LIE GROUPS AND THEIR LATTICES
- GRADED HOMEWORK -

Due date: **Friday, March 27**

1. Invariant measures, nilpotent lcsc groups and lattices

Exercise 1.1. Finite Haar measure.

Prove that an lcsc group G is compact if and only if any Haar measure on G is finite.

Exercise 1.2. Invariant measures on homogeneous spaces.

Let G be an lcsc group, and $H < F < G$ two closed subgroups. We assume that there exist:

- G-invariant Radon measures μ_H and μ_F on G/H and G/F, respectively.
- an F-invariant Radon measure $\mu_{F/H}$ on F/H.

From the exercise sheet on homogeneous spaces, we know that μ_H, μ_F and $\mu_{F/H}$ are unique up to proportionality.

1. Given $f \in C_c(G/H)$, prove that
$$\theta(f) : gF \in G/F \mapsto \int_{F/H} f(gx)\,\mathrm{d}\mu_{F/H}(x) \in \mathbb{R}$$
is a well defined function which belongs to $C_c(G/F)$. Observe that the map $\theta : C_c(G/H) \to C_c(G/F)$ defined this way is linear and positive (meaning that $\theta(f) \geq 0$ if $f \geq 0$).
2. Up to multiplying μ_F by a positve real number, prove that for all $f \in C_c(G/H)$,

$$\int_{G/H} f\,\mathrm{d}\mu_H = \int_{G/F} \left(\int_{F/H} f(gx)\,\mathrm{d}\mu_{F/H}(x) \right) \mathrm{d}\mu_F(g). \tag{1.1}$$

Observe that this equation is also non-trivial in the case where $H < G$ is the trivial subgroup.

Let us still assume that G is an lcsc group with two closed subgroups $H < F < G$ but we only assume that μ_H exists and is finite. In addition we assume that H is normal in F.

3. Check in this setting that μ_F and $\mu_{F/H}$ as above automatically exist and are finite.
4. Prove then that F/H is a compact group.

Exercise 1.3. Unimodularity.

1. Let G be an arbitrary lcsc group. Prove that if $G/\mathcal{Z}(G)$ is unimodular, then so is G.
Hint. Use (1.1).
2. Conclude that nilpotent lcsc groups are unimodular. Is it the case of solvable groups ?

Exercise 1.4. Lattices are co-compact.

1. Prove that $H_3(\mathbb{Z})$ is a co-compact lattice in $H_3(\mathbb{R})$, where for a ring R, $H_3(R)$ denotes the Heisenberg group over R:
$$H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in R \right\}.$$

We now aim to prove that lattices in nilpotent lcsc groups are always co-compact. More generally take a nilpotent lcsc group G and a closed subgroup H such that G/H carries a G-invariant Borel probability measure (we say that H is *co-finite* in G). We aim to prove that G/H is compact.

Denote by $F < G$ the closure of HZ where $Z = \mathcal{Z}(G)$ is the center of G.

(2) Prove that F is a subgroup of G containing H and such that F/H is compact.
(3) Prove that F/Z is co-finite in G/Z.
(4) Conclude.

2. Geometric growth rate of semi-simple lattices

In this second series of exercises we let G be our prototypical semi-simple Lie group: $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 2$, and we take a lattice Γ in G.

As usual, we consider the subgroups A, P_- of diagonal matrices and lower triangular matrices in G, respectively, and $K = \mathrm{SO}(d)$. We denote by $A^+ \subset A$ the subset of diagonal matrices $\mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0.$$

We set $X := G/\Gamma$, and denote by $x_0 \in X$ the base point $x_0 = \Gamma$. We then define $Y := Kx_0 \subset X$.

We denote by μ_X the G-invariant probability measure on $X = G/\Gamma$, and by $\lambda_Y \in \mathrm{Prob}(Y)$ the unique K-invariant probability measure on Y (so μ_Y is the image of the Haar measure of K under the orbit map $k \mapsto kx_0$).
We normalize the Haar measure λ_G of G so that for every $f \in C_c(G)$,

$$\int_G f \, \mathrm{d}\lambda_G = \int_X \left(\sum_{\gamma \in \Gamma} f(g\gamma) \right) \mathrm{d}\mu_X(g\Gamma).$$

This is possible thanks to equation (1.1).

Exercise 2.1. Classical decompositions.

1. Prove the refined version of the KAK-decomposition: $G = KA^+K$. This is called the *Cartan* decomposition.
2. The Iwasawa decomposition tells us that $G = P_-K$, i.e. the map $m : (w, k) \in P_- \times K \mapsto wk \in G$ is surjective. Prove that the Haar measure λ_G is the push-forward of the product of Haar measures on P_- and K under $\lambda_G = m_*(\lambda_{P_-} \otimes \lambda_K)$, where λ_K is normalized to be a probability measure and λ_{P_-} is normalized appropriately.
   Hint. Consider the left-right action of $P_- \times K$ on G and view G as a homogeneous space for this action.
3. (a) Prove that the map $m : P_- \times K \to G$ defined above is a local diffeomorphism.
   (b) Prove that there exists a small enough neighborhood W of e in P_- such that the map
   $$(w, y) \in W \times Y \mapsto wy \in X$$
   is injective. Deduce that this map is a diffeomorphism onto the open set WY.

Exercise 2.2. A mixing property.

We prove in this exercise that the action $G \curvearrowright (X, \mu_X)$ "mixes Y inside X".

1. Prove that there exists a basis of neighborhoods W_n, $n \geq 1$, of e in P_- such that

$$aW_na^{-1} \subset W_n, \text{ for all } n \geq 1, a \in A_+. \tag{2.1}$$

2. Prove that for any relatively compact open subset $W \subset P_-$, and any bounded measurable function $f \in L^\infty(X)$, supported on WY, we have
$$\int_{W\times Y} f(wy)\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\mu_Y(y) = \int_X f(gx_0)\,\mathrm{card}(WK \cap g\Gamma)\,\mathrm{d}\lambda_X(g\Gamma).$$
If W is as in question (3) of Exercise 2.1, prove that in fact
$$\int_{W\times Y} f(wy)\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\mu_Y(y) = \mathrm{card}(K\cap\Gamma)\int_X f\,\mathrm{d}\lambda_X. \tag{2.2}$$
3. Take a function $\varphi \in C_c(X)$. For all $a \in A_+$, and $n \geq 1$, define $I_a := \int_Y \varphi(ay)\,\mathrm{d}\mu_Y(y)$ and
$$J_{n,a} := \frac{1}{\lambda_{P_-}(W_n)}\int_{W_n\times Y} \varphi(awy)\,\mathrm{d}\lambda_{P_-}(w)\,\mathrm{d}\mu_Y(y),$$
   (a) Prove that there exists n_0 such that $|I_a - J_{n,a}| < \varepsilon$, for all $a \in A_+$, $n \geq n_0$.
   (b) Take $n \geq n_0$ large enough so that W_n is as in question (3) of Exercise 2.1. Prove that $J_{n,a}$ converges to $\int_X \varphi\,\mathrm{d}\mu_X$.
   Hint. Use Howe-Moore theorem.
   (c) Conclude that I_a converges to $\int_X \varphi\,\mathrm{d}\mu_X$ as $a \in A_+$ goes to infinity.
4. Prove that for every $\varphi \in C_c(X)$, the integral $\int_Y \varphi(gy)\,\mathrm{d}\mu_Y(y)$ converges to $\int_X \varphi\,\mathrm{d}\mu_X$ as $g \in G$ goes to infinity.

We thus have shown that the measures $g_*\lambda_Y \in \mathrm{Prob}(X)$ converge weakly to the unique G-invariant measure λ_X as g goes to infinity.

Exercise 2.3. Computation of the geometric growth.

The group $G = \mathrm{SL}_d(R)$ acts on $\mathbb{R}^d$. Endow $\mathbb{R}^d$ with its euclidian norm and define the corresponding operator norm on G: $\|g\| = \sup_{v\in\mathbb{R}^d\setminus\{0\}} \|gv\|/\|v\|$. Note that this norm is left K-invariant.

For every real number $r > 0$, denote by B_r the ball of radius r in G, with respect to the norm defined above, and by v_r its volume with respect to the Haar measure λ_G on G (normalized as in Exercise 2.2).

We will prove that $\mathrm{card}(\Gamma \cap B_r) \sim v_r$ as r goes to $+\infty$, i.e.
$$\lim_{r\to+\infty} \frac{\mathrm{card}(\Gamma\cap B_r)}{v_r} = 1.$$
This tells us that the number of points of Γ in a large ball grows like the Haar measure of the ball. We admit that $v_r = cr^{d(d-1)}$, for some constant $c > 0$. This allows to specify the above result, but this will also be needed in Question 2.b. below.

1. Consider the function $F_r : X \to \mathbb{N}$ defined by
$$F_r(g\Gamma) = \mathrm{card}(B_r \cap g\Gamma) = \sum_{\gamma\in\Gamma} \mathbf{1}_{B_r}(g\gamma), \text{ for every } g\Gamma \in X.$$
   (a) Prove that for all $r > 0$, and $f \in C_c(X)$, we have
$$\int_X F_r(x)f(x)\,\mathrm{d}\mu_X(x) = \int_{B_r}\left(\int_K f(gkx_0)\,\mathrm{d}\lambda_K(k)\right)\mathrm{d}\mu_X(g).$$
   Hint. Use the fact that B_r is right-K-invariant.
   (b) Deduce that
$$\lim_{r\to+\infty}\frac{1}{v_r}\int_X F_r(x)f(x)\,\mathrm{d}\mu_X(x) = \int_X f\,\mathrm{d}\mu_X.$$
2. For every $\varepsilon > 0$, define $G_\varepsilon := \{g \in G \mid \max(\|g\|, \|g^{-1}\|) \leq e^\varepsilon\}$.

(a) Prove that for every $r, \varepsilon > 0$ and every $g \in G_\varepsilon$, we have
$$F_{re^{-\varepsilon}}(g\Gamma) \leq F_r(x_0) \leq F_{re^{\varepsilon}}(g\Gamma).$$
(b) Take a non-negative function $f \in C_c(X)$ supported on $G_\varepsilon x_0$ and with integral 1. Prove that for every $r > 0$, we have
$$\int_X f(x) F_{re^{-\varepsilon}}(x)\, \mathrm{d}\mu_X(x) \leq \operatorname{card}(\Gamma \cap B_r) \leq \int_X f(x) F_{re^{\varepsilon}}(x)\, \mathrm{d}\mu_X(x).$$
(c) Conclude that $\operatorname{card}(\Gamma \cap B_r) \sim v_r$ as $r \to +\infty$.