LIE GROUPS AND THEIR LATTICES
- GRADED HOMEWORK -

Due date: **Friday, March 27**

1. Invariant measures, nilpotent lcsc groups and lattices

Exercise 1.1. Finite Haar measure.

Since a Haar measure is a Radon measure, so it is finite on compact subsets of G. So if G is compact, its Haar measure is finite.

Assume now that G is not compact. Then for any compact set $C \subset G$, the set CC^{-1} is still compact. So we may find $g \in G$, such that $g \notin CC^{-1}$. This implies that $gC \cap C$ is empty (otherwise we would find an elements $a, b \in C$ such that $a = gb$, which would imply $g = ab^{-1} \in CC^{-1}$). This implies that $\mu_G(G) \geq \mu_G(C \cup gC) = 2\mu_G(C)$, for any compact subset C of G. Since μ_G is a regular measure, this further implies that $\mu(G) \geq 2\mu(A)$ for any subset A of G. With $A = G$, this implies that $\mu(G) = +\infty$ (we recall that a Haar measure is non-zero).

Exercise 1.2. Invariant measures on homogeneous spaces.

1. Fix $f \in C_c(G/H)$. The map $\tilde{\theta}(f) : g \in \mathbb{G} \mapsto \int_{F/H} f(gx) \, \mathrm{d}\mu_{F/H}(x)$ is well defined, right-F-invariant because the measure $\mu_{F/H}$ is left F-invariant. It is also continuous by a standard argument of continuity of integrals with parameters (based on Lebesgue convergence theorem). So it indeed factorizes to a continuous map on G/F, the map $\theta(f)$. Let us check that $\theta(f) : G/F \to \mathbb{R}$ is compactly supported.

   Denote by $K \subset G/H$ the (compact) support of f. If $g \in G$ is such that $gx \notin K$ for all $x \in F/H$ then $\theta(f)(gF) = 0$. So $\theta(f)$ vanishes on the set $\{gF \in G/F \mid gF \cap K = \emptyset\} = p(K)^c$, where $p : G/H \to G/F$ is the projection map. So $\theta(f)$ is supported on $p(K)$ and since p is continuous, $p(K)$ is compact, as desired.
2. We may define a positive linear functional $C_c(G/H) \to \mathbb{R}$, by the formula
$$\phi : f \in C_c(G/H) \mapsto \int_{G/F} \theta(f) \, \mathrm{d}\mu_F.$$
   By Riesz representation theorem, this linear functional defines a Radon measure ν on G/H. One may observe that θ is G-equivariant with respect to the left action $G \curvearrowright G/H$ and $G \curvearrowright G/F$, both denoted by σ. This means that $\theta(f \circ \sigma_g) = \theta(f) \circ \sigma_g$ for all $f \in C_c(G/H)$, $g \in G$. Therefore, since μ_F is G-invariant, we find that for all $f \in C_c(G/H)$ and all $g \in G$:
$$\phi(f \circ \sigma_g) = \int_{G/F} \theta(f \circ \sigma_g) \, \mathrm{d}\mu_F = \int_{G/F} \theta(f) \circ \sigma_g \, \mathrm{d}\mu_F = \int_{G/F} \theta(f) \, \mathrm{d}(g_*\mu_F) = \phi(f).$$
   Therefore ν is G-invariant and must be proportional to μ_H, by uniqueness.
3. The push forward of μ_H under the projection map $G/H \to G/F$ is a G-invariant finite Borel measure on G/F. So μ_F exists and is finite.

   Moreover since H is normal and closed in F, F/H is a locally compact group. Its Haar measure is then an F-invariant Radon measure on F/H. So $\mu_{F/H}$ also exists. It only remains to show that $\mu_{F/H}$ is finite. Note by Lebesgue monotone convergence theorem that formula (1.1) applies to any positive measurable function f on G/H. In particular, with $f = 1$, the constant function equal to 1, we find $\mu_H(G/H) = \mu_{F/H}(F/H)\mu_F(G/F)$. Since μ_H is finite and μ_F is non-zero, we conclude that $\mu_{F/H}$ is also finite.

4. By the previous question, the Haar measure on F/H is finite. So F/H is compact by Exercise 1.1.

Exercise 1.3. Unimodularity.

1. We apply (1.1) with $H = \{e\}$, the trivial group and $F = \mathcal{Z}(G)$. We may do so because
   - $G/H = G$, which carries a G-invariant Radon measure (its Haar measure λ_G).
   - G/F is a lcsc group (since $F = \mathcal{Z}(G)$ is normal in G) so μ_F is nothing but the Haar measure on this group.
   - $F/H = F$ also carries an F-invariant measure, its Haar measure $\lambda_{\mathcal{Z}(G)}$.

   We find, for all $f \in C_c(G)$,
$$\int_G f \,\mathrm{d}\lambda_G = \int_{G/\mathcal{Z}(G)} \left(\int_{\mathcal{Z}(G)} f(ga) \,\mathrm{d}\lambda_{\mathcal{Z}(G)}(a) \right) \mathrm{d}\lambda_{G/\mathcal{Z}(G)}(g\mathcal{Z}(G)).$$
   Take now $f \in C_c(G)$ and $h \in G$. Define the function $f^h : g \mapsto f(gh)$. We want to show that $\int_G f^h \,\mathrm{d}\mu_G = \int_G f \,\mathrm{d}\mu_G$. Using the formula, we find
$$\begin{aligned}\int_G f^h \,\mathrm{d}\mu_G &= \int_{G/\mathcal{Z}(G)} \int_{\mathcal{Z}(G)} f(gah) \,\mathrm{d}\lambda_{\mathcal{Z}(G)}(a) \,\mathrm{d}\lambda_{G/\mathcal{Z}(G)}(g\mathcal{Z}(G)) \\ &= \int_{G/\mathcal{Z}(G)} \int_{\mathcal{Z}(G)} f(gha) \,\mathrm{d}\lambda_{\mathcal{Z}(G)}(a) \,\mathrm{d}\lambda_{G/\mathcal{Z}(G)}(g\mathcal{Z}(G)) \\ &= \int_{G/\mathcal{Z}(G)} \theta(f)(gh\mathcal{Z}(G)) \,\mathrm{d}\lambda_{G/\mathcal{Z}(G)}(g\mathcal{Z}(G))\end{aligned}$$
   where we used the fact that a belongs to the center of G to get to the second line. By unimodularity of $G/\mathcal{Z}(G)$, we now conclude that this last expression is equal to the same one with $h = e$, and so $\int_G f^h \,\mathrm{d}\lambda_G = \int_G f \,\mathrm{d}\lambda_G$, as desired.
2. Take an lcsc nilpotent group G, and denote by n its degree. This means that the central series $C_k(G)$, $k \geq 0$, defined inductively by $C_0(G) = G$ and $C_{k+1}(G) := [G, C_k(G)]$ satisfies $C_n(G) \neq \{e\}$ and $C_k(G) = \{e\}$, for all $k > n$. Recall two classical facts about this central series.
   - Observe that Since $[G, C_n(G)] = C_{n+1}(G) = \{e\}$, $C_n(G)$ is contained in $\mathcal{Z}(G)$.
   - By induction on k, one can check that $C_k(G/\mathcal{Z}(G)) = C_k(G)/C_k(G) \cap \mathcal{Z}(G)$.

   Combining these two facts shows that $G/\mathcal{Z}(G)$ is nilpotent of degree $n - 1$.

   We can now easily conclude by induction on the degree. If $n = 0$, G is abelian, hence it is unimodular. If $n > 1$, then $G/\mathcal{Z}(G)$ is of degree $n - 1$, so it is unimodular thanks to the induction hypothesis. Now the previous question immediately gives that G is unimodular.

   Of course, as we saw in class, the conclusion does not hold for solvable groups, since the group of all upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$ is solvable, but not unimodular.

Exercise 1.4. Lattices are co-compact.

1. It is clear that $\Gamma := H_3(\mathbb{Z})$ is discrete in $G := H_3(\mathbb{R})$. We claim that the set
$$\Omega := \left\{ M(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid 0 \leq a, b, c \leq 1 \right\}$$
   is such that $\Gamma\Omega = G$. Since Ω is compact, this implies that Γ is co-compact in G.
   Let us take arbitrary $x, y, z \in \mathbb{R}$ and check that $M(x,y,z)$ is in $\Gamma\Omega$. Take integers n_x and n_y such that $n_x \leq x < n_x + 1$ and $n_y \leq y < n_y + 1$. Then we have the computation
$$M(-n_x, -n_y, 0)M(x,y,z) = M(x - n_x, y - n_y, z - n_x y).$$
   Now take m such that $m \leq z - n_x y < m + 1$. We find that the matrix
$$M := M(0,0,-m)M(-n_x,-n_y,0)M(x,y,z) = M(x - n_x, y - n_y, z - n_x y - m)$$

belongs to Ω. Since $M(0,0,-m)M(-n_x,-n_y,0) \in \Gamma$, we are done.

2. Since Z is the center of G, it is clear that HZ is a group. So its closure F is a closed subgroup of G, which obviously contains H. Since the normalizer of H in G is closed and contains HZ, it contains F. So H is normal in F, and we are in the setting of Exercise 1.2., questions 3 and 4. So we immediately deduce that F/H is a compact group.
3. Since G/H carries a finite G-invariant measure, so does G/F, as we have seen in Exercise 1.2, question 3. Now observe that the action of G/Z on the homogeneous space $(G/Z)/(F/Z)$ is transitive, and can be viewed as a transitive G-action. The stabilizer of the trivial coset F/Z by this action is the subgroup F. By the homework on homogeneous spaces, we have a G-equivariant homeomorphism between $(G/Z)/(F/Z)$ and G/F. Since G/F carries a finite G-invariant Borel measure ν, so does $(G/Z)/(F/Z)$. Now the G action on this space factors to the action of G/Z, so ν is invariant under this action of G/Z. Thus F/Z is co-finite in G/Z.
4. Let us now assume that G is nilpotent of degree n. and let us proceed by induction on n.
   - If $n = 0$, G is abelian. Then for any cofinite closed subgroup H of G, we know that G/H is a lcsc group with finite Haar measure. So G/H must be compact by Exercise 1.1.
   - Assume now that $n > 0$ and that it is true for $n-1$. We use the notation from the previous questions. As we observed in the previous exercise, G/Z is nilpotent of degree $n-1$. By question 3, we know that F/Z is co-finite in G/Z. So by induction we deduce that F/Z is co-compact in G/Z. By the homeomorphism between $(G/Z)/(F/Z)$ and G/F found in the previous question, we deduce that G/F is compact. Moreover, we found in question 2 that F/H is compact as well. By the claim below applied to $L = F$, we find that there exists a compact set $C_1 \subset G$ such that $G = C_1F$. Applying now the claim to $G = F$ and $L = H$ we find that there exists a compact set $C_2 \subset F$ such that $F = C_2H$. Thus $G = C_1C_2H$, and since C_1C_2 is compact in G, we conclude that G/H is compact.

   We used the following claim mentioned in class.

   Claim. A closed subgroup $L < G$ is co-compact if and only if there exists a compact subset $C \subset G$ such that $CL = G$.

   Indeed, if such a C exists then the projection map $G \to G/L$ is surjective on the compact set C. So G/L must be compact (as the continuous image of a compact set). Conversely, if G/L is compact, take a non-empty relatively compact open set $U \subset G$. Of course we have $G = \bigcup_{g\in G} gU$ and hence $G/L = \bigcup_{g\in G} p(gU)$. Since the map p is open, $p(gU)$ is an open subset of G/L for every g. By compactness of G/L, we can extract a finite sub-cover $G/L = \bigcup_{g\in S} p(gU)$ for some finite set $S \subset G$. Then the closure C of $\bigcup_{g\in S} gU$ is a compact set such that $p(C) = G/L$. this last condition is equivalent to $CL = G$.

2. Geometric growth rate of semi-simple lattices

Exercise 2.1. Classical decompositions.

1. As we saw in class, the KAK-decomposition follows from the polar decomposition, and the fact that a positive matrix can be conjugated to a diagonal matrix by an orthogonal matrix. Take an element $g \in G$. Then we may write $g = kak'$, with $k, k' \in K$, $a \in A$. So it suffices to check that a belongs to KA^+K. We may find a diagonal matrix $b \in A \cap K$, with values ± 1, such that ba is still diagonal but has only positive entries. Now we may conjugate ba by a suitable permutation matrix ε to ensure that $\varepsilon ba \varepsilon^{-1} \in A^+$. Since b and ε belong to K, this proves the result.
2. We follow the hint. The action of $P_- \times K$ on G given by $(w,k)\cdot g := wgk^{-1}$ is transitive : any element is of the form wk^{-1} for some $(w,k) \in P_- \times K$. So we may view G as a homogeneous

space $(P_- \times K)/\operatorname{Stab}(\{e\})$. By the exercise sheet on homogeneous spaces there is at most one $(P_- \times K)$-invariant Radon measure on this homogeneous space G.

So we only need to check that the Haar measure λ_G on G is indeed $P_- \times K$-invariant. By definition it is of course left P_- invariant. Moreover since K-is compact, it has no character, so the modular function of G vanishes on K, which means that λ_G is right K-invariant, as desired. Of course we could have directly used the fact that G is unimodular.

3. (a) By homogeneity, the map m has constant rank. Indeed m satisfies $m(w_0 w, k k_0) = w_0 m(w,k) k_0$, for all $w_0, w \in P_-$, $k_0, k \in K$. This rewrites $m \circ (L_{w_0} \times R_{k_0}) = L_{w_0} \circ R_{k_0} \circ m$, where L_{w_0} denotes the left translation on P_- and on G, while R_{k_0} denotes the right translation on K and G. Since these translations are diffeomorphisms, derivating this relation at $(w,k) = (e,e)$ and using the chain rule gives that the derivative of m at (e,e) has the same rank as the derivative at any other point (w_0, k_0). So m indeed has constant rank.
   Since it is surjective, it is a submersion. Now we compare the dimensions of P_-, K and G:
   - $\dim(G) = d^2 - 1$.
   - $\dim(P_-) = d(d+1)/2 - 1$.
   - $\dim(K) = d(d-1)/2$.

   The first two computations are trivial. The third one can be seen in many ways. For instance, it can be observed that the tangent space of K at e is the set of antisymmetric matrices, which clearly has the required dimension. So $\dim(P_- \times K) = \dim(P_-) + \dim(K) = \dim(G)$. Since m is a submersion it must be an immersion, and therefore a local diffeomorphism at any point.

   (b) If the result does not hold then for any neighborhood W of e in P_-, we may find distinct (w,y) and (w',y') in $W \times Y$ such that $wy = w'y'$. Then taking a countable basis W_n of neighborhoods of e in P_-, we find two sequences of pairwise distinct elements (w_n, y_n) and (w'_n, y'_n) such that w_n, w'_n converge to e and $w_n y_n = w'_n y'_n$ for every n. For every n, write $y_n = k_n x_0$ and $y'_n = k'_n x_0$ for some $k_n, k'_n \in K$.
   This means that for every n, there exists $\gamma_n \in \Gamma$ such that $w_n k_n = w'_n k'_n \gamma_n$, as an equality inside G. By compactness of K we may assume (up to taking a subsequence) that k_n and k'_n converge to elements k and k' respectively, which forces γ_n to converge to some $\gamma \in \Gamma$. Since Γ is discrete in G, we find that $\gamma_n = \gamma$ for n large enough.
   Note now that taking the limit in the equality $w_n k_n = w'_n k'_n \gamma_n$ gives $k = k'\gamma$, which shows that $\gamma \in K$. Then for n large enough $\gamma_n = \gamma$, and thus ${w'_n}^{-1} w_n = k'_n \gamma k_n^{-1} \in P_- \cap K$. Thus for every n, ${w'_n}^{-1} w_n$ is a diagonal matrix with entries ± 1. So if n is large enough it must be the identity, and thus $w_n = w'_n$ for all n large enough. This then implies that $y_n = y'_n$, which is a contradiction.

Exercise 2.2. A mixing property.

1. For every integer $n \geq 1$, we define the subset
$$W_n = \{M \in P_- \mid |M_{i,j}| < \frac{1}{n} \text{ for every } i > j, \text{ and } |M_{k,k} - 1| < \frac{1}{n} \text{ for every } k\}.$$
   We note that if $a = \operatorname{diag}(\lambda_1, \ldots, \lambda_d) \in A^+$ and $M \in W_n$, then for every $i < j$, we have
$$|(aMa^{-1})_{i,j}| = |\frac{\lambda_i}{\lambda_j} M_{i,j}| \leq |M_{i,j}| < \frac{1}{n},$$
   while $(aMa^{-1})_{k,k} = M_{k,k}$ for every k. So $aMa^{-1} \in W_n$. Moreover, it is clear that W_n forms a basis of neighborhoods of e in P_-.
2. There was an issue in this question, all along, instead of P_-, we should have considered the open subgroup P_-^0 of elements in P_- with only positive diagonal entries. The advantage is that P_-^0 also satisfies all the above properties, but in addition, the Iwasawa map m :

$P_-^0 \times K \to G$ is bijective. Take $W \subset P_-$ and $f \in L^\infty(X)$ as in the question. Since P_-^0 is open in P_-, we may as well assume that $W \subset P_-^0$, by shrinking W if necessary. In this case we have $\mathbf{1}_{WK}(wk) = \mathbf{1}_{W\times K}(w,k)$, for every $w \in P_-^0$, $k \in K$. We have

$$\begin{aligned}
\int_{W\times Y} f(wy)\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\mu_Y(y) &= \int_{W\times K} f(wkx_0)\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\lambda_K(k)\\
&= \int_{P_-^0\times K} \mathbf{1}_{WK}(wk) f(wkx_0)\,\mathrm{d}\lambda_{P_-^0}(w)\,\mathrm{d}\lambda_K(k)\\
&= \int_G \mathbf{1}_{WK}(g) f(gx_0)\,\mathrm{d}\lambda_G(g)\\
&= \int_X \sum_{\gamma\in\Gamma} f(gx_0)\mathbf{1}_{WK}(g\gamma)\,\mathrm{d}\mu_X(g\Gamma)\\
&= \int_X f(gx_0)\,\mathrm{card}(WK\cap g\Gamma)\,\mathrm{d}\mu_X(g\Gamma).
\end{aligned}$$

Note that we used equation (1.1) to get the fourth line.

Assume now moreover that W is as in question (3) of Exercise 2.1. Since f is supported on WY, we find that for every g such that gx_0 is in the support of f, there always is at least one element $g\gamma_0$ in WK. The following claim then ensures that $\mathrm{card}(g\Gamma\cap WK) = \mathrm{card}(K\cap\Gamma)$.

Claim. For every $\gamma\in\Gamma$, we have that $g\gamma \in WK$ if and only if $\gamma_0^{-1}\gamma \in K\cap\Gamma$.

If $\gamma_0^{-1}\gamma\in K\cap\Gamma$ then $g\gamma = g\gamma_0 h$, for somme $h\in K\cap\Gamma$. So $g\gamma\in WK$. Conversely, assume that $g\gamma = wk$ for some $w\in W$, $k\in K$. Write also $g\gamma_0 = w_0k_0$. Then $wk\gamma^{-1} = w_0k_0\gamma_0^{-1}$ and thus $wkx_0 = w_0k_0x_0$. Since $W\times Y\to WY$ is a diffeomorphism, we find that $w = w_0$ and therefore $k\gamma^{-1} = k_0\gamma_0^{-1}$ which leads to $\gamma_0^{-1}\gamma = k_0^{-1}k\in K\cap\Gamma$, as wanted. Equation (2.1) follows.

3. (a) We first prove a version of Heine Thoerem in the setting of homogeneous spaces: Since φ is continuous with compact support on X, it is uniformly continuous: for every $\varepsilon > 0$, we may find a neighborhood U of the identity in G such that for every $x\in X$ and every $g\in U$, $|\varphi(gx)-\varphi(x)|\le\varepsilon$.
This can be checked as follows: for every $x\in\mathrm{supp}(\varphi)$, we may find an open neighborhood V_x of e in G such that $|\varphi(gx)-\varphi(x)|\le\varepsilon/2$, for every $g\in V_x$. Take an open subset $U_x\subset V_x$ such that $e\in U_x$ and $U_x^2\subset V_x$. Now the family $\{U_xx\}_{x\in\mathrm{supp}(\varphi)}$ forms an open cover of the compact set $\mathrm{supp}(\varphi)$. We may thus take a finite sub-cover indexed by a finite set S. Define $U := \bigcap_{y\in S} U_y$. Take $x\in\mathrm{supp}(f)$, $g\in U$. There exists $y\in S$ and $h\in U_x$ such that $x = hy$. Since $gh\in U_x^2\subset V_x$, we find that $|f(gx)-f(x)| = |f(ghy)-f(hy)| \le |f(ghy)-f(y)| + |f(hy)-f(y)|\le\varepsilon$. This proves this Heine theorem.
Keep this notation U. Take n large enough so that $W_n\subset U$. So we have for every $a\in A^+$,

$$|I_a - J_{n,a}| \le \frac{1}{\lambda_{P_-}(W_n)}\int_{W_n\times Y} |\varphi(ay)-\varphi(awy)|\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\mu_Y(y).$$

But observe that for every $w\in W_n$, we have $awy = awa^{-1}ay \in W_nay\subset Uay$. Thus $|\varphi(ay)-\varphi(awy)|\le\varepsilon$, and the result follows.

(b) We fix n large as in the statement of the question. Then we want to apply equation (2.2), but we cannot do it directly because the map $(w,y)\mapsto\varphi(way)$ needs not supported on $W_n\times Y$. So we arrange this by multiplying by $\mathbf{1}_{W_nY}(wy)$ (which is equal to 1 on the domain of integration we consider. We get

$$\begin{aligned}
J_{n,a} &= \frac{1}{\lambda_{P_-}(W_n)} \int_{W_n \times Y} \varphi(awy) \mathbf{1}_{W_n Y}(wy) \, \mathrm{d}\lambda_{P_-}(w) \, \mathrm{d}\mu_Y(y) \\
&= \frac{\mathrm{card}(K \cap \Gamma)}{\lambda_{P_-}(W_n)} \int_X \varphi(ax) \mathbf{1}_{W_n Y}(x) \, \mathrm{d}\mu_X(x) \\
&= \frac{\mathrm{card}(K \cap \Gamma)}{\lambda_{P_-}(W_n)} \langle \varphi(a\, \cdot\,), \mathbf{1}_{W_n Y} \rangle.
\end{aligned}$$

where the above scalar product is taken inside $L^2(X, \mu_X)$. By Howe-moore theorem, we know that the action of G on X is mixing, and hence so is the action of its closed subgroup A^+. Thus, as a goes to infinity, with n fixed, we find that $J_{n,a}$ converges to the product

$$\frac{\mathrm{card}(K \cap \Gamma)}{\lambda_{P_-}(W_n)} \int_X \varphi \, \mathrm{d}\mu_X(x) \int_X \mathbf{1}_{W_n Y} \, \mathrm{d}\mu_X.$$

On the other hand, applying (2.2) to the function $f = \mathbf{1}_{W_n Y}$ gives

$$\frac{\mathrm{card}(K \cap \Gamma)}{\lambda_{P_-}(W_n)} \int_X \mathbf{1}_{W_n Y} \, \mathrm{d}\mu_X = \frac{1}{\lambda_{P_-}(W_n)} \int_{W_n \times Y} \mathbf{1}_{W_n Y}(wy) \, \mathrm{d}\lambda_{P_-}(w) \, \mathrm{d}\mu_Y(y) = 1.$$

So indeed $J_{n,a}$ converges to $\int_X \varphi \, \mathrm{d}\mu_X$ as a goes to infinity inside A^+.

(c) Fix $\varepsilon > 0$. Take n_0 as in question (a) and choose $n \geq n_0$ large enough so that W_n satisfies the conclusion of Question 3 of Exercise 2.1. By the previous question, we may then find a compact $C \subset A^+$ such that $|J_{n,a} - \int_X \varphi \, \mathrm{d}\mu_X| < \varepsilon$ for all $a \notin C$. Then for such an a, we get that $|I_a - \int_X \varphi \, \mathrm{d}\mu_X| < 2\varepsilon$. This proves the desired convergence.

4. We will use the KA^+K-decomposition. We may write $G = \bigcup_n C_n$ as the increasing union of countably many compact sets C_n. Fix $f \in C_c(X)$. If the integral $\int_Y f(gy) \, \mathrm{d}\mu_Y(y)$ does not converges to $\int_X \varphi \, \mathrm{d}\mu_X$ as $g \in G$ goes to infinity, then we may find $\varepsilon > 0$ such that for every n, there exists $g_n \notin C_n$ such that

$$\left| \int_Y \varphi(g_n y) \, \mathrm{d}\mu_Y(y) - \int_X \varphi \, \mathrm{d}\mu_X \right| > \varepsilon.$$

For every n, write $g_n = k_n a_n k_n'$, with $k_n, k_n' \in K$, $a_n \in A^+$. Then since K is compact, we find that a_n goes to infinity inside A^+. Taking a subsequence if necessary we may also assume that k_n converges to some $k \in K$. Applying the version of Heine theorem described above and denoting by U the set given in question 3.(a), we have, for all n large enough,

$$|\varphi(k_n a_n k_n' y) - \varphi(k a_n k_n' y))| < \varepsilon.$$

So for n large enough we have

$$\begin{aligned}
\left| \int_Y \varphi(g_n y) \, \mathrm{d}\mu_Y(y) - \int_X \varphi \, \mathrm{d}\mu_X \right| &< \varepsilon + \left| \int_Y \varphi(k a_n k_n' y) \, \mathrm{d}\mu_Y(y) - \int_X \varphi \, \mathrm{d}\mu_X \right| \\
&= \varepsilon + \left| \int_Y \varphi(k a_n y) \, \mathrm{d}\mu_Y(y) - \int_X \varphi \, \mathrm{d}\mu_X \right|,
\end{aligned}$$

where the second line follows from the K-invariance of the measure μ_Y. Now we may apply question 3 to the function $\varphi' : x \in X \mapsto \varphi(kx)$ and thus get that

$$\lim_n \left| \int_Y \varphi(k a_n y) \, \mathrm{d}\mu_Y(y) - \int_X \varphi \, \mathrm{d}\mu_X \right| = 0.$$

Combining this with the previous computation contradicts the definition of g_n.

Exercise 2.3. Computation of the geometric growth.

1. (a) By equation (1.1), we have
$$\begin{aligned}\int_X F_r(x) f(x)\,\mathrm{d}\mu_X(x) &= \int_X \Big(\sum_{\gamma\in\Gamma} \mathbf{1}_{B_r}(g\gamma)\Big) f(g\gamma x_0)\,\mathrm{d}\mu_X(g\Gamma)\\ &= \int_G \mathbf{1}_{B_r}(g) f(gx_0)\,\mathrm{d}\lambda_G(g)\\ &= \int_{B_r} f(gx_0)\,\mathrm{d}\lambda_G(g).\end{aligned}$$
Since $B_r = B_r k$ for every $k \in K$, we also find that
$$\int_{B_r} f(gx_0)\,\mathrm{d}\lambda_G(g) = \int_K \int_{B_r} f(gkx_0)\,\mathrm{d}\lambda_G(g)\,\mathrm{d}\mu_K(k).$$
So the result follows by combining the two computations.

   (b) Take $\varepsilon > 0$. By Exercise (2.2), we may find a compact subset $C \subset G$ such that
$$I_g := \Big|\int_K f(gkx_0)\,\mathrm{d}\mu_K(k) - \int_X f\,\mathrm{d}\mu_X\Big| < \varepsilon/2, \text{ for all } g \in G \setminus C.$$
Take now r large enough so that $\mu(C\cap B_r) < \frac{\varepsilon v_r}{4\|f\|_\infty}$. Then we find
$$\begin{aligned}\Big|\frac{1}{v_r}\int_{B_r}\int_K f(gkx_0)\,\mathrm{d}\mu_K(k)\,\mathrm{d}\lambda_G(g) - \int_X f\,\mathrm{d}\mu_X\Big| &=\\ &= \Big|\frac{1}{v_r}\int_{B_r}\left(\int_K f(gkx_0)\,\mathrm{d}\mu_K(k) - \int_X f\,\mathrm{d}\mu_X\right)\mathrm{d}\lambda_G(g)\Big|\\ &\le \frac{1}{v_r}\int_{B_r} I_g\,\mathrm{d}\lambda_G(g)\\ &= \frac{1}{v_r}\int_{C\cap B_r} I_g\,\mathrm{d}\lambda_G(g) + \frac{1}{v_r}\int_{B_r\setminus C} I_g\,\mathrm{d}\lambda_G(g)\\ &< \sup_g I_g \frac{\varepsilon v_r}{4v_r\|f\|_\infty} + \frac{\varepsilon}{2}.\end{aligned}$$
Since for all g, we have $I_g \le 2\|f\|_\infty$, we find that the above quantity is majorized by $\varepsilon/2 + \varepsilon/2 = \varepsilon$. So the conclusion follows from question (a).

2. (a) Take $r, \varepsilon > 0$, $g \in G_\varepsilon$ and $\gamma \in \Gamma$.
If $\gamma \in B_r$, then $\|g\gamma\| \le \|g\|\|\gamma\| < e^\varepsilon r$. So $g\gamma \in B_{re^\varepsilon}$.
If now $g\gamma \in B_{e^{-\varepsilon}r}$, then $\|\gamma\| \le \|g^{-1}\|\|g\gamma\| < e^\varepsilon e^{-\varepsilon} r = r$ so $\gamma \in B_r$. We thus find
$$\mathbf{1}_{B_{re^{-\varepsilon}}}(g\gamma) \le \mathbf{1}_{B_r}(\gamma) \le \mathbf{1}_{B_{re^\varepsilon}}(g\gamma).$$
Summing up over $\gamma \in \Gamma$ gives the desired inequality.

   (b) Since f is supported on $G_\varepsilon x_0$, the previous question gives, for all $x \in X$,
$$f(x)F_{re^{-\varepsilon}}(x) \le f(x)F_r(x_0) \le f(x)F_{re^\varepsilon}(x).$$
Integrating over $x \in X$ gives the result.

   (c) Dividing by v_r the inequality from the previous question, and letting r go to infinity gives
$$\begin{aligned}\liminf_r \frac{1}{v_r}\int_X f(x)F_{re^{-\varepsilon}}(x)\,\mathrm{d}\mu_X(x) &\le \liminf_r \frac{1}{v_r}\operatorname{card}(\Gamma\cap B_r)\\ &\le \limsup_r \frac{1}{v_r}\operatorname{card}(\Gamma\cap B_r)\\ &\le \limsup_r \frac{1}{v_r}\int_X f(x)F_{re^\varepsilon}(x)\,\mathrm{d}\mu_X(x).\end{aligned}$$

As said in the instructions, we know that v_r is of the form $cr^{d(d-1}$ for some absolute constant d. So $v_r = v_{re^{-\varepsilon}} e^{\varepsilon d(d-1)}$, and thus

$$\liminf_r \frac{1}{v_r} \int_X f(x) F_{re^{-\varepsilon}}(x) \, \mathrm{d}\mu_X(x) = e^{-\varepsilon d(d-1)} \lim_r \frac{1}{v_{re^{-\varepsilon}}} \int_X f(x) F_{re^{-\varepsilon}}(x) \, \mathrm{d}\mu_X(x)$$
$$= e^{-\varepsilon d(d-1)}.$$

Likewise, $\limsup_r \frac{1}{v_r} \int_X f(x) F_{re^{\varepsilon}}(x) \, \mathrm{d}\mu_X(x) = e^{\varepsilon d(d-1)}$. So we conclude

$$e^{-\varepsilon d(d-1)} \leq \liminf_r \frac{1}{v_r} \operatorname{card}(\Gamma \cap B_r) \leq \limsup_r \frac{1}{v_r} \operatorname{card}(\Gamma \cap B_r) \leq e^{\varepsilon d(d-1)}.$$

As ε can be chosen arbitrarily small we get the conclusion.