 262. The latter correspond respectively to our Cases (1.a), (3.a) and (3.b). Our proof is different from Lipman's. In [14], §2.2 we deal with some more general rational singularities.

5. Dualizing sheaves

Let S be an integral excellent noetherian scheme of dimension ≤ 1 and let Z be an integral regular scheme, flat of finite type over S. Let let $Y \to Z$ be a normal double such that the extension $K(Y)/K(Z)$ is separable. We study in this section the dualizing sheaves of the Y_i's in a Lipman's desingularization sequence.

We refer to [12], §6.4 for basic properties of the dualizing sheaves ω_f for some classes of morphisms f of schemes.

Lemma 5.1. *Consider the commutative diagram*

$$\begin{array}{ccc} Y_1 & \xrightarrow{c_1} & Y_0 = Y \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ Z_1 & \xrightarrow{\mathrm{bl}_{q_0}} & Z_0 = Z \end{array}$$

as in Theorem 3.4(2). Let E be the exceptional divisor of $Z_1 \to Z_0$ and $r = [\lambda_{p_0}(Y_0)/2]$. Then we have canonical isomorphisms

$$\omega_{Y_1/Z_1} \simeq \psi_1^*(\mathcal{O}_{Z_1}(-rE)) \otimes c_1^*\omega_{Y_0/Z_0} \tag{5.1}$$

and

$$\omega_{Y_1/S} \simeq \psi_1^*(\mathcal{O}_{Z_1}(-(r-1)E)) \otimes c_1^*\omega_{Y_0/S} \tag{5.2}$$

Proof. As $Y_i \to Z_i$ is a local complete intersection, ω_{Y_i/Z_i} is an invertible sheaf on Y_i. We have

$$\omega_{Y_1/Z_1} \otimes (c_1^*\omega_{Y_0/Z_0})^\vee = \mathcal{O}_{Y_1}(D)$$

for some Cartier divisor D supported in the exceptional locus of $Y_1 \to Y_0$. The determination of D is local on Z_0. So we can suppose that Z_0 is affine and Y_0 is defined by an equation $y^2 + ay + b = 0$ with $\lambda_{p_0}(Y_0) = \lambda_{p_0}(y)$. We then have

$$\omega_{Y_0/Z_0} = \frac{1}{2y+a}\mathcal{O}_{Z_0}$$

([12], Example 6.4.15). Consider the affine open subset U_1 of Y_1 defined by the equation

$$y_1^2 + a_1 y + b_1 = 0$$

with $y_1 = y/s^r, a_1 = a/s^r, b_1 = b/s^{2r}$. Then $U_1 = \psi_1^{-1}(V_1)$ where $V_1 = D_+(s) \subset Z_1$ (Proposition 3.3). We have

$$\omega_{U_1/V_1} = \frac{1}{2y_1 + a_1}\mathcal{O}_{U_1} = s^r \frac{1}{2y+a}\mathcal{O}_{U_1} = s^r c_1^*(\omega_{Y_0/Z_0})|_{U_1}.$$

This implies that $D = \psi_1^*(-rE)$ and the first isomorphism is established.

The morphisms $Z_0 \to S$ and $Y_0 \to S$ are flat and local complete intersections ([12] Example 6.3.18 and Proposition 6.3.20(b)). The adjunction formula $\omega_{Y_i/S} = \psi_0^*(\omega_{Z_i/S}) \otimes \omega_{Y_i/Z_i}$ ([12], Lemma 6.4.26(b)) and the canonical isomorphism $\omega_{Z_1/S} \simeq (\mathrm{bl}_{q_0}^*\omega_{Z_0/S})(E)$ ([12], Proposition 9.2.24) combined with the isomorphism (5.1) imply immediately the second isomorphism. □

Now consider a sequence as in Theorem 3.4(2)

$$\begin{tikzcd} Y_n \arrow[r, "c_n"] \arrow[d, "\psi_n"] & Y_{n-1} \arrow[r] \arrow[d, "\psi_{n-1}"] & \cdots \arrow[r] & Y_1 \arrow[r, "c_1"] \arrow[d, "\psi_1"] & Y_0 \arrow[d, "\psi_0"] \\ Z_n \arrow[r] & Z_{n-1} \arrow[r] & \cdots \arrow[r] & Z_1 \arrow[r] & Z_0. \end{tikzcd}$$

For all $1 \leq i \leq n$, denote by $p_{i-1} \in Y_{i-1}$ the center of the normalized blowing-up $Y_i \to Y_{i-1}$, $E_i \subset Z_i$ the exceptional divisor of $Z_i \to Z_{i-1}$ as well as its strict transform in any Z_j, $j \geq i$. Define inductively a sequence of non-negative integers $(m_i)_{1\leq i\leq n}$ as follows

(1) $m_1 = [\lambda_{p_0}(Y_0)/2] - 1$;
(2) for all $1 < i \leq n$,

$$m_i = ([\lambda_{p_{i-1}}(Y_{i-1})/2] - 1) + \sum_{0<j<i,\ E_j \ni q_{i-1}} m_j.$$

Remark 5.2 The index set in the above summation can be empty (if $p_{i-1} \in Y_0 \setminus \{p_0\}$), or contains one E_j, or contains two components (if q_{i-1} is an intersection point in the exceptional locus of $Z_{i-1} \to Z_0$).

Proposition 5.3. *Let $Y \to Z \to S$ be as at the beginning of this section and let $(Y_i)_i$ be a Lipman's desingularization sequence. Denote the composition*

$$\phi = \phi_n\colon\ Y_n \to Y_{n-1} \to \cdots \to Y_0$$

and

$$D_n = \sum_{1\leq i\leq n} m_i E_i.$$

(1) *We have two canonical isomorphisms*

$$\omega_{Y_n/S} \simeq \psi_n^*(\mathcal{O}_{Z_n}(-D_n)) \otimes_{\mathcal{O}_{Y_n}} \phi^*(\omega_{Y_0/S}),$$

$$\phi_*\omega_{Y_n/S} \simeq \phi_*(\psi_n^*(\mathcal{O}_{Z_n}(-D_n))) \otimes_{\mathcal{O}_{Y_0}} \omega_{Y_0/S}.$$

(2) *We have canonically*

$$H^0(Y_n, \omega_{Y_n/S}) \subseteq H^0(Y_{n-1}, \omega_{Y_{n-1}/S}) \subseteq \cdots \subseteq H^0(Y_0, \omega_{Y_0/S}).$$

Proof. (1) The first isomorphism is easily proved by induction on n. Indeed, the desired isomorphism is just Lemma 5.1 when $n = 1$. To pass from n to $n+1$, we just need to notice that the pull-back of $E_j \subseteq Z_n$ to Z_{n+1} is E_j if $p_n \notin E_j$, and is $E_j + E_n$ otherwise.

The second isomorphism is obtained by applying the projection formula to the first one, noting that $\omega_{Y_0/S}$ is invertible.

(2) It is enough to consider the case $n = 1$. As ψ_1 is flat, we have

$$\phi_*(\psi_1^*(\mathcal{O}_{Z_1}(-D_1))) \subseteq \phi_*\mathcal{O}_{Y_1} = \mathcal{O}_{Y_0},$$

the equality at the end holds because ϕ is proper and birational, and Y_0 is normal. □

Remark 5.4 Suppose that the generic fiber of $Y_0 \to S$ is smooth.

(1) The isomorphisms in the above proposition are equalities of subsheaves of the sheaf of rational differential forms $\Omega^1_{Y_0/S} \otimes_{\mathcal{O}_{Y_0}} K(Y_0)$.

(2) The global sections of $\omega_{Y_n/S}$ can be found as follows. For all $1 \leq i \leq n$, denote by η_i the generic point of E_i and $\mathfrak{m}_{\eta_i}$ the maximal ideal of $\mathcal{O}_{Z_n,\eta_i}$. Let $\omega \in H^0(Y_0, \omega_{Y_0/S})$. Then $\omega \in H^0(Y_n, \omega_{Y_n/S})$ if and only if for all singular points $p \in Y_0$ and for all η_i lying over $\psi_0(p) \in Z_0$, we have

$$\omega \in \mathfrak{m}_{\eta_i}^{m_i}(\omega_{Y_0/S})_p.$$

This follows from the Hartogs type lemma ([12], Lemma 9.2.17(a)) and the second isomorphism of Proposition 5.3(1). We will come back to this question when $Z_0 = \mathbb{P}^1_S$ in [14].

6. Desingularization algorithm

The aim of this section is to explain how to construct a Lipman's desingularization sequence concretely. We hope that this will be helpful for computer program implementation.

Starting with a double cover of a regular surface, in § 6.1 we explain how to find its normalization, then how to find the singular points in § 6.2 afterward. In § 6.3 we summary Lipman's desingularization process. Finally in § 6.4 we discuss a technique to simplify the computation of the order function in a regular local domain.

Let us fix the settings of this section. Let A be a noetherian excellent regular domain of dimension 2. We will assume A is factorial (*e.g.* local). This assumption implies that locally free A-algebras B of rank 2 are free (because the quotient A-module B/A is locally free of rank 1 by Lemma 2.2). Let B be a domain which is a free A-algebra of rank 2. We denote as before $Z = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$.

6.1. **Normalization.** We determine the integral closure B' of B when $\Delta_{B/A} \neq 0$. Write $B = A[y]$ with $y^2 + ay + b = 0$. Let t be a prime divisor of $a^2 - 4b$. Denote by ν_t the normalized valuation on $\operatorname{Frac}(A)$ induces by tA.

As A (hence B) is excellent, B' is finite over B. By Serre's criterion B' is Cohen-Macaulay, hence finite flat of rank 2 over A. Therefore B' has a basis $\{1, z\}$ over A. There exist $u, v \in A$ such that

$$y = uz + v. \tag{6.1}$$

In particular $B = A[y]$ is normal at primes lying over tA if and only if $t \nmid u$. Note that $u \mid a + 2v, b + av + v^2$, hence $u \mid b - v^2$.

Algorithm 6.1 Normalization algorithm. Let y and t be as above.

(1) Let $r = [(\min\{2\nu_t(a), \nu_t(b)\})/2]$. Let $y_1 = y/t^r$, $a_1 = a/t^r$, $b_1 = b/t^{2r}$. Then $a_1, b_1 \in A$ and
$$y_1^2 + a_1 y_1 + b_1 = 0$$
with $B \subseteq B_1 := A[y_1] \subset L := \operatorname{Frac}(B)$ and $\min\{2\nu_t(a_1), \nu_t(b_1)\} \leq 1$.

(2) Now replace y with y_1, we can assume $\min\{2\nu_t(a), \nu_t(b)\} \leq 1$. Then $A[y]$ is normal at primes above tA except possibly when $\nu_t(a) > 0$, $\nu_t(b) = 0$ and $\operatorname{char}(A/tA) = 2$ (Lemma 3.2).

(3) Assume that the above conditions hold. By the discussions following Equation (6.1), we see that $A[y]$ is not normal at some prime lying over tA if and only if there exists $c \in A$ such that $t \mid b - c^2$ (equivalently b is a square in A/tA).

(4) Suppose that such a $c \in A$ exists. Let $y_1 = y - c$. Then $A[y] = A[y_1]$ with

$$y_1^2 + (a + 2c)y_1 + (b + ac + c^2) = 0$$

and $\nu_t(a+2c), \nu_t(b+ac+c^2) > 0$. Go back to Step (1) with this new equation.

When the above algorithm terminates (this always happens eventually because the valuation $\nu_t(u)$ of the factor u in Equation (6.1) decreases strictly when we pass through Step (4) twice.), we go to another prime divisor of $a^2 - 4b$. At the end we get B' together with a basis $\{1, z\}$.

6.2. **Finding the singular points.** We are given a normal $B = A[y]$ with

$$y^2 + ay + b = 0.$$

6.2.1. *Singular points of Y.* Even though there are only finitely many singular points, there is no obvious way to find them for general A.

- If A (hence B) is a finite type algebra over some perfect field, we can use Jacobian criterion.

- Suppose that Z is smooth with integral fibers over some Dedekind domain R (*e.g.* $A = R[x]$) and that $B \otimes_A \mathrm{Frac}(R)$ is regular. We can look for singular points in the closed fibers of $Y \to \operatorname{Spec} R$ (which are curves over fields). The singular points of Y are among them.

However, if $B \otimes_R k(w)$ is non-reduced for some closed point $w \in \operatorname{Spec} R$, all points of the fiber Y_w are singular. The worst situation is when $\Delta_{B/A}(w) = 0$. Suppose R is local of dimension 1 with a uniformizing element π. Then $\pi \mid a$ and b is a square in $A/\pi A$. Let $b \equiv c^2 \mod \pi$. Replace y with $y + c$, then we get an equation with $\pi \mid a, b$. As B is normal, $\pi^2 \nmid b$. The singular points of Y are exactly those lying over the zeros of b/π in the curve $\operatorname{Spec}(A \otimes_R k(w))$.

6.2.2. *Singular points in the normalized blowing-ups.* Let $q_0 \in Z$ be such that $\lambda_{q_0}(Y) \geq 2$. Let $k = k(q_0)$.

Let $\tilde{Y} \to Y$ be the normalized blowing-up at the point p_0 lying over q_0 (see § 3). We search for singular points of $\tilde{Y}$ lying over the exceptional locus $E \subset Z' \xrightarrow{\mathrm{bl}_{\mathfrak{m}}} Z$. Let $a_1, b_1 \in \widetilde{A}_{(\tilde{s})}$ be as in Proposition 3.3. We first look for singular points lying over $D_+(\tilde{s})$. There are two cases.

(1) $s \nmid a_1^2 + 4b_1$ in $\widetilde{A}_{(\tilde{s})}$ (ramification index 1). Then any singular point of $\tilde{Y}$ lying over $E \cap D_+(\tilde{s})$ must correspond to a multiple factor $f(t_1)$ of the polynomial $\overline{a_1^2 + 4b_1} \in \widetilde{A}_{(\tilde{s})}/(s) \simeq k[t_1]$. Lift $f(t_1)$ to an $F \in \widetilde{A}_{(\tilde{s})}$. Then (s, F) is a maximal ideal. Compute $\lambda_{(s,F)}(\widetilde{A}_{(\tilde{s})}[y_1])$ to see whether we really get a singular point.

(2) $s \mid a_1^2 + 4b_1$. Then we have $s \mid a_1$ (we take $a = 0$ if $\mathrm{char}(k) \neq 2$).

(2.1) If $s \mid b_1$ then $s^2 \nmid b_1$. The singular points correspond exactly to the zeros of $\overline{s^{-1}b_1} \in \widetilde{A}_{(\tilde{s})}/(s) \simeq k[t_1]$.

(2.2) If $\mathrm{char}(k) = 2$, and $\bar{b}_1 \in \widetilde{A}_{(\tilde{s})}/(s)$ is a square. Translate y_1 by an element of $\widetilde{A}_{(\tilde{s})}$ to make $s \mid b_1$. Go back to (2.1).

(2.3) If $\mathrm{char}(k) = 2$, and $\bar{b}_1 \in \widetilde{A}_{(\tilde{s})}/(s)$ is not a square. A necessary condition for (s, F), $F \in \widetilde{A}_{(\tilde{s})}$, to define a singular point is that $\bar{F}(t_1) \in k[t_1]$ divides the derivative of $\bar{b}_1(t_1)$.

(3) It remains the points lying over $V_+(\tilde{s}) \cap E$. Let $t \in \mathfrak{m}$ be such that s, t generate $\mathfrak{m}A_{\mathfrak{m}}$. Then $V_+(\tilde{s}) \cap E = V_+(\tilde{s}) \cap D_+(\tilde{t}) \cap E$ is the point $s_1 = t = 0$ in $D_+(\tilde{t})$ (where $s_1 = s/t$). Check similarly whether it induces a singular point of $\tilde{Y}$.

6.3. **Desingularization algorithm.** Let Z be an integral noetherian excellent regular surface. Let $Y \to Z$ be a double cover. We work locally on Z and suppose $Z = \operatorname{Spec} A$ is affine.

(1) Find the normalization of Y (§ 6.1), then suppose Y is normal.

(2) Find the singular points of Y (§ 6.2).

(3) Fix a singular point lying over a closed point $q_0 \in Z$. Compute $\lambda_{q_0}(Y)$ using Lemma 2.10.

(4) Consider the normalized blowing-up $\tilde{Y} \to Y$ along the point p_0 lying over q_0. Use the affine equation of $\tilde{Y}$ given Proposition 3.3 to find the singular points of $\tilde{Y}$ lying over p_0.

(5) Repeat the above process for all singular points of Y. At the end we get a new normal double cover of an integral noetherian excellent regular surface.

(6) Go back to (2).

6.4. **Presentation of a regular ring and of its Rees algebras.** The most computationally complicated part of our algorithm should be the determination of $\operatorname{ord}_{\mathfrak{m}}$ and of the initial form of an element of A (Lemma 2.10(2.b)). For these purposes we need an as simple as possible presentation (by generators and relations) of A and of the rings appearing after blowing-up $\operatorname{Spec} A$ along maximal ideals.

We will restrict ourselves to the case when A is of finite type over some principal ideal domain R (*e.g.* $\mathbb{Z}$, $\mathbb{F}_p[T]$). Write

$$A = R[U_1, \dots, U_n]/I$$

for some $n \geq 2$. Let $\mathfrak{m} \subset A$ be a maximal ideal of height 2 generated by s, t. Then

$$\widetilde{A}_{(\tilde{s})} = A[T_1]/(sT_1 - t) = R[U_1, \dots, U_n, T_1]/(I, ST_1 - T)$$

where $S, T \in R[U_1, \dots, U_n]$ are liftings of s, t respectively. Thus in general we need to add a new variable and a new relation to represent the R-algebra $\tilde{A}_{(\tilde{s})}$. However in some particular cases we can find a small number of generators.

Lemma 6.2. *Let R be a PID, let $\pi \in R$ be an irreducible element and let $k = R/(\pi)$. Consider an R-algebra*

$$A = R[S, T]/(f - \pi)$$

with $f \in (S, T)A[S, T]$. Let $\mathfrak{m} \subset A$ be a maximal ideal of A such that $\mathfrak{m} \cap R = \pi R$ and $k = A/\mathfrak{m}$. Then the blowing-up of $\operatorname{Spec} A$ *along* $\mathfrak{m}$ *is covered by two regular affine schemes of the form*

$$\operatorname{Spec}\left(R[S_1, T_1]/(f_1 - \pi)\right)$$

with $f_1 \in (S_1, T_1)R[S_1, T_1]$.

Proof. We first reduce to the case $\mathfrak{m} = (S, T)A$. There exist $\alpha, \beta \in R$ such that $\mathfrak{m}$ is generated by the classes of $\pi, S - \alpha$ and $T - \beta$ in A. But $\pi \in (S, T)A$, so $f(\alpha, \beta) \in \pi R$. Hence $A = R[U, V]/(g - \pi)$ with $g(U, V) = f(U + \alpha, V + \beta) \in (U, V)R[U, V]$.

So assume that $\mathfrak{m}$ is generated by s, t, the respective classes of S, T in A. With the notation of Proposition 3.1, we have

$$\widetilde{A}_{(\tilde{s})} = R[S, T, T_1]/(f - \pi, ST_1 - T) = R[S, T_1]/(f_1 - \pi)$$

where $f_1 = f(S, ST_1) \in (S, T_1)R[S, T]$. Similarly

$$\tilde{A}_{(\tilde{t})} = R[S, T, S_1]/(f - \pi, TS_1 - S) = R[T, S_1]/(f_2 - \pi)$$

with $f_2(S_1, T) = f(S_1T, T) \in (S_1, T)R[S_1, T]$. □

Corollary 6.3. *Let R be a PID, let $\pi \in R$ be an irreducible element and let $k = R/(\pi)$. Then for any sequence of blowing-ups*

$$Z_N \to \cdots \to Z_1 \to \mathbb{A}^1_R$$

at k-rational singular points, each Z_i is covered by spectra of R-algebras of the form $R[S, T]/(f(S, T) - \pi)$ with $f(S, T) \in (S, T)R[S, T]$.

Proof. We have $R[x] = R[S, T]/(f - \pi)$ with $S = x$ and $f = T$. It is enough to apply Lemma 6.2 repeatedly. □

7. A complete example of resolution of singularities

We show with a concrete example of Lipman's desingularization sequence.

Fix an odd integer n. Consider the genus 2 curve C over $\mathbb{Q}$ defined by the affine equation

$$y^2 = (s^2 - 2)^3 + 16n^2$$

(note that the discriminant of the right-hand side polynomial is $2^{25}3^6n^4(1 - 2n) \neq 0$). We will show that the reduction mod 2 of the minimal regular model of C over $\mathbb{Z}$ has type $[2\mathrm{I}_0 - 3]$ (with the notation of [10], pp. 71-72).

Let $Z = \mathbb{P}^1_{\mathbb{Z}_{2\mathbb{Z}}}$ with coordinate function s, let $Y \to Z$ be the normalization of Z in $\mathbb{Q}(C)$. We will find a Lipman's desingularization sequence (Definition 3.6) to find a regular model, and then the minimal regular model of C over $\mathbb{Z}_{2\mathbb{Z}}$ by contracting the (-1)-curves. Replacing y with $y + s^3 + 4n$, we get

$$y^2 + 2(s^3 + 4n)y + 2(3s^4 + 4ns^3 - 6s^2 + 4) = 0. \tag{7.1}$$

By Algorithm 6.1(2), this equation defines an integrally closed domain, finite over $\mathbb{Z}_{2\mathbb{Z}}[s]$. Therefore it is an affine equation of Y. The closed fiber of Y is a double (multiplicity 2) projective line over $\mathbb{F}_2$. Denote it by $2\Theta_0$. Denote by $\boxed{t = 2}$, a uniformizing element of $\mathbb{Z}_{2\mathbb{Z}}$.

(I) Let $x = 1/s$. We first deal with the singular points of $Y \times_Z \mathrm{Spec}(\mathbb{Z}_{2\mathbb{Z}}[x])$. The latter is given by the equation

$$z^2 + 2(1 + 4nx^3)z + 2x^2(3 + 4nx - 6x^2 + 4x^4) = 0 \tag{7.2}$$

where $z = y/s^3$. Using § 6.2.1, we find a unique singular point $p_0 = \{z = t = x = 0\}$. We have $\boxed{\lambda_{p_0}(Y) = 2}$.

(I.1) *Blowing-up $Y_1 \to Y$ at p_0.* Let $Z_1 \to Z$ be the blowing-up at q_0, the image of p_0. Then $Y_1 \times_{Z_1} D_+(\tilde{x})$ is given by the equation

$$z_1^2 + t_1(1 + 4nx^3)z_1 + t_1x(3 + 4nx - 6x^2 + 4x^4) = 0 \tag{7.3}$$

with $t_1 = t/x, z_1 = z/x$. Using § 6.2.2, we find, in this affine piece, a unique singular point $p_1 = \{t_1 = x = z_1 = 0\}$ with multiplicity $\lambda_{p_1}(Y_1) = 2$.

The scheme $Y_1 \times_{Z_1} D_+(\tilde{t})$ is given, by the equation

$$w_1^2 + (1 + 4nt^3x_1^3)w_1 + tx_1^2(3 + 4ntx_1 - 6t^2x_1^2 + 4t^4x_1^4) = 0,$$

where $x_1 = x/t, w_1 = z/t$, so it is étale over $D_+(\tilde{t})$, hence regular.

Using the description of Z_1 by Proposition 3.1(2) and (5.a), we find that the exceptional locus Θ_1 (with the reduced structure) of $Y_1 \to Y$, equal to the pre-image of the exceptional divisor E_1 of $Z_1 \to Z$, is union of two projective lines $\Theta_{1,1}, \Theta_{1,2}$ over $\mathbb{F}_2$, of multiplicity 1, intersecting at p_1; and the strict transform of Θ_0 intersects Θ_1 at $t_1 = 0$ which is p_1.

(I.2) *The blowing-up $Y_2 \to Y_1$ along p_1.* We use Equation (7.3). Let $Z_2 \to Z_1$ be the blowing-up at $q_1 = (t_1 = x = 0) \in Z_1$. Then $Y_2 \times_{Z_2} D_+(\tilde{x})$ is given by

$$z_2^2 + t_2(1 + 4nx^3)z_2 + t_2(3 + 4nx - 6x^2 + 4x^4) = 0$$

with $t_2 = t_1/x, z_2 = z_1/x$. On the other hands, $Y_2 \times_{Z_2} D_+(\tilde{t}_1)$ is given by

$$w_2^2 + (1 + 4nt_1^3x_1^3)w_2 + x_1(3 + 4nt_1x_1 - 6t_1^2x_1^2 + 4t_1^4x_1^4) = 0$$

with $x_1 = x/t_1, w_2 = z_1/t_1$. We see that the exceptional locus Θ_2 of $Y_2 \to Y_1$ is the smooth conic of equation $w_2^2 + w_2 + x_1 = 0$ over $\mathbb{F}_2$, of multiplicity 2 (we have $t = x^2t_2$). Its intersection with the strict transform of Θ_0 is the point $t_2 = z_2 = 0$. The intersection with the strict transform of Θ_1 are the points $x_1 = 0, w_2 = 0, 1$.

The closed fiber of the commutative diagram

$$\begin{tikzcd} Y_2 \arrow[r] \arrow[d, "\psi_2"'] & Y_1 \arrow[r] \arrow[d, "\psi_1"'] & Y \arrow[d, "\psi"'] \\ Z_2 \arrow[r] & Z_1 \arrow[r] & Z \end{tikzcd}$$

is represented by Figure 1. For a given prime divisor, we use the same symbol for all its strict transforms. The component $E_i \subset Z_i$ is exceptional divisor of $Z_i \to Z_{i-1}$ for $i = 1, 2$.

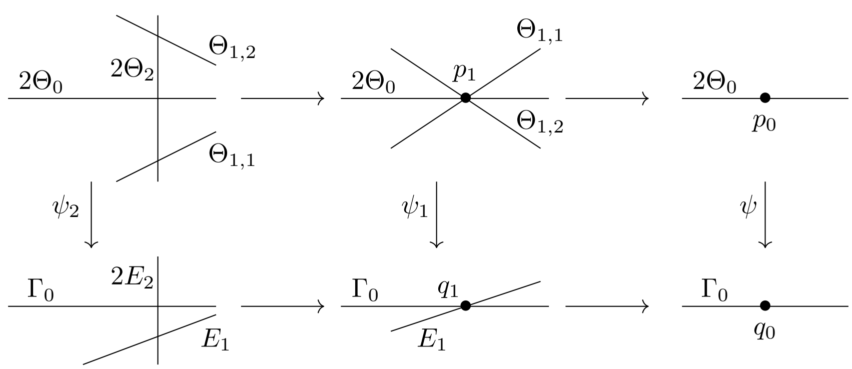


FIGURE 1. The closed fibers of Y, Y_1, Y_2 and Z, Z_1, Z_2.

(II) Now we desingularize $Y \times_Z \mathrm{Spec}(\mathbb{Z}_{2\mathbb{Z}}[s])$. We will use § 6.2.2 and Proposition 3.1(2) and (5.a) without mentioning it explicitly. The above scheme has a unique singular point p_2, lying over $q_2 = \{s = t = 0\}$ with $\boxed{\lambda_{p_2}(Y) = 3}$. As $Y_2 \to Y$ is an isomorphism over $Y \setminus \{p_0\}$, we can view p_2 as the unique singular point of Y_2, and q_2 as a point of Z_2 as well.

In what follows, for $i = 3, 4, 5$, we will show that the normalized blowing-up $Y_i \to Y_{i-1}$ at p_{i-1} has a unique singular point p_i. We will denote by $q_i \in Z_i$ the image of p_i and $Z_{i+1} \to Z_i$ the blowing-up at q_i.

(II.1) *Normalized blowing-up* $Y_3 \to Y_2$ *at* p_2. We use Equation (7.1). The equation of $Y_3 \times_{Z_3} D_+(\tilde{s})$ is

$$y_3^2 + st_1(s^2 + 2nt_1)y_3 + st_1(3s^2 + 4ns - 3st_1 + t_1^2) = 0 \tag{7.4}$$

where $t_1 = t/s$, $y_3 = y/s$. The point at infinity, $s/t = 0$ in $D_+(\tilde{t})$, is regular in Y_3. The exceptional locus Θ_3 is $\mathbb{P}^1_{\mathbb{F}_2}$ of multiplicity 2. It intersects the strict transform of Θ_0 at $p_3 = \{s = t_1 = y_3 = 0\}$, with $\boxed{\lambda_{p_3}(Y_3) = 4}$.

(II.2) *Normalized blowing-up* $Y_4 \to Y_3$ *at* p_3. We use Equation (7.4). Then the equation of $Y_4 \times_{Z_4} D_+(\tilde{s})$ is

$$y_4^2 + st_2(s + 2nt_2)y_4 + t_2(3 + 2nst_2 - 3t_2 + t_2^2) = 0 \tag{7.5}$$

where $t_2 = t_1/s$, $y_4 = y_3/s^2$. The point at infinity $s/t_1 = 0$, is regular in Y_4. The exceptional locus Θ_4 is a rational curve over $\mathbb{F}_2$ of multiplicity 2, with a cusp p_4 at $t_2 = 1$. The intersection of Θ_4 with the strict transform of Θ_0 is $t_2 = 0$, and with that of Θ_3 is $s_2 = 0$. The strict transforms of Θ_0 and of Θ_3 do not intersect each other.

The only possible singular point of Y_4 is p_4. Let us compute $\lambda_{p_4}(Y_4)$. We make the change of variables $t_2 = t_4 + 1$ and $y_4 = z_4 + 1$, then

$$z_4^2 + ns^3(1 + t_4)^3 z_4 + t_4^3 = 0. \tag{7.6}$$

So $\boxed{\lambda_{p_4}(Y_4) = 3}$. The strict transform of Θ_3 in Y_4 is a projective line over $\mathbb{F}_2$, with self-intersection -1. It is an exceptional divisor.

(II.3) *Normalized blowing-up* $Y_5 \to Y_4$ *at* p_4. We use Equation (7.6) with $p_4 = \{s = t_4 = z_4 = 0\}$. The equation of $Y_5 \times_{Z_5} D_+(\tilde{s})$ is

$$z_5^2 + ns^2(1 + st_5)^3 z_5 + t_5^3 s = 0, \tag{7.7}$$

with $t_5 = t_4/s, z_5 = z_4/s$. This affine piece has a unique singular point p_5, defined by $s = t_5 = z_5 = 0$, with multiplicity $\boxed{\lambda_{p_5}(Y_5) = 4}$. The affine piece $Y_5 \times_{Z_5} D_+(\tilde{t}_4)$ is regular. The exceptional locus Θ_5 is isomorphic to $\mathbb{P}^1_{\mathbb{F}_2}$ and has multiplicity 4.

(II.4) *Final normalized blowing-up* $Y_6 \to Y_5$ *at* p_5. We use Equation (7.7). The equation of $Y_6 \times_{Z_6} D_+(\tilde{s})$ is

$$z_6^2 + n(1 + s^2 t_6)^3 z_6 + t_6^3 = 0,$$

with $t_6 = t_5/s,\ z_6 = z_5/s^2$. We see immediately that the exceptional divisor Θ_6 is the elliptic curve over $\mathbb{F}_2$ defined by the equation

$$\Theta_6 : \quad z_6^2 + z_6 + t_6^3 = 0$$

and there is no more singular point in Y_6.

In summary, the closed fiber of the regular model Y_6 of C is represented by Figure 2, and that of Z_6 by Figure 3. All the irreducible components of $(Y_6)_{\mathbb{F}_2}$ are isomorphic to $\mathbb{P}^1_{\mathbb{F}_2}$ except Θ_6. The intersection points are rational over $\mathbb{F}_2$ and transverse. The self-intersection numbers of the irreducible components are given by the formula of [12], Proposition 9.1.21(b). In particular, $\Theta_3^2 = \Theta_5^2 = -1$.

After contracting the exceptional divisors in Y_6, we can see that we get the minimal regular model. The latter has type $[2\mathtt{I}_{0-2}]$ in [17], page 159 (denoted by $[2\mathtt{I}_0 - 3]$ in [10], pp. 71-72).

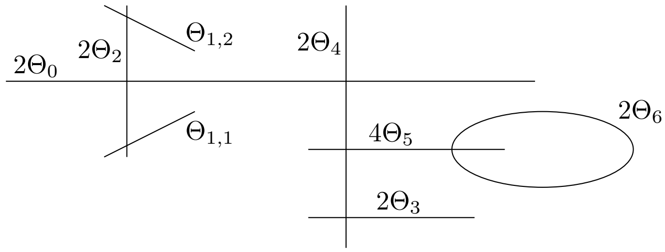


FIGURE 2. The closed fiber of Y_6.

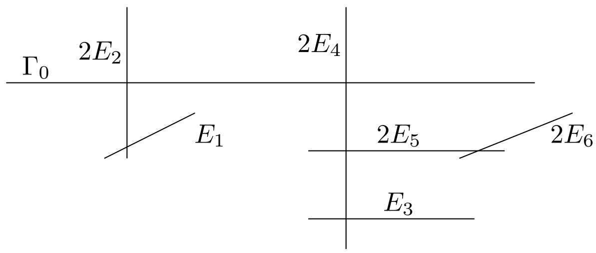


FIGURE 3. The closed fiber of Z_6.

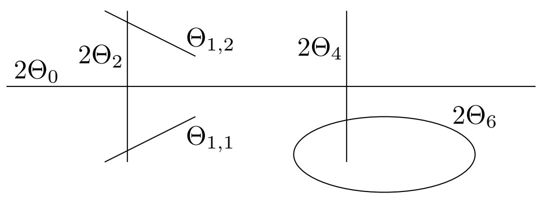


FIGURE 4. The closed fiber of the minimal regular model of C.

Remark 7.1 The genus 2 curve in the above example is a special case of that of [10], Exemple 1 page 74. But the resolution of singularities there is incorrect. Nevertheless, one can check that the integer d defined in *op. cit.*, Théorème 1, page 70, is equal to 5. It still satisfies the inequalities stated at the beginning of that example.

Competing interests: the author declares none.

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