Reduction and lifting of finite covers of curves

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November 14, 2003

1 Introduction

This note is based on a talk given in September 2003 at Chuo University during the COE Workshop "Cryptography and related mathematics". It is a non exhaustive survey on the reduction and lifting of finite covers of curves.

Let us fix a discrete valuation field K (e.g. $K = \mathbb{Q}_p$ or $\mathbb{F}_p((x))$), with valuation ν and residue field k. Let X_K be a smooth (i.e. non-singular) projective variety over K. There is a process of "**reduction**" which gives rise to a (not necessarily smooth) projective variety X_k over k. If $f_K : X_K \to Y_K$ is a finite morphism of smooth projective varieties, one can as well construct a finite morphism $f_k : X_k \to Y_k$ which is a "reduction" of f_K.

The inverse process is called the **lifting**: a projective variety Z over k is "*lifted*" to a (smooth projective) variety X_K over K if Z is isomorphic to a reduction X_k of X_K. One defines similarly liftings of finite morphisms $Z \to W$ of projective varieties over k to K.

Some motivations

- Starting with X_K, the reduction gives a variety defined over a smaller field (e.g. finite field), thus "simpler" to study.
- To understand the arithmetic properties of varieties defined over a global field, one needs to understand the reduction of the variety at all finite places. The construction of L-functions is a typical example.
- If $f_K : X_K \to Y_K$ is a finite cover, and if Y_K has known nice reduction and if f_K has nice properties, one expects to find nice properties for the reductions of X_K.

*I would like to thank the organizers of the workshop to give me the opportunity to give a talk, and Sylvain Maugeais, Michel Matignon for helpful discussions.

- Lifting a variety Z defined over a field of positive characteristic to a variety defined over a field of characteristic 0 allows to benefit from results known in characteristic 0 and transcendental methods. The same idea applies for the lifting of singular varieties to smooth varieties.
- Applications to the fundamental group of curves in positive characteristics; applications to the compactification of moduli spaces; study of the field of moduli of Galois covers ...

Convention. All varieties we consider over K are supposed to be smooth, projective and geometrically connected. Curves over the residue field k are not necessarily smooth, but assumed to be geometrically connected.

2 Reduction

Denote by $R := \{a \in K | \nu(a) \geq 0\}$ the valuation ring of K (e.g. $\mathbb{Z}_p$, $\mathbb{F}_p[[x]]$), and $p = \mathrm{char}(k)$ the characteristic of the residue field k of R ($k = \mathbb{F}_p$ in the above examples). In this section, it is not necessary to suppose R complete.

2.1 Basic properties

Let us start by the definition of reduction. To give a projective variety X_K is to give some homogeneous polynomials $F_1, ..., F_m \in K[T_0, ..., T_n]$ with coefficients in K. Different sets of polynomials can define the same variety.

Definition 2.1 A **model** X of X_K over R is a projective scheme over R (defined by homogeneous polynomials $G_1, ..., G_r \in R[S_0, ..., S_m]$) such that the projective variety over K associated to the G_i's (considered as polynomials with coefficients in K) is isomorphic to X_K. We require furthermore the technical condition that X is a "**flat**" scheme over R. This condition is satisfied if for instance X is integral (e.g. the ideal generated by the G_i's is a prime ideal).

Definition 2.2 Let $\overline{G}_i$ be the canonical image of G_i in $k[S_0, ..., S_m]$ (obtained by reducing the coefficients modulo the maximal ideal of R). Then the $\overline{G}_i$'s define a projective variety over k denoted by X_k. It is called a **reduction** of X_K. It depends on the choice of a model X. This is a projective variety over k. The flatness hypothesis on X insures that X_k has the *same dimension* as X_K.

Remark 2.3 If X_K is an arbitrary algebraic variety (= separated scheme of finite type) over K, one can define a model of X_K over R as a flat scheme X over R, with generic fiber isomorphic to X_K and such that X_k is non-empty. In particular, a projective variety X_K may have non-projective models. However we are only concerned with projective ones here.

Remark 2.4 Any variety X_K, except if $\dim X_K = 0$, has infinitely many models and infinitely many reductions. Therefore, we would like to find a "canonical model" when it is possible.

- If X_K is a smooth projective curve of genus $g \geq 1$, then X_K has a so called minimal regular model which is unique ([28], §9.3.3). The reduction defined by the minimal regular model is in general nasty: it may be reducible (i.e. have several irreducible components) and not reduced[1]. See Example 2.6 below.
- For abelian varieties over K, there are canonical smooth models (Néron models), but they are not projective in general. See [5].
- In higher dimension, there are no canonical regular models in general. If a certain type of singularities are allowed, there exists a minimal model for surfaces over K (cf. Mori's program on minimal models).

Definition 2.5 We say that X_K has **good reduction** if there exists a model X such that X_k is (projective and) smooth. Note that X_k is then unique if X_K is a curve or an abelian variety. If there exists a finite extension K'/K such that the curve $X_{K'} := X_K \times \mathrm{Spec}K'$ has good reduction, we will say that X_K has **potentially good reduction**.

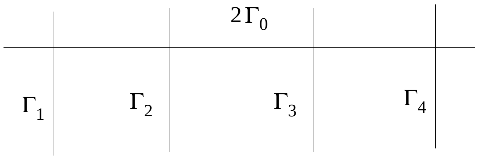
Example 2.6 Suppose that $p = \mathrm{char}(k) \neq 3$. Let X_K be the elliptic curve defined by

$$y^2 z = x^3 + az^3$$

with $1 \leq \nu(a) \leq 5$. The same equation defines a plane model P of X_K over R whose reduction is a singular curve:

$$y^2 z = x^3.$$

One can show that actually X_K can not have good reduction. It is done by looking at the minimal regular model X of X_K which would be smooth over R if X_K had good reduction. If $\nu(a) = 1$, then $X = P$ and X_k is singular. The type of X_k is respectively IV, I_0^*, IV^*, II^* if $\nu(a) = 2, 3, 4, 5$. See for instance [55] for the explanation of these (Kodaira) symbols. Type I_0^* means that X_k is union of five projective lines over k as follows:



the horizontal component is not reduced and has multiplicity 2 (i.e. the uniformizing element of R vanishes with order 2 at Γ_0). This reduction is rather complicated. However, if we enlarge the base field K to $K' = K[\alpha]$ by including a 6-th root α of a, then

$$(y/\alpha^3)^2 z = (x/\alpha^2)^3 + z^3$$

defines a model of $X_{K'}$ with good reduction. Hence X_K has potentially good reduction.

[1] The mathematical terminology is unfortunately poor in this context: we *reduce* a variety and we can get something which is still *reducible* and not *reduced*...

2.2 Finite covers and stable reduction

In Example 2.6 above, we see that the reduction is better understood if we allow extensions of K. We can ask whether every curve has potentially good reduction. The answer is definitely no (see Example 2.8). The general situation is described by the stable reduction theorem below.

A (geometrically connected) projective curve C over a field k is said to be **semi-stable** if over $\bar{k}$ it is reduced and has only ordinary double points (nodes) as singularities. It is **stable** if furthermore $C_{\bar{k}}$ is connected, with arithmetical genus $p_a(C) \geq 2$, and if any irreducible component of $C_{\bar{k}}$ isomorphic to $\mathbb{P}^1$ meets the other components in at least 3 points.

Theorem 2.7 (Deligne-Mumford, [13]) *Let X_K be a (geometrically connected) projective smooth curve over K. Then there exists a finite separable extension K'/K such that $X_{K'}$ has semi-stable reduction (resp. stable reduction if $g(X_K) \geq 2$).*

Alternative proofs of this very important theorem were given later by Artin-Winters [3], Bosch-Lütkebohmert [6], van der Put [44], and T. Saito [51]. Note that if $g(X_K) \geq 2$, then $X_{K'}$ has semi-stable reduction if and only if $X_{K'}$ has stable reduction.

Example 2.8 If $g(X_K) = 0$, then X_K has good reduction isomorphic to $\mathbb{P}^1$ (after a quadratic extension if necessary). If X_K is an elliptic curve, after an extension of degree dividing 24, it has either good reduction (if $\nu(j(X_K)) \geq 0$) which is again an elliptic curve, or the reduction of the minimal regular model is semi-stable and equal to the union of $n = -\nu(j(X_K))$ projective lines.

Remark 2.9 Suppose that K is strictly henselian (e.g. complete with separably closed residue field k) and, $g(X_K) \geq 2$ or $X_K(K) \neq \emptyset$. Then there exists an extension L/K such that X_L has semi-stable reduction and which is contained in any extension K'/K realizing semi-stable reduction for X_K. This extension is Galois. If $g(X_K) = 0$ (hence $X_K \simeq \mathbb{P}^1_K$ by our hypothesis), then $L = K$. If X_K is an elliptic curve, then L/K is at most quadratic when $\nu(j(X_K)) < 0$. When $\nu(j(X_K)) \geq 0$, let E be the good reduction of X_L, extended to $\bar{k}$. Then $\mathrm{Gal}(L/K)$ is isomorphic to a subgroup of $\mathrm{Aut}_k(E)$. In the remainder case where $g(X_K) \geq 2$, we denote by C the stable reduction of X_L extended to $\bar{k}$, then $\mathrm{Gal}(L/K)$ is isomorphic to a subgroup of $\mathrm{Aut}_k(C)$. A similar result for abelian variety is proved in [14], 5.15, and the case of curves can be deduced from the case of abelian varieties [29], Prop. 4.

Remark 2.10 Let $f_K : X_K \to Y_K$ be a finite morphism of smooth projective curves. After enlarging K if necessary, f_K induces a finite morphism of semi-stable reductions $f_k : X_k \to Y_k$. If f_K is Galois of group G, then we can choose respective semi-stable models X, Y of X_K, Y_K in such a way that $X \to Y$ is also Galois of group G. Note that $X_k/G \to Y_k$ is in general not an isomorphism. It is purely inseparable on each irreducible component of X_k/G.

If $g(Y_K) \geq 2$, then f_K induces a (not necessarily finite) morphism between the stable reductions of X_K and Y_K ([30], Cor. 4.7).

2.11 Tame covers. The above remark says that if a semi-stable reduction of X_K is known, then in some sense one can recover a semi-stable reduction of Y_K. The converse is much harder. Let $f_K : X_K \to Y_K$ be a Galois cover with Galois group G. Suppose that a semi-stable reduction of Y_K is known (e.g. if $Y_K = \mathbb{P}^1_K$), we can ask whether a semi-stable reduction of X_K can be computed. If $|G|$ is prime to p (including the case $p = 0$), the problem is rather well understood.

Theorem 2.12 (Grothendieck) *Let $X_K \to Y_K$ be an étale Galois cover of order prime to p. If Y_K has good reduction, then X_K has potentially good reduction.*

The proof is easy: let Y be a smooth model of Y_K. Let X be the integral closure of Y in the function field $K(X_K)$. By Abhyankar's lemma (here the hypothesis $|G|$ prime to p is used), after (tamely ramified) finite extension of R, X_k is reduced. Since $X_k \to Y_k$ is generically étale (again because $|G|$ is prime to p), $X \to Y$ is ramified at finitely many closed points. By purity of ramification locus (Y is regular of dimension 2), $X \to Y$ is étale. In particular X is smooth. Following the same principle, one has (the first part of) the next theorem.

Theorem 2.13 (Saïdi, [47], 4.3) *Let $X_K \to Y_K$ be an étale cyclic cover of order prime to p. Let Y be a semi-stable model of Y_K. After (tamely ramified) finite extension of R, the integral closure X of Y in $K(X_K)$ is semi-stable and $X_k \to Y_k$ is étale outside the singular locus of Y_k. Moreover, the ramification over the singular locus is described explicitly in terms of currents on the intersection graph of Y_k.*

Theorem 2.14 (Liu-Lorenzini, [30], 2.3) *Let $f_K : X_K \to Y_K$ be a (not necessarily étale) Galois cover of order prime to p. Let us suppose that Y_K has semi-stable reduction and that the branch locus B_K of f_K is contained in $Y_K(K)$ (this can be done after finite separable extension of K). Let Y be a semi-stable model of Y_K such that the Zariski closure B of B_K in Y is contained in the smooth locus of Y and étale over R (this is alway true after blowing up Y if necessary). Then after (tamely ramified) finite extension of R, the integral closure X of Y in $K(X_K)$ is semi-stable, and $X \to Y$ is étale outside $B \cup (Y_k)_{\mathrm{sing}}$.*

As an easy consequence of the study of the tame fundamental group ([20], Theorem 2.3.2) or directly by [30] 2.1(b), we have a generalization of 2.12:

Proposition 2.15 (Fulton, [16], 3.3, 3.4) *Let $X_K \to Y_K$ be a Galois cover of order prime to p. Suppose that Y_K has good reduction and there exists a smooth model Y of Y_K such that B is étale over R, then X_K has potentially good reduction.*

Remark 2.16 Conversely, if X_K has good reduction and if $g(X_K) \geq 1$, then there exists a smooth model Y of Y_K such that B is étale over R ([46], 2.3.4 or [30], 3.5(a)). When $B_K \subset Y_K(K)$, the condition B being étale over R means that the points of B_K specialize to pairwise distinct points of Y_k.

Corollary 2.17 *Let $C \to \mathbb{P}^1_F$ be a Galois cover of degree d, defined over a number field F, and ramified only above $0, 1, \infty$. Then C has potentially good reduction at every finite place of K prime to d.*

2.18 Wild covers. When $p \mid |G|$, the situation is completely different and is much more complicated. Even the very simple case of hyperelliptic curves with $p = 2$ is not known in general except when $g(X_K) = 2$ ([29], Théorème 1). We notice that the branch locus no longer determines the stable reduction of X_K. Next is an analog of 2.15.

Theorem 2.19 (Raynaud, [45], Théorème 1') *Let $X_K \to Y_K$ be a Galois cover of order a power of p. Suppose that Y_K has a smooth model Y such that the Zariski closure B of the branch locus B_K in Y is étale over R (B might be empty), and that $X_K \to Y_K$ is tamely ramified (i.e. the inertia groups at points of X_K are of order prime to* $\mathrm{char}(K)$*). Then the Jacobian of X_K has potentially good reduction.*

Remark 2.20 One can not weaken the hypothesis on Y_K to, for instance, Jacobian of Y_K having potentially good reduction. Actually, it is easy to construct cyclic (of any order d) étale covers $X \to Y$ such that Y_k is the union of two ordinary elliptic curves meeting each other at one point, and such that X_k is the union of two elliptic curves meeting each other at d points. See [45], Remark at page 189.

Remark 2.21 In [31], Matignon describes an effective algorithm to finding the stable reduction of p-cyclic covers $X_K \to \mathbb{P}^1_K$ whose branch locus satisfies the conditions of 2.19. An earlier tentative algorithm without condition on the branch locus was proposed by Coleman [9], §6. Stable reduction of quotients of Fermat curves (they are wild Galois covers of $\mathbb{P}^1$ ramified at $\{0, 1, \infty\}$) are explicitly computed in [11] with interesting arithmetic applications [11], [12].

Let us finish this section by indicating some other recent works on the stable reduction of Galois covers $X_K \to Y_K$ in the wild case: [46] with extra assumption on G and the absolute ramification index of K; [7], [8], [57] when $Y_K = \mathbb{P}^1_K$ with special configuration of the branch locus and p divides exactly $|G|$; [24], [48] for p-cyclic covers.

3 Lifting projective varieties

From now on, and unless expressly mentioned, we will suppose that R is a *complete* discrete valuation ring. A uniformizing element of R will be denoted by π. Recall that $K = \mathrm{Frac}(R)$ and k is the residue field.

Let Z be a projective variety over k. We want to lift Z to a smooth projective variety X_K over K, i.e. to find a projective scheme X over R with smooth generic fiber and such that $X_k \simeq Z$. When such a X exits, we will say for short that Z can be lifted to K.

One can try to lift naively Z in the following manner: write Z as a closed subscheme of some projective space $\mathbb{P}^n_k$ defined by homogeneous polynomials $\bar{F}_1(T_0, ..., T_n), ..., \bar{F}_m(T_0, ..., T_n)$, lift the $\bar{F}_i$'s to homogeneous polynomials $F_1, ..., F_m$ with coefficients in R such that $\deg F_i = \deg \bar{F}_i$, and take $X = \operatorname{Proj} R[T_0, ..., T_n]/(F_1, ..., F_m)$. Sometimes this scheme is a good candidate for lifting Z.

Proposition 3.1 *Let Z be a complete intersection in a projective space $\mathbb{P}^n_k$, then Z can be lifted to K. (The field K needs not be complete in this proposition.)*

Proof. Keep the above notation. By hypothesis, Z is defined by m homogeneous polynomials in $n+1$ variables, where $m = n - \dim Z$. It is easy to see that the scheme X defined as above is then flat over R. It remains to show that we can choose the liftings F_i so that X_K is smooth.

We can suppose $m > 0$. Let $d_i = \deg F_i$. Consider $P := \mathbb{P}^{r_1}_K \times ... \times \mathbb{P}^{r_m}_K$, where $r_i = \dim_K H^0(\mathbb{P}^n_K, \mathcal{O}(d_i))$ is the dimension of the K-vector space of the homogeneous polynomials of degree d_i. To any rational point $x \in P(K)$ one can associate obviously m homogeneous polynomials $H_1, ..., H_m$ and a projective subvariety $\varphi(x) := V_+(H_1, ..., H_m)$ of $\mathbb{P}^n_K$. In the same manner we can define naturally a projective scheme $C = C(n, d_1, ..., d_m) \subset \mathbb{P}^n \times P$ over P such that for any rational point $x \in P(K)$, $C_x = \varphi(x)$. The generic fiber of $C \to P$ is a smooth variety of dimension $n - m$ over $K(P)$ (use Bertini's theorem). Thus there exists a dense open subset U of P such that $C \times_P U \to U$ is smooth. (Using Jacobian criterion and resultants, one can actually show that the set of $x \in P$ such that C_x is smooth of dimension $n - m$ is open in P.)

Let us go back to our $F_i's$. Write $F_i = \sum_\nu a_{i\nu} T^\nu$ with $a_{i\nu} \in R$. The $a_{i\nu}'s$ (when ν varies) correspond to a rational point of $\mathbb{P}^{r_i}_K$. Since K is infinite, $U(K)$ contains a point of the form $(a_{i\nu} + \pi b_{i\nu})_{i,\nu}$, with $b_{i\nu} \in R$. This point defines a smooth lifting of Z over K.

Remark 3.2 Note that in general (when Z is not complete intersection), the naive lifting X may not be flat over R. In fact, X_K may have smaller dimension than $X_k \simeq Z$.

Theorem 3.3 (Grothendieck, [SGA1], III.7.3) *Let Z be a smooth projective variety over k. If*

$$H^2(Z, \Omega^\vee_{Z/k}) = H^2(Z, \mathcal{O}_Z) = 0,$$

then Z can be lifted to K.

More precisely, the vanishing of the first cohomology group insures that Z is the special fiber of some projective formal scheme $\mathfrak{X}$ over R. The vanishing of the second one implies that $\mathfrak{X}$ is algebraizable.

Note that when Z is smooth, the lifting problem is trivial if R has equal characteristics, as we can take $X = Z \times_{\operatorname{Spec} k} \operatorname{Spec} R$.

Corollary 3.4 *Any projective smooth curve over k can be lifted to K.*

Actually it is enough to suppose K henselian in this corollary.

Proposition 3.5 *Let Z be a geometrically reduced and locally complete intersection curve over k (e.g. if Z is semi-stable). Then Z can be lifted to a smooth curve over K.*

Proof: Use formal patching (Lemma 5.6(a)) or show that $\mathrm{Ext}^2(\Omega_{Z/k}, \mathcal{O}_Z) = 0$ and use theory of deformations of locally complete intersections ([27] or the more accessible [56], 4.4).

Remark 3.6 There exists a *curve Z over k with no lifting* to any discrete valuation field K with residue field k. Let Z be a projective curve over k containing $\mathrm{Spec}\, k[u,v]/(uv, v^2)$ as an affine open subscheme. Suppose that there exists a flat projective scheme X over R such that X_K is integral and that $X_k \simeq Z$. Let us show that this leads to a contradiction.

By flatness, X is integral. There exists an affine open subscheme U of X with special fiber isomorphic to $\mathrm{Spec}\, k[u,v]/(uv, v^2)$. Denote by x_0 the point $u = v = 0$. Then U is smooth at all points of $U_k \setminus \{x_0\}$. Let $V \to U$ be the normalization of U. Then $V_k \to U_k$ is an isomorphism outside x_0. Since V is normal, hence (S_2), V_k is (S_1). Therefore V_k is reduced. The map $V_k \to U_k$ then factorizes into a finite birational map $V_k \to (U_k)_{\mathrm{red}} \simeq \mathbb{A}^1_k$ (which is then an isomorphism) followed by the closed immersion $(U_k)_{\mathrm{red}} \to U_k$. Since V is smooth at x_0, $\hat{\mathcal{O}}_{V,x_0} \simeq R[[t]]$ for any $t \in \mathcal{O}_{V,x_0}$ which reduces to a parameter of V_k at x_0. We then deduce immediately that $\hat{\mathcal{O}}_{U,x_0} \to \hat{\mathcal{O}}_{V,x_0}$ is surjective, and hence an isomorphism. This is impossible, because U_k is not smooth (even not reduced).

Remark 3.7 There exists a projective smooth surface over k with no lifting to characteristic 0 (Mumford).

Remark 3.8 (*Abelian varieties*) Let Z be an abelian variety over k and suppose that $p = \mathrm{char}(k) > 0$. We can ask whether Z can be lifted to an abelian variety over K of characteristic 0. The answer is that it is always possible after possibly a finite extension of K. See [40] for a survey on this subject. Note that the canonical lifting of abelian varieties (Serre-Tate) gained interest in cryptography recently.

Remark 3.9 (*Embedding into regular surfaces*) Let Z be a connected projective curve over k algebraically closed. One can ask whether Z is the special fiber of a regular relative curve X over R. There are two obvious necessary conditions. The first one is local: $\dim T_{Z,z} \leq 2$ for the tangent space at any point $z \in Z$. It implies that Z is locally a divisor in a regular curve over R. The second one is global and related to the intersection theory on Z. When $\mathrm{char}(k) = 0$, G. Winters [60] proved that these conditions are sufficient to embed Z into a regular curve over R. The general case is open.

4 Lifting Galois covers

4.1 Smooth case

Let $f_k : X_k \to Y_k$ be a Galois cover of projective smooth curves over k, with Galois group G. Suppose R complet, k perfect, $\mathrm{char}(k) = p > 0$, and $\mathrm{char}(K) = 0$ (otherwise lifting problem is trivial). A lifting of f_k will be by definition a Galois cover $X \to Y$ of smooth curves over R with group G and which special fiber is the given cover f_k. Several surveys are available in the literature on this topic ([32], [41]...).

Theorem 4.1 (Grothendieck, [SGA1], I.8.3) *If $f_k : X_k \to Y_k$ is étale, then there exists a lifting $X \to Y$ to R. The lifting is unique if Y is fixed.*

More generally (Grothendieck): if $f_k : X_k \to Y_k$ is tamely ramified (i.e. the inertia groups at all $x \in X_k$ are of order prime to p), then there exists a lifting to R. This lifting is unique if we fix a lifting of Y_k and a lifting of the branch locus of f_k. If no condition on the ramification is required, there exists a "birational lifting":

Theorem 4.2 (Garuti, [17]) *There exists a Galois cover $f'_k : X'_k \to Y_k$ of group G such that X'_k has only cuspidal singularities, with normalization equal to X_k, and such that, after finite extension of K, f'_k lifts to a Galois cover $X'_K \to Y_K$ of smooth curves over K:*

$$\begin{tikzcd} & X' \arrow[r, "G"] & Y \\ X_k \arrow[r, "\text{normalization}"] & X'_k \arrow[r, "G"] \arrow[u] & Y_k \arrow[u] \end{tikzcd}$$

Remark 4.3 In general, one can not expect to lift every Galois cover from characteristic $p > 0$ to characteristic 0. A simple reason is that for a given genus $g \geq 2$, a curve X_k of genus g in characteristic p may have a bigger automorphism group G than in characteristic 0, hence $X_k \to X_k/G$ can not be lifted to characteristic 0. Nakajima [38] proved that abelian subgroups of the automorphism group of a smooth projective curve of genus $g \geq 2$ have the same bound in any characteristic ($= 4g + 4$). However, there are abelian covers with no lifting to characteristic 0 (see e.g. [18], 5.3 with $G = (\mathbb{Z}/p\mathbb{Z})^2$, $p > 2$). So the bound on the order of G is not the real obstruction to the lifting problem. As showed in [4], [18], the obstruction is of local nature on X_k (deformation of finite groups of automorphisms of power series). Pagot ([43], or [34]) proves that the lifting problem always has a solution if $G = (\mathbb{Z}/2\mathbb{Z})^2$.

4.4 Cyclic covers. Now we consider cyclic covers $X_k \to Y_k$ with group G, and suppose that k is *algebraically closed.*

Theorem 4.5 (Sekiguchi-Oort-Suwa, [53]) *If p divides exactly $|G|$, then $f_k : X_k \to Y_k$ can be lifted to a Galois cover over $W(k)[\xi_p]$, where ξ_p is a primitive p-th root of unit.*

Green and Matignon established a local-global principle and, using an explicit form in degree p^2 of Sekiguchi-Suwa's theory on deformation of Artin-Schreier-Witt covers to Kummer covers [54], they proved:

Theorem 4.6 (Green-Matignon, [18]) *If p^e, $e = 1, 2$, divides exactly $|G|$, then $X_k \to Y_k$ can be lifted to a Galois cover over $W(k)[\xi_{p^e}]$.*

Conjecture 4.7 (Oort, [42], 4A?) Any cyclic cover of smooth projective curves over a field k (not necessarily algebraically closed) has a lifting to characteristic 0.

Here "lifting to characteristic 0" means there exists an integral domain A of characteristic 0, a homomorphism $A \to k$, and a Galois cover $X \to Y$ of smooth proper curves over A which induces the Galois cover $X_k \to Y_k$ by the base change $\operatorname{Spec} k \to \operatorname{Spec} A$. This implies that there exists a discrete valuation ring R' with residue field k', finite extension of k, such that $X_{k'} \to Y_{k'}$ lifts to R'. When k is algebraically closed, a more optimistic version of this conjecture can be found in [33], p. 94, with $A = W(k)[\xi_{p^e}]$ if $|G| = p^e r$, $(p, r) = 1$.

Remark 4.8 (*Minimal extensions*) Let $X_k \to Y_k$ be a *ramified* cyclic cover of degree d divisible by p. It would be interesting to find the minimal extensions R_{m} of $W(k)$ (for the inclusion relation) such that $X_k \to Y_k$ can be lifted to R_{m}. Let us consider the simplest situation where p divides exactly d. By [53], $R_{\mathrm{m}} = W(k)$ if $p = 2$. Using the structure of the versal deformation ring R_σ of $(\operatorname{Spec} k[[T]], G)$ given in [4], 4.2.8, we have :

- If $p > 2$ and if the Hasse conductor m at every ramification point equals to 1, then R_{m} is the sub-extension $W(k)[\xi_p + \xi_p^{-1}]$ of $W(k)[\xi_p]$ of index 2. In particular, $R_{\mathrm{m}} = W(k)$ if $p = 3$ (op. cit. top of page 217).
- If $p = 5$ and one of the conductors is equal to 2, then $R_{\mathrm{m}} = W(k)[\xi_p]$.

If $p > 3$, it is known that $R_{\mathrm{m}} \neq W(k)$ ([10], Proposition 5; [39], Corollary, page 94). Coleman states ([10], Remark 2, page 247) that $[R_{\mathrm{m}} : W(k)] \geq (p-1)/2$. This has an interesting arithmetic application (loc.cit.):

Let K be a discrete valuation field of mixed characteristics $(0, p)$, let $X_K \to Y_K$ be a ramified p-cyclic cover. If Y_K has good reduction (e.g. $Y_K = \mathbb{P}^1_K$) and if the absolute ramification index of K is $< (p-1)/2$, then X_K has bad reduction.

Note that it is easy to construct plenty of p-cyclic covers of $\mathbb{P}^1_K$ for any field K (see [52]). Suppose now that the conductor of $X_k \to Y_k$ at some point is equal to 2. Using [19], III.3.1, one can show easily the following:

- If $p \not\equiv 1 \bmod 3$, then $R_{\mathrm{m}} = W(k)[\xi_p]$;
- If $p \equiv 1 \bmod 3$ and R_{m} is a minimal extension contained in $W(k)[\xi_p]$, then $R_{\mathrm{m}} = W(k)[\xi_p]$;

- If $p \equiv 1 \bmod 3$ and R_{m} is not contained in $W(k)[\xi_p]$, then $[R_{\mathrm{m}} : W(k)]$ is a multiple of $2(p-1)/3$. However, we have no example of lifting with such a R_{m}.

4.9 Deformation theory. Lifting a Galois cover $X_k \to Y_k$ of group G is equivalent to lifting the curve X_k endowed with the action of G. Instead of looking for the lifting of (X_k, G) for each individual couple X_k and G, Bertin and Mézard [4] consider the space of all possible liftings of (X_k, G). They give a global-local principle ([4], 3.3.4 and 3.3.5) which is more precise then that of Green-Matignon. They also study the structure of the versal deformation ring of $(\operatorname{Spec} k[[T]], G)$ when G is p-cyclic (op. cit., 4.2.8). The study of such deformation rings is extended to semi-stable curves (Maugeais, [36]) and to finite group schemes (instead of constant groups G) in Wewers [59].

4.10 Lifting torsors. Let G be a finite flat group scheme over R. Let $X_k \to Y_k$ be a G_k-torsor over Y_k. Andreatta and Gasbarri [2] solve the lifting problem of $X_k \to Y_k$ to a G-torsor $X \to Y$, where Y is some smooth projective curve lifting Y_k. Note that X_k is not smooth (even not reduced) if G_k is not étale. When R is ramified enough, they prove (op. cit., Theorem 6.1) that the lifting is possible for a class of finite group schemes $\{\mathcal{G}_{\lambda,n}\}$ of order p^n. This class contains the finite group schemes of order p.

4.2 Case of semi-stable curves over k

Let $f_K : X_K \to Y_K$ be a Galois cover of smooth projective curves over K. Since X_K does not have necessarily good reduction, to understand what happens to the reductions of f_K, it is natural to consider finite (Galois) cover of semi-stable curves over k.

Note that in the singular case, the lifting to equal characteristics R is not at all trivial contrarily to the case of smooth curves.

Lemma 4.11 *Let $f : X \to Y$ be a finite morphism of semi-stable curves over R with smooth generic fibers. Let $y \in Y_k$ be a singular point and let $x \in f^{-1}(y)$. Then x is singular in X_k. Suppose that x is rational.*

(a) *Write $\hat{\mathcal{O}}_{X_k,x} \simeq k[[s,t]]/(st)$, $\hat{\mathcal{O}}_{Y_k,y} \simeq k[[u,v]]/(uv)$ with $u \in s\hat{\mathcal{O}}_{X_k,x}$, then $(u) = s^n\hat{\mathcal{O}}_{X_k,x}, (v) = t^n\hat{\mathcal{O}}_{X_k,x}$ for some $n \geq 1$.*

(b) *Assume that $X \to Y$ is Galois of group G, generically étale on X_k, and suppose that the inertia group I_x has order prime to p. Then I_x is cyclic of order n, and $\sigma(u) = \xi_n u$, $\sigma(v) = \xi_n^{-1} v$ for a generator σ of I_x and a primitive n-th root ξ_n of unit.*

Proof: (a) The fact that x is singular is contained in the proof of [45], Prop. 5. The second condition comes from the fact that $X \setminus \{x\} \to Y \setminus \{y\}$ is finite free of some degree n in a neighborhood of x (because $Y \setminus \{y\}$ is regular of dimension 2 in a neighborhood of y).

(b) The special fiber of Spec $\hat{\mathcal{O}}_{X,x}$ has two smooth irreducible components Γ_1, Γ_2. Since y is singular, I_x leaves each Γ_i stable and it is also the inertia group of $\Gamma_1 \to \Gamma_1/I_x$ at x, hence I_x is cyclic. To prove the second condition, one can replace $X \to X/G$ by $X \to X/I_x$ as $X/I_x \to X/G$ is étale in a neighborhood of the image of x in X/I_x. By [30], 3.6, $X \to Y$ is étale cyclic in a neighborhood of x except at x. Then Part (b) is contained in [47], 5.7.

4.12 Suppose that k is separably closed. Let us start again with tamely ramified covers. Recall that a finite morphism $f_k : X_k \to Y_k$ of semi-stable curves is a **tame admissible** cover if the following conditions are satisfied.

(1) Let X'_k and Y'_k be the respective normalizations of X_k, Y_k. The morphism $X'_k \to Y'_k$ induced by f_k is tamely ramified.

(2) Let $y \in Y_k$ be a singular point. Then every $x \in f_k^{-1}(y)$ is singular. We furthermore require that the conclusion of Lemma 4.11(a) holds.

Note that the integer n in 4.11(a) will be prime to char(k) by the first condition. On the other hand, we can see that above the smooth part of Y_k, f_k is locally free of constant degree. Over an arbitrary field k, f_k is tame admissible if $f_{k^{\mathrm{sep}}}$ is tame admissible.

A cover $X_k \to Y_k$ is called a **tame admissible Galois cover** of group G if G acts on X_k, freely on a dense open subset, such that $X_k \to Y_k$ can be identified with the quotient map $X_k \to X_k/G$, is a tame admissible cover, and satisfies the conclusion of 4.11(b) above singular points. A tame admissible Galois cover is also called *Kummerian cover* ([50], 2.1). The action of G is then said to be *balanced* in [1], 2.1.3.

Remark 4.13 Note that the usual definition of tame admissible (Galois) covers requires that f_k maps the singular points of X_k to singular points of Y_k. This means exactly that the singular points of X_k are isolated in the ramification locus of $X \to Y$, or equivalently, that the Zariski closure in X of the ramification locus of $X_K \to Y_K$ is contained in the smooth part of $X \to \operatorname{Spec} R$. This is natural when we study finite covers of marked semi-stable curves. But this condition is not necessary for the lifting purpose (see 5.4). Our definition is more natural in this context because every finite (Galois) cover of semi-stable curves as in 4.11 induces a tame admissible (Galois) cover if char(k) is zero or bigger than the degree of $X_K \to Y_K$.

The next theorem is well-known and is due to various authors (see references in [58] and [50], 3.7):

Theorem 4.14 *Let k be separably closed. Let $f_k : X_k \to Y_k$ be a tame admissible (Galois) cover (of group G). Then f_k can be lifted to a (Galois) cover $X_K \to Y_K$ of smooth projective curves (of group G) over K.*

See also Proposition 5.4. Note that the same statement holds for tame admissible covers over an arbitrary field k ([58], 3.1.2). Theorem 4.14 is useful for the study of Hurwitz moduli schemes (see e.g. [23], [37], [1]).

4.15 On the **wild ramification** side, there are some recent works on the p-cyclic case using ideas of Raynaud. Suppose k algebraically closed. Let $X_K \to Y_K$ be a p-cyclic cover with group G and suppose that X_K has a semi-stable model X over R endowed with an action of G. Let $Y = X/G$ and let $f : X \to Y$ be the canonical morphism. Then $X_k \to Y_k$ is a finite morphism generically flat of degree p. Let Γ be an irreducible component of Y_k, then either $f^{-1}(\Gamma) \to \Gamma$ is generically étale, or $f^{-1}(\Gamma)$ is irreducible and purely inseparable over Γ. In fact, some open dense subset U of $f^{-1}(\Gamma)$ comes with a structure of torsor under a group scheme G_Γ of order p (hence isomorphic to $\mathbb{Z}/p\mathbb{Z}$, α_p or μ_p over k) over $f(U)$. The structure of these torsors induce some numerical data on the intersection graph of X_k satisfying some compatibility conditions ([24], [26], [48], §2.2). Such a graph is called an admissible Hurwitz graph.

Theorem 4.16 ([48], [26]) *Conversely, let $X_k \to Y_k$ be a finite surjective morphism of semi-stable curves over k, generically flat of degree p, endowed with a structure of admissible Hurwitz graph. If* $\mathrm{char}(K) = 0$*, then after finite extension of K, $X_k \to Y_k$ can be lifted to a p-cyclic cover of smooth projective curves $X_K \to Y_K$.*

When Y_k has genus 0, then such an admissible Hurwitz graph exists (after modifying X_k and Y_k if necessary). By establishing this property, Maugeais was able to show:

Theorem 4.17 (Maugeais, [35], 5.4, 5.6) *Let X_k be a semi-stable curve over k endowed with a faithful action of $G \simeq \mathbb{Z}/p\mathbb{Z}$. Suppose that $p_a(X_k/G) = 0$. Then after a finite extension of R, the pair (X_k, G) can be lifted to (X, G) over R with X_K smooth.*

This theorem is first proved in equal characteristic case, and the mixed characteristics case is obtained with Theorem 4.5.

Corollary 4.18 (Maugeais, [35]) *Let X_k be a stable curve over k. Then X_k is the stable reduction of some smooth hyperelliptic curve over (a finite extension of) K if and only if there exists an involution τ on X_k such that $X_k/\langle\tau\rangle$ has arithmetical genus* 0 *and, either* $\mathrm{char}(k) = 2$*, or* $\mathrm{char}(k) \neq 2$ *and $X_k \to X_k/\langle\tau\rangle$ is tame admissible.*

4.19 Let $X_k \to Y_k$ be a generically étale Galois cover such that for all $x \in X_k$, the inertia group I_x is cyclic and has order not divisible by p^3. Then, under some mild conditions on the inertia groups of singular points of X_k, the cover $X_k \to Y_k$ can be lifted to a Galois cover after finite extension of K (Henrio, [25], 3.7).

5 Lifting finite covers of curves

Let R be a complete discrete valuation ring with residue field k. Let $f_k : X_k \to Y_k$ be a finite, surjective morphism of projective curves over k. We ask whether

f_k can be lifted to finite morphism $f : X \to Y$ of projective curves over R with smooth generic fibers. We do not consider Galois actions in this section.

Proposition 5.1 (Inseparable morphisms) *Let $f_k : X_k \to Y_k$ be a finite inseparable morphism of projective smooth curves over k, with $g(X_k) \geq 2$.*

(a) *The morphism $X_k \to Y_k$ can not be lifted to a Galois cover over K.*

(b) *If $X_k \to Y_k$ is purely inseparable, then every lifting of f_k is purely inseparable. In particular, f_k can not be lifted to characteristic 0.*

(c) *There are inseparable $X_k \to Y_k$ which can be lifted to separable $X_K \to Y_K$.*

Proof: (a) comes from the injectivity of $\mathrm{Aut}_K(X_K) \to \mathrm{Aut}_k(X_k)$ ([13], 1.11, or [28], 10.3.38). (b) We have $g(X_K) = g(X_k) = g(Y_k) = g(Y_K)$. So any lifting f_K is purely inseparable. (c) Let E_K be an elliptic curve with good reduction at p, let $X_K \to E_K$ be a double cover such that X_K has good reduction (we can take X_K of genus 2). After enlarging K, we can suppose that E_K contains a p-torsion point $P \in E_K(K)$ which specializes to 0 in the reduction of E_K. Let $E_K \to Y_K := E_K/H$ be the quotient by the subgroup H of $E_K(K)$ generated by P. Then $X_K \to Y_K$ is separable, and its reduction $X_k \to Y_k$ factorizes through $E_k \to Y_k$, and is therefore inseparable.

From now on, we will only consider generically étale morphisms.

Proposition 5.2 *Let $f_k : X_k \to Y_k$ be a finite, surjective, and generically étale morphism of projective curves over k. Let $Y \to \mathrm{Spec}\, R$ be a lifting of Y_k (when it exists, see 3.5).*

(a) *If $X_k \to Y_k$ is flat locally complete intersection (e.g. if X_k and Y_k are normal), then $X_k \to Y_k$ can be lifted to $X \to Y$.*

(b) *If Y_k is smooth, and if $\hat{\mathcal{O}}_{X_k,x}$ can be lifted to a flat local complete R-algebra for all closed $x \in X_k$ (e.g. if X_k is smooth), then f_k can be lifted to $X \to Y$.*

Proof: (a) Let A be an R-algebra (will be equal to $\hat{\mathcal{O}}_{Y,y}$). It is easy to see that a finite flat complete intersection over $A \otimes k$ lifts to a finite flat complete intersection over A with smooth generic fiber (see e.g. proof of 3.1). One applies 5.6(b) to conclude. See also [SGA1], III.5.6 when X_k and Y_k are smooth. It should be possible to prove (a) using deformation theory as in 3.5.

(b) Let $x \in X_k$ and $y = f_k(x)$. Let $\hat{\mathcal{O}}_x$ be a lifting of $\hat{\mathcal{O}}_{X_k,x}$ to R. Since Y is smooth, $\hat{\mathcal{O}}_{Y_k,y} \to \hat{\mathcal{O}}_{X_k,x}$ can be lifted to some $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_x$ which is necessarily finite because it is finite when tensored by k, and $\hat{\mathcal{O}}_x$ is complete. As in (a), we can now use 5.6(b).

5.3 Suppose now that X_k and Y_k are semi-stable. Then $X_k \to Y_k$ is in general not flat nor l.c.i at singular points of X_k. So Proposition 5.2 can not be applied to lift f_k. Let us call a finite surjective generically étale morphism $X_k \to Y_k$ **admissible** if every point $x \in X_k$ lying over a singular point $y \in X_k$ is also singular and if the conclusion of 4.11(a) is satisfied.

Proposition 5.4 *Suppose k is algebraically closed. Let $f_k : X_k \to Y_k$ be a finite surjective generically étale morphism of semi-stable curves over k. Then f_k can be lifted to $X \to Y$ if and only if it is admissible.*

Proof: The "only if" part is just 4.11(a). Suppose that $X_k \to Y_k$ is admissible. For every singular $y \in Y_k$, let e_y be an integer multiple of the ramification index n_x (this is the integer n of 4.11(a)) of $X_k \to Y_k$ at x, for all $x \in f_k^{-1}(y)$. Let $Y \to \operatorname{Spec} R$ be a lifting of Y_k such that for every singular $y \in Y_k$, $\hat{\mathcal{O}}_{Y,y} \simeq R[[U,V]]/(UV-b)$, $\nu(b) = e_y$. Such a lifting always exists (see e.g. [25], 2.6).

By Lemma 5.6(b) below, it is enough to show that $\hat{f}_x : \hat{\mathcal{O}}_{Y_k,y} \to \hat{\mathcal{O}}_{X_k,x}$ can be lifted to $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_x$ over R for all $x \in X_k$ and $y = f_k(x)$. If y is smooth, then $\hat{f}_x$ can be lifted as we saw in the proof of 5.2(b). So suppose that y is singular. Let us identify $\hat{f}_x$ with $\rho : k[[u,v]]/(uv) \to k[[s,t]]/(st)$, let $n = n_x$ and write

$$\rho(u) = s^n(1 + s\bar{P}(s)), \quad \rho(v) = t^n(1 + t\bar{Q}(t))$$

where $P \in R[[S]], Q \in R[[T]]$, and $^-$ means reduction modulo π. This is possible by changing s,t if necessary, because k is algebraically closed.

Let $\hat{\mathcal{O}}_x := R[[S,T]]/(ST-a)$ where $a \in R$ is such that $c := ba^{-n} \in R^*$ and $\overline{c} = 1$. Let $G(T) \in R[[T]]$ be such that

$$1 + TG(T) = (1 + TQ(T))^{-1}.$$

Put $\rho(U) = S^n(1 + SP(S) + TG(T))$ and $\rho(V) = cT^n(1 + SP(S) + TG(T))^{-1}$. Then

$$\rho(U)\rho(V) = b, \quad \overline{\rho(U)} = \rho(u), \quad \overline{\rho(V)} = \rho(v).$$

because

$$t^n(1 + s\bar{P}(s) + t\bar{G}(t))^{-1} = t^n(1 + t\bar{G}(t))^{-1} = t^n(1 + t\bar{Q}(t)).$$

Therefore we found a lifting of ρ to $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_x$.

5.5 Formal patching. To attack the lifting problem, there are mainly two approaches: one is the (global) infinitesimal deformation theory as in [SGA1], III.5, [27], [56]; the second is the local formal patching method (including rigid analytic method).

Lemma 5.6 (Formal patching) *Let R be a complete noetherian local ring with residue field k. Let $f_k : X_k \to Y_k$ be a finite surjective and generically étale morphism of projective curves.*

(a) *Suppose that Y_k is geometrically reduced. So there exists a dense open subset V of Y_k, smooth over k. Suppose moreover that for all $y \in Y_k \setminus V$, there exists a complete noetherian R-algebra $\hat{\mathcal{O}}_y$, flat over R, such that $\hat{\mathcal{O}}_y \otimes_R k \simeq \hat{\mathcal{O}}_{Y_k,y}$. Then there exists a projective curve $Y \to \operatorname{Spec} R$ lifting Y_k such that $\hat{\mathcal{O}}_{Y,y} \simeq \hat{\mathcal{O}}_y$ for all $y \in Y_k \setminus V$.*

(b) *Let* $Y \to \operatorname{Spec} R$ *be a lifting of* Y_k. *Let* V *be a dense open subset of* Y_k *such that* $U := f_k^{-1}(V) \to V$ *is étale. Suppose that for any* $x \in X_k \setminus U$, *there exists a finite* $\hat{\mathcal{O}}_{Y,y}$*-algebra* $\hat{\mathcal{O}}_x$ *(where* $y = f_k(x)$*) which lifts the finite* $\hat{\mathcal{O}}_{Y_k,y}$*-algebra* $\hat{\mathcal{O}}_{X_k,x}$. *Then there exists a unique lifting* $X \to Y$ *over* R *such that* $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_x$ *for all* $x \in X_k \setminus U$.

Proof: The lemma follows easily from the formal patching method (Ferrand-Raynaud [15], Harbater [21], [22]). Let us however give some details here.

(a) We will use [35], 4.7. Let $\mathcal{V} \to \operatorname{Spec} R$ be a smooth formal scheme lifting V ([SGA1], III.6.3). Let $y \in Y_k \setminus V$, let ξ be a generic point of

$$\operatorname{Spec}(\hat{\mathcal{O}}_y \otimes_R k) = \operatorname{Spec}(\hat{\mathcal{O}}_{Y_k,y})$$

(it corresponds to an "analytic branch" of Y_k at y) and let η be its canonical image in Y_k. Since $\mathcal{V}$ is formally smooth, the canonical homomorphism $\mathcal{O}_{V,\eta} \to (\hat{\mathcal{O}}_{Y_k,y})\widehat{_\xi} = (\hat{\mathcal{O}}_y)\widehat{_\xi} \otimes k$ lifts to a homomorphism $\mathcal{O}_{\mathcal{V},\eta} \to (\hat{\mathcal{O}}_y)\widehat{_\xi}$. In other words, we lift the canonical morphism $\operatorname{Spec}(\hat{\mathcal{O}}_y)\widehat{_\xi} \otimes k \to V$ to $\operatorname{Spec}(\hat{\mathcal{O}}_y)\widehat{_\xi} \to \mathcal{V}$. Now one can apply [35], 4.7 to conclude.

(b) A proof can be found in [49], Proposition 1.6, see also [25], 2.8 and an equivariant version in [35], 4.7. Let $\mathcal{Y}$ be the completion of Y along Y_k, let $\mathcal{V}$ be the open formal subscheme of $\mathcal{Y}$ with underlying space equal to V. The generic étaleness implies the uniqueness of the lifting $\mathcal{U} \to \mathcal{V}$ of $U \to V$, and allows to "patch" $\mathcal{U}$ and the $\operatorname{Spec} \hat{\mathcal{O}}_x$'s along the generic points of $\operatorname{Spec} \hat{\mathcal{O}}_{X_k,x}$.

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