HOMEWORK: AMENABILITY

A general tool: barycenter map on compact convex spaces. This exercise aims to give some details about the construction and properties of the barycenter map. Let $\mathcal{C}$ be a compact convex space in a locally convex topological vector space E. By definition an affine function on $\mathcal{C}$ is the restriction to $\mathcal{C}$ of a continuous affine map $f : E \to \mathbb{R}$. We denote by $\mathrm{Aff}(\mathcal{C})$ the set of all such affine functions on $\mathcal{C}$.

(1) We first prove the existence of the map $\mathrm{Bar} : \mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$ such that $f(\mathrm{Bar}(\mu)) = \int_{\mathcal{C}} f(x)\,\mathrm{d}\mu(x)$, for all $f \in \mathrm{Aff}(\mathcal{C})$, $\mu \in \mathrm{Prob}(\mathcal{C})$. Fix $\mu \in \mathrm{Prob}(\mathcal{C})$.
  (a) Fix finitely many continuous affine functions $f_1, \ldots, f_n$ on E and define the affine map
  $$\alpha : x \in E \mapsto (f_1(x), \ldots, f_n(x)) \in \mathbb{R}^n.$$
  Prove that the point $p := (\int_X f_1\,\mathrm{d}\mu, \cdots, \int_X f_n\,\mathrm{d}\mu)$ belongs to the convex set $\alpha(\mathcal{C}) \subset \mathbb{R}^n$. *Hint.* Argue by contradiction and use a separation theorem in $\mathbb{R}^n$.
  (b) For each $f \in \mathrm{Aff}(\mathcal{C})$, consider the affine hyperplane $H_f := \{y \in E \mid f(y) = \int_{\mathcal{C}} f d\mu\}$. Deduce from the previous question that the sets $H_f \cap X$ have the finite intersection property: any finite intersection of such sets is non-empty.
  (c) Conclude that $\mathrm{Bar}(\mu)$ exists. Prove that it is unique.

(2) Prove that the barycenter map is continuous and affine.

(3) Let $g : \mathcal{C} \to \mathcal{C}$ be a continuous affine map. Of course g induces a continuous affine map $\mathrm{Prob}(\mathcal{C}) \to \mathrm{Prob}(\mathcal{C})$ (simply given by the push-forwad of measures). Prove that the barycenter map is equivariant for these actions, i.e. $\mathrm{Bar}(g_*\mu) = g(\mathrm{Bar}(\mu))$, for all $\mu \in \mathrm{Prob}(\mathcal{C})$.

Invariant means on discrete groups. A *mean* on a discrete group Γ is a finitely additive measure $m : \mathcal{P}(\Gamma) \to [0, 1]$ (as opposed to σ-additive) such that $m(\Gamma) = 1$. Just as for measures, we can define the notion of a left *invariant mean* on Γ to be a mean m such that $m(gA) = m(A)$ for every $g \in \Gamma$, $A \subset \Gamma$. We aim to prove that Γ is amenable if and only if it has an invariant mean.

(1) Check that a positive linear functional $\phi : \ell^\infty(\Gamma) \to \mathbb{R}$ such that $\phi(1) = 1$ always gives rise to a mean m on Γ, in such a way that $m(A) = \phi(\mathbf{1}_A)$ for all $A \subset \Gamma$. Prove that every mean on Γ arises in this way (from a unique ϕ). *Hint.* First define ϕ on the linear span E of $\{\mathbf{1}_A \mid A \subset \Gamma\}$, in order to obtain a positive linear functional ϕ_0 on E. Then check that, being positive, ϕ_0 is automatically continuous on E and use an extension theorem.

(2) Deduce that if Γ is amenable, then it has a left invariant mean.

(3) Conversely, assume that Γ has a left invariant mean m, and consider a continuous action $\Gamma \curvearrowright X$ on a compact space X. Denote by ϕ the linear functional associated with m as in the first question. Fix $x \in X$, and consider the orbit map $\theta : g \in \Gamma \mapsto gx \in X$. Prove that the flowing map is a well defined positive linear functional on $C(X)$, which is Γ-invariant:
$$f \in C(X) \mapsto \phi(f \circ \theta) \in \mathbb{R}.$$
Conclude that Γ is amenable.

In the last question, what we did was to define the push forward of the invariant mean on Γ under the orbit map θ. The functional analytic perspective allows to see that this push forward is in fact a σ-additive Borel probability measure on X (which is Γ-invariant).

Free groups. Denote by $F(a, b)$ the free group on two letters a, b. As a set, this is the set of all reduced words in the letters a, a^{-1}, b, b^{-1}, where reduced means that the word does not contain a succession aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$. Note that any word in these letters can be reduced to reach a (unique) reduced form. The product law of $F(a, b)$ is then given by concatenation and reduction. The trivial element is the empty word.

Given a letter $x \in \{a, a^{-1}, b, b^{-1}\}$, denote by F_x the subset of all elements whose reduced form starts with the letter x.

(1) Check that $F(a,b) = \{e\} \sqcup F_a \sqcup F_{a^{-1}} \sqcup F_b \sqcup F_{b^{-1}}$ and also that
$$F(a,b) = aF_{a^{-1}} \cup F_a.$$
(2) Deduce from the previous exercise that $F(a,b)$ is non-amenable.

Closed subgroups. We aim to prove that a closed subgroup of an amenable group is amenable. For simplicity we restrict to discrete groups, but the ideas are the same for the general case. Let Γ be a discrete group and $\Lambda < \Gamma$ be a subgroup.

- Prove that we may choose a set $I \subset \Gamma$ such that $\Lambda I = \Gamma$ and $\Lambda x \cap \Lambda y = \emptyset$ for all $x, y \in I$, $x \neq y$.
- Assuming that Γ has a left invariant mean m, then the map $A \in \mathcal{P}(\Lambda) \mapsto m(AI)$ is a left invariant mean on Λ. Conclude.
- Prove that a discrete group Γ containing a free group is non-amenable.

This last fact is not true if Γ is non-discrete (unless the free subgroup is closed): there exists free groups in SO(3).