LIE GROUPS AND THEIR LATTICES
- FINAL EXAM -

Université de Bordeaux, M2 ALGANT, 2019-2020
26 Mai 2020, 14h00.
Durée : **3 heures**

Exercise 1. Finite index subgroups and commensurability

(1) Prove that if $\Sigma < \Lambda < \Gamma$ are three groups then Σ has finite index inside Γ if and only if both inclusions $\Sigma < \Lambda$ and $\Lambda < \Gamma$ have finite index. Prove the formula for the indices:

$$[\Gamma : \Sigma] = [\Gamma : \Lambda][\Lambda : \Sigma].$$

Here we recall that by definition, the index $[\Gamma : \Lambda]$ is the cardinal of Γ/Λ.

(2) Prove that now that if $\Lambda_1, \Lambda_2 < \Gamma$ are two subgroups of finite index, then $\Lambda_1 \cap \Lambda_2$ also has finite index in Γ.

(3) Deduce that if $\Lambda < \Gamma$ is a subgroup of finite index then there is a closed subgroup $\Lambda_0 < \Lambda$ which still has finite index inside Γ, and is normal inside Γ. Assuming that Γ is in fact a topological group, and that Λ is closed in Γ, check that we may choose Λ_0 to be closed as well.

(4) Assume that Γ is a topological group with a finite index closed subgroup $\Lambda < \Gamma$. Prove that Λ is amenable if and only if Γ is amenable.

Exercise 2. A ping-pong game

Consider a group G acting on a set X. Take two elements $g, h \in G$. The universal property of free groups gives us a morphism $\pi : F_2 = F(a, b) \to G$ which maps a to g and b to h. Assume that there are disjoint sets $A, B \subset X$ such that for all $n \in \mathbb{Z} \setminus \{0\}$,

$$g^n(B) \subset A \quad \text{and} \quad h^n(A) \subset B. \tag{0.1}$$

(1) Let $w \in F(a, b)$ be an element whose expression as a reduced word starts and ends with letters in $\{a, a^{-1}\}$. Prove that $\pi(w)$ is non-trivial.

(2) Prove that π is injective.
Hint. Prove that any element of $F(a, b)$ can be conjugated to an element as in the previous question.

Likewise, one can show that if (0.1) holds only for positive integers n, then π is injective on the semi-group generated by a and b. In this case we say that g and h *generate a free semi-group.* We admit this fact.

Exercise 3. Free semi-groups in linear groups

Consider a linear group $G \subset \mathrm{GL}(V)$ over a finite dimensional real vector space V. Denote by $X := \mathbb{P}(V)$ and view naturally G as acting on X by projective transformations. Assume that the action of G on V is strongly irreducible and proximal. recall that the later means that there exist a rank one matrix A, a sequence $(g_n)_n \subset G$ and scalar numbers $(\lambda_n)_n$ such that $\lambda_n g_n$ converges to A as n goes to infinity.

(1) Prove that we may choose $(g_n)_n$, $(\lambda_n)_n$ and A to ensure that $\mathrm{Im}(A) \cap \mathrm{Ker}(A) = \{0\}$.

(2) Assume now that $\mathrm{Im}(A) \cap \mathrm{Ker}(A) = \{0\}$. Prove that for any neighborhood $U_+ \subset X$ and $U_- \subset X$ of $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$, respectively, there exists n such that

$$g_n^k \cdot (X \setminus U_-) \subset U_+ \text{ for every positive integer } k \geq 1.$$

(3) A finite union of subspaces of V will be called a "quasi-linear variety".
   (a) Prove that any subset S of V is contained in a unique minimal quasi-linear variety $V(S)$. Check that if S is G invariant, then either $S = \{e\}$ or $V(S) = V$.
   (b) Deduce that there exists $h \in G$ such that
$$h\,\mathrm{Im}(A) \cap (\mathrm{Im}(A) \cup \mathrm{Ker}\, A) = \{0\}.$$
   (c) Prove that for this choice of h and for n large enough, g_n and hg_n generate a free semi-group.

(4) Conclude that lattices in connected semi-simple Lie groups contain free sub-semigroups.

Exercise 4. Around superrigidity

In this exercise, the term "Lie group topology" is used to make the distinction with the Zariski topology.

(1) Is there a surjective group homomorphism from $\mathrm{PSL}_4(\mathbb{Z})$ onto $\mathrm{PSL}_3(\mathbb{Z})$?
(2) Consider a linear group $\Gamma \subset \mathrm{GL}(V)$ whose closure (in the Lie group topology) is non-amenable.
   (a) Consider the Zariski closure G of Γ, and view it as a Lie group. Prove that the identity component G^0 of G, in the Lie group topology, is a non-amenable Lie group.
   Hint. Recall that the identity component (in the Lie group topology) of a Zariski closed group has finite index in this group.
   (b) Denote by $\Gamma_0 := \Gamma \cap G^0$, which is a finite index subgroup of Γ. Prove that there exists a connected semi-simple Lie group H with trivial center and without compact factor, and a surjective Lie group homomorphism $\pi : G^0 \to H$ such that the image of Γ_0 in H has non-amenable closure.
(3) Let H be a non-compact, connected simple Lie group with trivial center and rank at least 2 and let $\Gamma < H$ be a lattice. Using question 2, prove that for any connected semi-simple Lie group G with $\dim(G) < \dim(H)$, the image of any group homomorphism $\pi : \Gamma \to G$ has amenable closure. In particular the image of any morphism $\mathrm{PSL}_4(\mathbb{Z}) \to \mathrm{PSL}_3(\mathbb{Z})$ is amenable.
   Hint. Argue by contradiction and use a minimality argument.

Remark. It is a fact, based on property (T), that any amenable quotient of $\mathrm{PSL}_4(\mathbb{Z})$ is finite. So in fact, any morphism $\mathrm{PSL}_4(\mathbb{Z}) \to \mathrm{PSL}_3(\mathbb{Z})$ has finite image.