Lattices in semi-simple Lie groups

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This course aims to present various constructions and properties of lattices in Lie groups.

I thank Yves Benoist for allowing me to use his lecture notes [**Ben08**]. Most of the time, I did nothing more than translate his notes into english. Sometimes I added minor details to proofs and I take full responsibility if mistakes or inconsistencies appear. No originality is claimed.

Contents

CHAPTER 1

Locally compact groups and lattices

1. The Haar measure

Definition 1.1. A *topological group* is a group G endowed with a topology for which the map $(g,h) \in G \times G \to gh^{-1}$ is continuous. G is said to be *locally compact* if every point admits a compact neighborhood, or equivalently if the trivial element admits a compact neighborhood.

In the above terminology, the term compact includes the Hausdorff axiom, according to the French convention.

Example 1.2. We will encounter many locally compact groups.

- All Lie groups are examples of locally compact groups.
- Any group can be made locally compact by considering its discrete topology. This is the topology we will usually use for countable groups.
- The additive group $(\mathbb{Q}_p, +)$ is locally compact (for the topology given by its ultrametric norm), and the subgroup $\mathbb{Z}_p \subset \mathbb{Q}_p$ is a compact neighborhood of 0.

Definition 1.3. A *Haar measure* on a locally compact group is a (non-zero) Radon measure λ on G which is invariant under left translation, in the sense that for all $g \in G$, and all Borel set $A \subset G$, $\lambda(gA) = \lambda(A)$.

We recall that a Radon measure is a Borel measure which is finite on compact sets and *regular*, meaning that for any Borel set A,

$$\lambda(A) = \sup\{\lambda(K) \mid K \subset A \text{ compact}\} = \inf\{\lambda(U) \mid U \text{ open set containing } A\}.$$

Example 1.4. If $G = (\mathbb{R}, +)$, the Lebesgue measure is a Haar measure. The counting measure on a discrete group is a Haar measure.

The following observation follows from standard considerations on measurable functions.

Lemma 1.5. *A Radon measure λ on a locally compact group is a Haar measure if and only if for every $f \in C_c(G)$ and every $g \in G$, we have $\int_G f(x)d\lambda(x) = \int_G f(gx)d\lambda(x)$.*

Theorem 1.6. *If G is a locally compact group it always admits a Haar measure. Moreover, any two Haar measures on G are proportional.*

We will not prove the existence part in this theorem because, one, it is rather long to do and not much more instructive than the usual construction of the Lebesgue measure, and two, because for our examples a Haar measure can often be found by more concrete methods. For example, one can easily construct a left invariant volume form on a Lie group: just pick an n-form on the tangent space at the identity of the Lie group G (where $n = \dim(G)$) and propagate it using the left translations L_g, $g \in G$. We shall see how to concretely compute a Haar measure for $G = \mathrm{SL}_2(\mathbb{R})$. Moreover, for all discrete groups we already explained that the counting measure is a Haar measure.

PROOF. Let us prove the proportionality statement. Fix a Haar measure λ.

Claim 1. For all non-zero function $f \in C_c(G)$ such that $f \geq 0$ we have $\int_G f d\lambda \neq 0$.

To prove this claim, note that given such a function f, there exists $\varepsilon > 0$ such that $U := \{x \in G \mid f(x) \geq \varepsilon\}$ is non-empty, and of course we have $f \geq \varepsilon 1_U$. So it suffices to prove that $\lambda(U) > 0$ for any non-empty open set U in G. By assumption, λ is non-zero so there exists a Borel set A such that $\lambda(A) \neq 0$. Since λ is moreover regular we may actually find a compact subset $K \subset A$ such that $\lambda(K) \neq 0$.

If $U \subset G$ is a non-empty open set, the collection of translates gU, $g \in G$ is an open cover of G, and in particular, of K. By compactness of K, we may extract from it a finite sub-cover. In other words we find a finite set $F \subset G$ such that $K \subset \cup_{g \in F} gU$. But this leads to an inequality on measures:

$$0 < \lambda(K) \leq \sum_{g \in F} \lambda(gU) = |F|\lambda(U).$$

So we indeed arrive at the conclusion that $\lambda(U) \neq 0$, and Claim 1 follows.

Fix two non-zero functions $f, g \in C_c(G)$, such that $f, g \geq 0$. For notational simplicity, we denote by $\lambda(f) := \int_X f \, \mathrm{d}\lambda$ and $\lambda(g) = \int_X g \, \mathrm{d}\lambda$.

Claim 2. The ratio $\lambda(f)/\lambda(g)$ makes sense thanks to Claim 1. It does not depend on λ.

Assume that μ is another Haar measure on G. We consider the function $h : G \times G \to \mathbb{R}$ given by the formula

$$h(x, y) = f(x)a(x)g(yx), \text{ where } a(x) = (\int_G g(tx) \, \mathrm{d}\mu(t))^{-1}.$$

This is easily seen to be a compactly supported continuous function. Moreover, our choice of normalization by $a(x)$ gives $\int_G h(x, y) d\mu(y) = f(x)$ for all $x \in G$. Therefore,

$$\int_G \int_G h(x, y) \, \mathrm{d}\mu(y) \, \mathrm{d}\lambda(x) = \lambda(f).$$

On the other hand, since h is compactly supported and continuous, it is integrable, and Fubini theorem applies. Combining it with the fact that λ and μ are left invariant we get

$$\begin{aligned}
\int_{G \times G} h(x, y) \, \mathrm{d}\mu(y) \, \mathrm{d}\lambda(x) &= \int_{G \times G} f(x)a(x)g(yx) \, \mathrm{d}\lambda(x) \, \mathrm{d}\mu(y) \\
&= \int_{G \times G} f(y^{-1}x)a(y^{-1}x)g(x) \, \mathrm{d}\lambda(x) \, \mathrm{d}\mu(y) \\
&= \int_{G \times G} f(y^{-1})a(y^{-1})g(x) \, \mathrm{d}\mu(y) \, \mathrm{d}\lambda(x).
\end{aligned}$$

So we arrive at

$$\lambda(f) = \int_G g(x) \, \mathrm{d}\lambda(x) \int_G f(y^{-1})a(y^{-1}) \, \mathrm{d}\mu(y) = C\lambda(g),$$

where $C = \int_G f(y^{-1})a(y^{-1}) \, \mathrm{d}\mu(y)$, which does not depend on λ. This proves Claim 2.

Now if λ' is another Haar measure, we get for any two non-negative, non-zero functions $f, g \in C_c(G)$,

$$\lambda(f) = \frac{\lambda(g)}{\lambda'(g)} \lambda'(f).$$

Fixing g once and for all and setting $\alpha := \lambda(g)/\lambda'(g)$, we arrive at $\lambda(f) = \alpha\lambda'(f)$ for every $f \in C_c(G)$. So λ and $\alpha\lambda'$ coincide as linear functionals on $C_c(G)$, which implies that these two Radon measures are equal. $\square$

Example 1.7. Consider the group $P := \{M(a,b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$, where

$$M(a,b) = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}.$$

This group is parametrized by $\mathbb{R}^* \times \mathbb{R}$. The Haar measure on this group is given by $da\,db/a^2$. One easily checks that indeed this formula defines an invariant measure. Observe that if we only allow positive a's then the group that we obtain acts transitively by homography on the upper half plane, and the stabilizer of every point is trivial. In fact the Haar measure that we gave above is exactly the one coming from the usual hyperbolic metric on the upper half plane (for which the group actually acts by isometries).

Example 1.8. The Haar measure on $G := \mathrm{SL}_2(\mathbb{R})$ can be described as follows. Observe that every element of G can be uniquely written as a product hk, where $h \in P^0 := \{M(a,b) \mid a \in \mathbb{R}^*_+, b \in \mathbb{R}\}$ and $k \in K := \mathrm{SO}(2)$. This follow for instance by considering the action by homography of G on the upper half plane. The action is transitive and the stabilizer of the point i is K. So the decomposition follows from Example 1.7.

More explicitly, for any $g \in G$, we can find $a > 0$, $b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$ such that

$$g = M(a,b) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

With this parametrization, one checks that the measure $\frac{1}{a^2} d\theta\, da\, db$ is a Haar measure on G.

Let's get back to the general setting where G is an arbitrary locally compact group. Then a (left) Haar measure on G as we defined above needs not be right invariant, that is, $\lambda(Ag)$ needs not be equal to $\lambda(A)$ for all $g \in G$, $A \subset G$. But there is a nice way to measure this defect.

Fixing $g \in G$, the Radon measure defined by $A \mapsto \lambda(Ag)$ is again left invariant, because $h(Ag) = (hA)g$ for all Borel set A. So it is again a Haar measure and by the previous theorem, it follows that there exists a constant $\Delta(g) \in \mathbb{R}^*_+$, depending on g such that

$$\Delta(g)\lambda(Ag) = \lambda(A), \text{ for every set } A \subset G.$$

Applying the above formula to Ag^{-1} gives $\Delta(g)\lambda(A) = \lambda(Ag^{-1})$, and thus we see that $\Delta(g)$ is characterized by the formula

$$\int_G f(xg)\,\mathrm{d}\lambda(x) = \Delta(g) \int_G f(x)\,\mathrm{d}\lambda(x), \text{ for all } f \in C_c(G).$$

This equation shows that Δ is a continuous map from G to $\mathbb{R}^*_+$, and it is readily seen that it is in fact a group homomorphism. Moreover, It does not depend on a choice of the Haar measure λ.

Definition 1.9. The group homomorphism $\Delta : G \to R^*_+$ is called the *modular function* on the locally compact group G. G is called *unimodular* if this homomorphism is trivial.

Example 1.10. We make the following observations.

- Discrete groups are obviously unimodular since the counting measure is clearly both left and right invariant.

- Compact groups are unimodular. To see this observe that any Haar measure λ on such a group G is finite. Thus we have $\Delta(g)\lambda(G) = \lambda(Gg^{-1}) = \lambda(G)$ for all $g \in G$, showing that Δ has constant value 1.
- Since the modular function is a character $G \to R_+^*$, any group that does not admit a character is unimodular. For instance simple groups are unimodular. This also gives another proof that compact groups are unimodular.

Remark 1.11. One should be careful that unimodularity doesn't pass to subgroups. In particular, the restriction of the modular function Δ_G of a group G to a subgroup H needs not be the modular function Δ_H of the subgroup.

For example, we deduce from the previous exemple that $\mathrm{SL}_2(\mathbb{R})$ is unimodular, since it has no character. On the other hand it contains the subgroup $H := \{M(a,b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$, where

$$M(a,b) = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}.$$

But one verifies that the modular function on H is given by $\Delta_H(M(a,b)) = a^2$, which is non-trivial.

2. Lattices in locally compact groups

Definition 1.12. If G is a topological group, a *discrete* subgroup $\Gamma < G$ is a subgroup which is discrete in G for the induced topology. This amounts to saying that there is a neighborhood $U \subset G$ of the trivial element $e \in G$ such that $U \cap \Gamma = \{e\}$.

We will restrict our attention to locally compact, second countable groups (i.e. those admitting a countable basis of open sets). We write lcsc for short.

Lemma 1.13. *For any discrete group Γ in a lcsc group there always exists a Borel fondamental domain, i.e. a Borel subset $\mathcal{F} \subset G$ such that $\mathcal{F}\Gamma = G$ and $\mathcal{F}g \cap \mathcal{F}h = \emptyset$ for all distinct elements $g, h \in \Gamma$.*

Proof. Fix a neighborhood $V \subset G$ of e such that $V \cap \Gamma = \{e\}$, and pick a neighborhood $U \subset G$ of e such that $U^{-1}U \subset V$. This is possible because the map $(g,h) \mapsto g^{-1}h$ is continuous on $G \times G$. Since G is second countable, there exists a countable set of elements $(g_n)_{n\geq 1}$ in G such that $G = \bigcup_{n\geq 1} g_n U$.

Define inductively a sequence of Borel sets $\mathcal{F}_n \subset G$, $n \geq 1$ as follows. Set $\mathcal{F}_1 = g_1 U$ and

$$\mathcal{F}_{n+1} = g_{n+1}U \setminus \left(g_{n+1}U \cap \bigcup_{k=1}^{n} g_k U\Gamma \right).$$

These are disjoint sets and better, for every $n \neq m$ we have $\mathcal{F}_n\Gamma \cap \mathcal{F}_m\Gamma = \emptyset$, while

$$\bigcup_{k=1}^{n} \mathcal{F}_k\Gamma = \bigcup_{k=1}^{n} g_k U\Gamma.$$

So the set $\mathcal{F} = \bigcup_n \mathcal{F}_n$ is a Borel set such that $\mathcal{F}\Gamma = G$. Assume now that $g, h \in \Gamma$ are two elements such that $\mathcal{F}g \cap \mathcal{F}h \neq \emptyset$. There exist two indices such that $\mathcal{F}_n g \cap \mathcal{F}_m h \neq \emptyset$. By construction this forces n and m to be equal. Since $\mathcal{F}_n \subset g_n U$ we can then find $x, y \in U$ such that $g_n x g = g_n y h$, which leads to $x^{-1}y = gh^{-1} \in V \cap \Gamma = \{e\}$. So we conclude that $g = h$.

Thus $\mathcal{F}$ is indeed a Borel fundamental domain. $\square$

DEFINITION 1.14. A discrete subgroup Γ in a locally compact group G is called a *lattice* if it admits a Borel fundamental domain with finite Haar measure.

EXAMPLE 1.15. It is trivial to see that $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$, for which a fundamental domain is given by $[-1,1[^n$.
A less trivial example is that of $\mathrm{SL}_2(\mathbb{Z})$ inside $\mathrm{SL}_2(\mathbb{R})$. To see that this is indeed a lattice, one can use the fact that the action of $\mathrm{SL}_2(\mathbb{Z})$ on the hyperbolic half-plane admits a Borel fundamental domain with finite measure. We leave the details as an exercise. In the same spirit, the fundamental group of any compact surface with negative curvature embeds as a lattice inside $\mathrm{PSL}_2(\mathbb{R})$.

LEMMA 1.16. *Let Γ be a discrete subgroup in a lcsc group G. Then any two Borel fondamental domains for Γ have the same right-Haar measure.*

PROOF. Let $\mathcal{F}$ and $\mathcal{F}'$ be two fundamental domains and denote by λ a right-Haar measure on G. Then we have

$$\lambda(\mathcal{F}) = \sum_{g\in\Gamma} \lambda(\mathcal{F}\cap\mathcal{F}'g) = \sum_{g\in\Gamma} \lambda(\mathcal{F}g^{-1}\cap\mathcal{F}') = \lambda(\mathcal{F}').$$

We used implicitly the fact that Γ is countable, which is automatic if it is discrete and G is lcsc. □

COROLLARY 1.17. *If G is a lcsc group admitting a lattice Γ then it is unimodular. Moreover, there exists a G-invariant Borel probability measure on G/Γ.*

PROOF. Let λ be a right Haar measure on G, and denote by $\mathcal{F}$ a fundamental domain for Γ. Then for every $g\in G$, $g\mathcal{F}$ is also a fundamental domain for Γ. So $\lambda(\mathcal{F}) = \lambda(g\mathcal{F})$, hence λ is left invariant, which implies that G is unimodular.

Moreover, let us restrict the projection map $p: G\to G/\Gamma$ to the Borel subset $\mathcal{F}$. Then the restriction of λ to $\mathcal{F}$ is a finite measure (and we may normalize it so that it is a probability measure). Its push forward on G/Γ is also a finite measure $\nu_{\mathcal{F}}$ which depends a priori on $\mathcal{F}$. Let us check that this is not the case. Take another fundamental domain $\mathcal{F}'$. Take $A\subset G/\Gamma$, and denote by $B := p^{-1}(A)$. We have

$$\nu_{\mathcal{F}}(A) = \lambda(B\cap\mathcal{F}) = \sum_{g\in\Gamma} \lambda(B\cap\mathcal{F}\cap\mathcal{F}'g).$$

In the last inequality above, we used the fact that G is the disjoint union of the sets $\mathcal{F}g$, $g\in\Gamma$. Now since the measure λ is right-invariant and since B is globally right Γ-invariant, we further find

$$\nu_{\mathcal{F}}(A) = \sum_{g\in\Gamma} \lambda(B\cap\mathcal{F}g^{-1}\cap\mathcal{F}') = \lambda(B\cap\mathcal{F}') = \nu_{\mathcal{F}'}(A).$$

Now observe that for $g\in G$, the left translation by g maps the fundamental domain $\mathcal{F}$ to $\mathcal{F}' = g\mathcal{F}$, and since it preserves the Haar measure λ, we find that $g_*\nu_{\mathcal{F}} = \nu_{g\mathcal{F}}$. Since $\nu_{\mathcal{F}}$ does not depend on $\mathcal{F}$, we get that it is g-invariant. □

PROPOSITION 1.18. *Let G be a unimodular lcsc group and take a discrete subgroup Γ in G. The following facts are equivalent.*

(i) Γ *is a lattice in* G*;*
(ii) There exists a Borel set $\Omega\subset G$ with finite right Haar measure such that $\Omega\Gamma = G$;
(iii) There exists a G-invariant Borel probability measure on G/Γ.

Proof. The implication $(i) \Rightarrow (ii)$ is tautologic and its converse is proved similarly to Lemma 1.13 (one just needs to intersect all the sets with Ω all along the proof of that lemma). The previous corollary showed $(i) \Rightarrow (iii)$. Let us prove the converse. This is where we will use the unimodularity assumption.

Denote by $\nu \in \mathrm{Prob}(G/\Gamma)$ a G-invariant probability measure. The choice of a Borel fundamental domain $\mathcal{F}$ for Γ gives rise to an isomorphism of measure spaces

$$G/\Gamma \times \Gamma \simeq \mathcal{F} \times \Gamma \simeq G.$$

We only need to check that the measure $\nu \times c$, where c is the counting measure on Γ, is transported to a left invariant measure on G. Once we do this, we will conclude that the fundamental domain $\mathcal{F} \subset G$ has finite left Haar measure, and hence finite right Haar measure by unimodularity.

Exercise. Prove that the above isomorphism transports the left action of G on itself to the action of G on $G/\Gamma \times \Gamma$ given by the formula $g(x, \gamma) = (gx, c(g, x)\gamma)$, for all $g \in G$, $x \in G/\Gamma$, $\gamma \in \Gamma$, where $c(g, x)$ is the element such that $gh \in \mathcal{F}c(g, x)$, where $h \in \mathcal{F}$ is the representative of $x \in \mathcal{F}$. Prove that the measure $\nu \times c$ is invariant under this action. □

3. Some extra examples

A sufficient condition to be a lattice is to be co-compact.

Definition 1.19. A closed subgroup H in a lcsc group G is said to be *co-compact* if the quotient G/H is compact for the quotient topology.

Lemma 1.20. *If Γ is a co-compact discrete subgroup of G, then it is a lattice.*

Proof. As we already mentioned, the quotient map $p : G \to G/\Gamma$ is open. Let U be a non-empty open set in G with compact closure. Then we may cover G by left translates of U. Then the family $p(gU)$, $g \in G$ is an open cover of G/Γ. So if Γ is co-compact in G, we may find a finite set $F \subset \Gamma$ such that $G/\Gamma = \bigcup_{g \in F} p(gU)$. The subset $\Omega := \bigcup_{g \in F} gU$ has compact closure (so it must have finite measure) and satisfies $\Omega\Gamma = G$. We conclude that Γ is a lattice in G. □

In the exercise sheets it is explained how to prove that $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$.

Example 1.21. $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ is not co-compact in $G = \mathrm{SL}_n(\mathbb{R})$. Here is why.

- The unipotent element $\gamma = \mathrm{id} + E_{1,2} \in \Gamma$ and the sequence of diagonal matrices $g_n = \mathrm{diag}(1/n, n, 1, \dots, 1) \in G$ satisfy $\lim_n g_n \gamma g_n^{-1} = \mathrm{id}$ (also denoted by e).
- Assume by contradiction that G/Γ is compact, and thus, that some subsequence $g_{n_k}\Gamma$ converges to an element $h\Gamma \in G/\Gamma$. Then we may write $g_{n_k} = h_k \gamma_k$ with $h_k \in G$, $\gamma_k \in \Gamma$ and $\lim_k h_k = h \in G$.
- Then $\gamma_k \gamma \gamma_k^{-1} = h_k^{-1}(g_{n_k} \gamma g_{n_k}^{-1}) h_k$ converges to $h^{-1} e h = e$. But this is a sequence of elements in Γ. Since Γ is discrete in G, this forces $\gamma_k \gamma \gamma_k^{-1} = e$ for k large enough, and hence $\gamma = e$. This is absurd.

Nevertheless, any simple Lie group, such as $\mathrm{SL}_n(\mathbb{R})$, has plenty of co-compact and non co-compact lattices. The main construction is of arithmetic nature. More precisely, Borel and Harish-Chandra proved that if $\mathbb{G}$ is an algebraic group defined over $\mathbb{Q}$, that admits no $\mathbb{Q}$-character, then its set of integer points $\mathbb{G}(\mathbb{Z})$ is a lattice in its set of real points (which is a Lie group). We will not give details, but some of the definitions will be given in the chapter on superrigidity.

Moreover, a criterion of Godement says exactly when $\mathbb{G}(\mathbb{Z})$ is co-compact in $\mathbb{G}(\mathbb{R})$: it is exactly when $\mathbb{G}(\mathbb{Z})$ does not have unipotent element. One direction is easy and follows essentially the argument that we used to show that $\mathrm{SL}_n(\mathbb{Z})$ is not co-compact.

Example 1.22. We admit the following examples.

(1) Denote by $q : \mathbb{R}^n \to \mathbb{R}$ a non-degenerate quadratic form with rational entries. Then $\mathrm{SO}(q, \mathbb{Z})$ is a lattice in $\mathrm{SO}(q, \mathbb{R})$.
(2) $\mathbb{Z}[\sqrt{2}]$ embeds into $\mathbb{R}$ in two ways. This gives two embeddings of $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}])$ in $\mathrm{SL}_n(\mathbb{R})$. Then the diagonal embedding $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}]) \subset \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ gives a lattice. This case follows from restriction of scalars.
(3) The theorem of Borel and Harish Chandra also generalizes to the more general framework of *S-adic algebraic integers*. This allows to say that the diagonal embedding of $\mathrm{SL}_n(\mathbb{Z}[1/p])$ into $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$ is a lattice.

Note that if $\Gamma_1 < G_1$ and $\Gamma_2 < G_2$ are lattices, then $\Gamma_1 \times \Gamma_2 < G_1 \times G_2$ is also a lattice. This case is called *reducible*.

Definition 1.23. A lattice Γ in a product of groups $\prod_{i=1}^n G_i$ is said to be *irreducible* if its projection on a sub-product $\prod_{i \in J} G_i$ has dense image as soon as $J \subsetneq \{1, n\}$ is a proper subset.

For example, the examples (2) and (3) given above are irreducible.

CHAPTER 2

Some facts on Lie groups

1. Basic definitions and examples

1.1. Lie groups.

Definition 2.1. A *Lie group* is a smooth manifold G over $\mathbb{R}$ which admits a group structure such that the corresponding structure map $m : (x, y) \in G \times G \mapsto xy^{-1} \in G$ is smooth. A *morphism* between two Lie groups will be by definition a smooth group homomorphism.

We will only focus on *real* Lie groups. But many results from these notes also hold for analytic Lie groups over $\mathbb{Q}_p$.

Among the first obvious examples we can think of, the groups $\mathbb{R}^n$ and its quotient $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are Lie groups. More interestingly, the multiplicative group $\mathrm{GL}_n(\mathbb{R})$ is an open set inside $M_n(k)$. As such, it may be endowed with the corresponding smooth structure. It is of dimension n^2 as a manifold. The other examples we will describe are actually realized as closed subgroups of $\mathrm{GL}_n(\mathbb{R})$.

Example 2.2. It is a theorem of Cartan and von Neumann that a closed subgroup H of a Lie group G is a Lie group (and naturally, the manifold structure on H is so that the inclusion $H \subset G$ is a smooth embedding). So all the following classical examples are clearly Lie groups (but this can also be checked using the submersion theorem).

- The **Heisenberg group** is the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of upper triangular matrices with 1's on the diagonal. It is a *nilpotent* Lie group. The larger group of all upper triangular matrices is also a Lie group, and it is *solvable*.
- The **special linear group** $\mathrm{SL}_n(\mathbb{R})$, consisting of matrices with determinant 1.
- The **orthogonal group** $O(p, q)$, associated to a non-degenerate quadratic form of signature (p, q). We will denote by $\mathrm{SO}(p, q)$ the intersection of $O(p, q)$ with $\mathrm{SL}_n(\mathbb{R})$, and we use the notation $O(n) = O(n, 0)$ and $\mathrm{SO}(n) = \mathrm{SO}(n, 0)$.
- The isometry group $\mathrm{SO}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is a Lie group (endowed with the product structure as a manifold). It is the Lie group of orientation preserving isometries of $\mathbb{R}^n$. It can be realized as a subgroup of $\mathrm{SL}_{n+1}(\mathbb{R})$:
$$\mathrm{SO}(n, \mathbb{R}) \ltimes \mathbb{R}^n \simeq \left\{ \begin{pmatrix} A & u \\ 0 & 1 \end{pmatrix} \,\middle|\, A \in \mathrm{SO}(n, \mathbb{R}), u \in \mathbb{R}^n \right\}.$$

More generally we can define unitary and symplectic analogues to orthogonal groups.

1.2. The Lie algebra of a Lie group. The advantage with Lie groups is that they come with a so-called Lie algebra, giving all the tools from linear algebra to study them.

Definition 2.3. A *Lie algebra* is a finite dimensional vector space V, endowed with a bilinear map $[\cdot, \cdot] : V \times V \to V$ (the so-called *Lie bracket*) which satisfies the two axioms:

- $[X, Y] = -[Y, X]$ for all $X, Y \in V$ (anti-symmetry);

- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in V$ (Jacobi identity).

In order to define the Lie algebra of a Lie group G we introduce some terminology. Recall that a *vector field* on the manifold G is a derivation of the algebra $C^\infty(G)$, that is, a linear map $X : C^\infty(G) \to C^\infty(G)$ such that $X(f_1 f_2) = f_1 X(f_2) + f_2 X(f_1)$, for all $f_1, f_2 \in C^\infty(G)$. Besides, for any $g \in G$, the translation map $L_g : h \in G \mapsto gh \in G$ induces a map $\tau_g : f \mapsto f \circ L_g$ on $C^\infty(G)$. A vector field X on G is called *left-invariant* if it commutes with τ_g for all $g \in G$, i.e. $X(f \circ L_g) = (X(f)) \circ L_g$, for all $f \in C^\infty(G)$. Note that this formula is equivalent to

$$(X(f))(g) = (X(f \circ L_g))(e) \text{ for all } g \in G.$$

We write $X_g(f)$ instead of $X(f)(g)$. We denote by $\mathcal{L}(G)$ the vector space of all left-invariant derivations of $C^\infty(G)$. Then the map $X \in \mathcal{L}(G) \mapsto X_e \in T_e(G)$ is a linear isomorphism from $\mathcal{L}(G)$ into the tangent space to G at e. In particular $\mathcal{L}(G)$ is finite dimensional.

Exercise 2.4. Given two left invariant vector fields X, Y on $C^\infty(G)$, we may define their composition $XY : C^\infty(G) \to C^\infty(G)$. While this map still commutes with τ_g, $g \in G$, it is no longer a derivation in general.

- However, check that $[X, Y] := XY - YX$ is again a left invariant derivation.
- Check that $(\mathcal{L}(G), [\cdot, \cdot])$ is a Lie algebra.

Definition 2.5. The Lie algebra of left invariant derivations on $C^\infty(G)$ with the above bracket is called the *Lie algebra of* G. It is denoted by $\mathfrak{g}$.

Example 2.6. The Lie algebra of $G = \mathrm{GL}_n(k)$ is $M_n(k)$, endowed with the Lie bracket

$$[X, Y] = XY - YX.$$

Let us indicate how to prove this fact. First observe that since $\mathrm{GL}_n(k)$ is open inside $M_n(k)$, the tangent space at every point is naturally identified with $M_n(k)$. Now given a vector $X \in M_n(k)$, we view it as a left-invariant vector field by the formula $g \in G \mapsto gX \in M_n(k) \simeq T_g G$. Its effect on $C^\infty(G)$ is given by $X(f)(g) = (df)_g(gX)$ for all $f \in C^\infty(G)$, $g \in G$. Differentiating further, we see that for all $X, Y \in M_n(k)$, $Y(X(f))(g) = (d^2 f)_g(gX, gY) + (df)_g(gYX)$. Since the second derivative $d^2(f)_g$ is a symmetric bilinear form, we get $(Y(X(f))(g) - X(Y)(f))(g) = (df)_g(gXY) - (df)_g(gYX) = (XY - YX)(f)(g)$, as desired.

Now recall that for any sub-manifold $N \subset M$ defined by a submersion $\phi : M \to M'$ as $N = \phi^{-1}(\{x\})$, the tangent space of N at a point is just the kernel of the derivative of ϕ at that point. In particular we may compute the Lie algebras of all the standard examples mentioned above.

Example 2.7. We have the following computations. We only describe the underlying vector space, because the Lie bracket is simply the restriction of the Lie bracket on $M_n(k)$.

- The Lie algebra of the Heisenberg group consists of all upper triangular matrices, with 0's one the diagonal.
- Since the derivative of the determinant map is the trace, the Lie algebra of $\mathrm{SL}_n(k)$ is the vector space of trace 0 matrices in $M_n(k)$;
- The Lie algebra of $O(n)$ is the subspace of matrices X such that $X^T + X = 0$ (anti-symmetric matrices). More generally the Lie algebra of $O(p, q)$ is the subspace of matrices such that $I_{p,q} X^T I_{p,q} + X = 0$, where $I_{p,q}$ is the diagonal matrix $I_{p,q} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, where 1 appears p times and -1 appears q times.

1.3. Functoriality. The following lemma asserts that taking the Lie algebra of a Lie group is a functor.

LEMMA 2.8. *Consider two Lie groups G and H, with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\phi : G \to H$ is a differentiable homomorphism then its derivative $d\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

PROOF. Here we need to define properly the derivative $d\phi$. Note that ϕ defines by pre-composition an algebra homomorphism $\phi_* : f \in C^\infty(H) \mapsto f \circ \phi \in C^\infty(G)$. Then if $X \in \mathfrak{g}$ is a left invariant derivation, $X(f \circ \phi)$ is again in $C^\infty(G)$ and we would like to check that it is of the form $X'(f) \circ \phi$, and ensure that the map $X' : C^\infty(H) \to C^\infty(H)$ makes the following diagram commute.

$$\begin{array}{ccc} C^\infty(H) & \xrightarrow{\phi_*} & C^\infty(G) \\ {\scriptstyle X'}\downarrow & & \downarrow{\scriptstyle X} \\ C^\infty(H) & \xrightarrow{\phi_*} & C^\infty(G) \end{array}$$

For every $f \in C^\infty(H)$ and $g \in G$, we have $X(f \circ \phi)(g) = X(f \circ \phi \circ L_g)(e)$. Moreover, since ϕ is a group homomorphism, we have $\phi \circ L_g = L_{\phi(g)} \circ \phi$. So,

$$X(f \circ \phi)(g) = X(f \circ L_{\phi(g)} \circ \phi)(e) = X'(f)(\phi(g)), \tag{1.1}$$

where X' is the derivation on H defined by the formula $X'(f) : h \in H \mapsto X(f \circ L_h \circ \phi)(e)$. It is checked to be a left invariant derivation. Then we see that $d\phi : X \mapsto X'$ is a linear map between the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ making the above diagram commute.

We now check that this is in fact a Lie algebra homomorphism. Take $X, Y \in \mathfrak{g}$. Denote by $X' := d\phi(X)$ and $Y' := d\phi(Y)$. For all $f \in C^\infty(H)$ and $g \in G$, equation (1.1) gives

$$X'(Y'(f))(\phi(g)) = X(Y'(f) \circ \phi)(g) = X(Y(f \circ \phi))(g).$$

This immediately gives the computation

$$(X'Y' - Y'X')(f)(\phi(g)) = (XY - YX)(f \circ \phi)(g) = [(d\phi)(XY - YX)](f)(\phi(g)).$$

So the vector fields $X'Y' - Y'X'$ and $(d\phi)(XY - YX)$ coincide on $\phi(G)$ and in particular at e. Since they are both left invariant, they must coincide, which proves that $d\phi$ is a Lie algebra homomorphism. □

In the above proof, observe that the definition of $d\phi$ that we gave coincides with the classical differential $(d\phi)_e : T_eG \to T_eH$ when we identify the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ with T_eG and T_eH as vector spaces. Using the structure theorems for sub-immersions one can prove the following property.

PROPOSITION 2.9. *Consider two Lie groups G and H and a Lie group homomorphism $\phi : G \to H$. Then*

(1) The kernel of ϕ is a closed Lie subgroup of G. Its Lie algebra is the kernel of $d\phi$;
(2) The image of ϕ is a (not necessarily closed) Lie subgroup of H; its Lie algebra is the image of $d\phi$;
(3) If $K \subset H$ is a closed subgroup of H with Lie algebra $\mathfrak{k}$, then $\phi^{-1}(K)$ is a closed subgroup of G whose Lie algebra is $(d\phi)^{-1}(\mathfrak{k})$.

EXAMPLE 2.10. Pick a point in $a \in \mathbb{R}^2$ which is not a multiple of a point in $\mathbb{Q}^2$ e.g. $a = (1, \sqrt{2})$. Then the image of the homomorphism $t \in \mathbb{R} \mapsto ta \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a Lie subgroup which is dense in $\mathbb{T}^2$ (hence not closed).

Item (3) above implies the following fact.

COROLLARY 2.11. *If $\pi : G \to \mathrm{GL}(V)$ is a smooth representation of a connected Lie group G on a finite dimensional vector space V, then a subspace $W \subset V$ is globally invariant under $\pi(G)$ if and only if it is invariant under $(d\pi)(\mathfrak{g})$.*

PROOF. The computation of Example 2.6 gives that $H = \mathrm{GL}(V)$ is a Lie group with Lie algebra $\mathfrak{h} = \mathrm{End}(V)$. The subgroup $K := \{g \in \mathrm{GL}(V) \mid g(W) = W\}$ is a closed subgroup with Lie algebra $\mathfrak{k} = \{X \in \mathrm{End}(V) \mid X(W) \subset W\}$.

If W is $\pi(G)$ invariant, then $\pi^{-1}(K) = G$ and Proposition 2.9.(3) implies that $(d\pi)^{-1}(\mathfrak{k}) = \mathfrak{g}$, which means exactly that W is invariant under $d\pi(\mathfrak{g})$. Conversely, if W is invariant under $d\pi(\mathfrak{g})$, then Proposition 2.9.(3) implies that $\pi^{-1}(K)$ has Lie algebra $\mathfrak{g}$. So the subgroup $\pi^1(K)$ of G has the same dimension as G. This implies that it is open in G, hence it is also closed in G, and by connectedness, it is equal to G. This means that W is G invariant. □

1.4. How much the Lie algebra remembers. We want to know to what extend groups with the same Lie algebra are the same. Obviously, a Lie group and its identity connected component have the same Lie algebra, because the latter is open inside the former and the Lie algebra is a local data. But even in the connected setting, there exist different Lie groups with the same Lie algebra. For instance this is the case of $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$. In the same spirit $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ have the same Lie algebra.

LEMMA 2.12. *The universal cover $\widetilde{G}$ of a connected Lie group G is naturally endowed with a Lie group structure such that the covering map $\widetilde{G} \to G$ is a Lie group homomorphism. Its kernel is contained in the center of $\widetilde{G}$.*

PROOF. Denote by $\pi : \widetilde{G} \to G$ the covering map. Fix a lift e of the identity element e_G. We define the product on $\widetilde{G}$ as follows. For $g, h \in \widetilde{G}$, choose paths $t \mapsto g_t, h_t \in \widetilde{G}$ between e and g, h: $g_0 = h_0 = e$, $g_1 = g$, $h_1 = h$. The product path $t \mapsto \pi(g_t)\pi(h_t)$ in G is a path between e_G and $\pi(g)\pi(h)$. Lift it to a path γ inside $\widetilde{G}$ starting at e. We set $gh := \gamma(1)$. One checks that this definition is independent of the choices of paths that we made.

It is then easy to verify that this product law is associative, that e is a neutral element, and that the inverse g^{-1} of g is the end point of a lift starting at e of the path $t \mapsto \pi(g_t)^{-1}$. Moreover the covering map π is clearly a group homomorphism. Since it is also a local homeomorphism, we may define the analytic structure of $\widetilde{G}$ by declaring that π is locally analytic. The fact that G is a Lie group implies that the structure map $(g, h) \mapsto gh^{-1}$ is analytic.

Finally, take $g, h \in \widetilde{G}$, with $\pi(g) = e$. Take a path $t \mapsto g_t$ from e to g such that $g_t = g$ for all $t \geq 1/2$ and a path $t \mapsto h_t$ from e to h such that $h_t = e$ for all $e \leq 1/2$. We have:

- $\pi(g_t)\pi(h_t) = \pi(g_t) = \pi(h_t)\pi(g_t)$ if $t \leq 1/2$;
- $\pi(g_t)\pi(h_t) = \pi(g)\pi(h_t) = \pi(h_t) = \pi(h_t)\pi(g_t)$ if $t \geq 1/2$.

So $gh = hg$, as desired. □

A connected Lie group is always locally isomorphic to its universal cover in the following sense.

Definition 2.13. Two Lie groups G and H are said to be *locally isomorphic* if there exist open neighborhoods $U \subset G$ and $V \subset H$ of the identity elements e_G and e_H and an analytic diffeomorphism φ from U onto V such that $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in U$ such that $gh \in U$.

Theorem 2.14. *Consider two Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Denote by $\widetilde{G}$ and $\widetilde{H}$ the universal covers of the identity components of G and H respectively. The following are equivalent.*

(i) G and H are locally isomorphic;
(ii) $\widetilde{G}$ and $\widetilde{H}$ are isomorphic;
(iii) $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.

Proof. To prove that $(i) \Leftrightarrow (ii)$ first observe that being locally isomorphic is an equivalence relation and that G (resp. H) is locally isomorphic with $\widetilde{G}$ (resp. $\widetilde{H}$). Then standard topological considerations show that two simply connected groups are locally isomorphic if and only if they are isomorphic.

The implication $(ii) \Rightarrow (iii)$ follows from Lemma 2.8. The hard part is the converse, which we admit. □

In particular if two simply connected Lie groups have the same Lie algebra, they must be isomorphic. There is also another situation where we can avoid the covering noise.

Corollary 2.15. *If G, H are connected groups with trivial center and isomorphic Lie algebras, then they are isomorphic.*

Proof. Theorem 2.14 implies that G and H have isomorphic universal cover : $\widetilde{G} \simeq \widetilde{H}$. But recall that the covering map $p : \widetilde{G} \to G$ is a group homomorphism whose kernel is contained in the center of $\widetilde{G}$. Since G has trivial center, we conclude that the kernel of p is exactly the center of $\widetilde{G}$ (which thus must be discrete in $\widetilde{G}$). Therefore $G \simeq \widetilde{G}/Z(\widetilde{G})$. Likewise $H \simeq \widetilde{H}/Z(\widetilde{H})$, which implies that $G \simeq H$. □

1.5. The adjoint representation. There are in fact two adjoint representations.

Lie group setting. Given a Lie group G, we may define for each $g \in G$ a smooth automorphism $I(g) : h \in G \mapsto ghg^{-1} \in G$. Such group automorphisms are called *inner automorphisms*. The differential of $I(g)$ at e is then an invertible endomorphism of the Lie algebra $\mathfrak{g}$ of G, denoted by $\mathrm{Ad}(g) \in \mathcal{L}(\mathfrak{g})$; it is even a Lie algebra automorphism. The mapping $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is then a linear representation of G, called the *adjoint representation*.

Lie algebra setting. A *derivation* of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that $D([X, Y]) = [DX, Y] + [X, DY]$ for all $X, Y \in \mathfrak{g}$. One checks that if D and D' are two derivations, then so is $DD' - D'D$. This operation turns the vector space $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ into a Lie algebra.

It follows from the Jacobi identity that for any $X \in \mathfrak{g}$, the endomorphism $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is actually a derivation, called an *inner derivation*. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is actually a Lie algebra homomorphism, called the *adjoint representation*. The term *representation* of a Lie algebra refers to a Lie algebra homomorphism from a given Lie algebra into the Lie algebra $\mathcal{L}(V)$ of all endomorphisms of a finite dimensional vector space V, endowed with the Lie bracket of $[X, Y] = XY - YX$. Since ad is a representation, its image is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$.

Lemma 2.16. *Take a Lie group G with Lie algebra $\mathfrak{g}$. The group $\mathrm{Aut}(\mathfrak{g})$ of all automorphisms of $\mathfrak{g}$ is a Lie group. Its Lie algebra is $\mathrm{Der}(\mathfrak{g})$. The derivative of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ of G is the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ of $\mathfrak{g}$.*

We admit this fact. It is based on properties of the exponential mapping.

Lemma 2.17. *The kernel of the adjoint representation of a connected Lie group is its center.*

Proof. Observe that the differential of the map $p : (g, h) \mapsto gh^{-1}$ at the point (e, e) is given by $dp_{(e,e,)}(X, Y) = X - Y$, for all $X, Y \in \mathfrak{g}$. By the chain rule, this implies that for a fixed $g \in G$, the derivative of the map $f : h \mapsto ghg^{-1}h^{-1} = p(I(g)(h), h)$ at e is given by $X \mapsto \mathrm{Ad}(g)(X) - X$.

If $g \in \mathrm{Ker(Ad)}$, then $\mathrm{Ad}(g)(X) = X$, for all $X \in \mathfrak{g}$. The above computation shows that the map f has zero derivative at e. Let us check that it has 0 derivative at any point $h_0 \in G$. We will use translation maps. Denote by $L_0 : x \in G \mapsto h_0x \in G$ the left multiplication map by h_0, and by $dL_0 : T_e(G) \to T_{h_0}(G)$ its derivative at e, which is a vector space isomorphism. The map $f \circ L_0 : G \to G$ is given by

$$f \circ L_0(x) = f(h_0x) = g(h_0x)g^{-1}(x^{-1}h_0^{-1}) = (gh_0g^{-1})f(x)h_0^{-1}.$$

The map $\psi : x \in G \mapsto (gh_0g^{-1})xh_0^{-1}$ is a diffeomorphism, so its derivative at any point is invertible. The above relation reads as $f \circ L_0 = \psi \circ f$. Derivating this equality at e gives

$$(df)_{h_0} \circ dL_0 = (d\psi)_e \circ (df)_e = 0.$$

So $(df)_{h_0}$ vanishes on the range of dL_0, i.e. on all the tangent space $T_{h_0}(G)$. We conclude that the derivative of f at any point is zero, which implies by connectedness of G that f is constant, equal to $f(e) = e$. So g lies in the center of G. This argument can clearly be reversed to show the converse inclusion. □

Corollary 2.18. *The Lie algebra of the center of a connected Lie group G is the center of its Lie algebra $\mathfrak{g}$, that is, the kernel of* ad.

Proof. This follows from combining the above two lemmas with Proposition 2.9. □

2. Semi-simple Lie groups

2.1. Solvable Lie groups. The definition of a nilpotent (resp. solvable) Lie group is as expected: it is a Lie group which is nilpotent (resp. solvable) as an abstract group.

Example 2.19. The key examples are the following ones:

- The group U_n of upper triangular matrices of size n, with 1's on the diagonal is nilpotent.
- The group P_n of all upper triangular matrices of size n is solvable.

Although we won't need it, let us mention the famous structure theorem of Lie. For a proof we refer to the book of Serre [**Ser06**].

Theorem 2.20 (Lie). *If G is a connected solvable Lie group and if $\pi : G \to \mathrm{GL}(n, \mathbb{R})$ is a continuous representation of G on $\mathbb{R}^n$, then $\pi(G)$ is conjugate to a subgroup of S_n.*

We will be interested in the solvable radical of a connected Lie group.

LEMMA 2.21. *If G is a connected Lie group, then it admits a greatest solvable normal connected Lie subgroup. It is a closed subgroup that we call the* solvable radical *of G, and denote by $R(G)$.*

PROOF. Denote by $\mathcal{S}$ the class of all normal connected and solvable subgroups of G. There are two observations about this class of groups :

- If $H, H' \in \mathcal{S}$ then the group HH' generated by H and H' is again connected and normal in G. Moreover it is solvable, because there is a short exact sequence
$$1 \to H \to HH' \to H'/H \cap H' \to 1,$$
where H and $H'/H \cap H'$ are both solvable.
- If $H \in \mathcal{S}$, then its closure is again in $\mathcal{S}$.

Let us take a closed subgroup $H \in \mathcal{S}$ of maximal dimension, and prove that $H' \in \mathcal{S}$ is contained in H. By the above observations, we may assume that H' is closed. Then $HH' \in \mathcal{S}$ and it is also a closed group. By maximality, it must have the same dimension as H. Thus H must be open in HH' (and hence closed) and by connectedness, we must have equality $H = HH'$. This forces $H' \subset H$, as wanted. □

Semi-simple Lie groups.

DEFINITION 2.22. A connected Lie group is called *semi-simple* if its solvable radical is trivial.

EXERCISE 2.23. Prove that a connected Lie group is semi-simple if and only if it has no non-trivial connected abelian normal subgroup.

There are many beautiful theorems about the structure of semi-simple Lie groups, essentially due to Cartan. We will not present them but any book on Lie groups contains this material (see e.g. [**Hal15**]). The following result is also based on a trick of Weyl, the so-called *unitary trick*.

THEOREM 2.24. *Let G be a connected semi-simple Lie group, and let $\pi : G \to \mathrm{GL}(V)$ be a continuous representation of G on a finite dimensional vector space V. Then any G-invariant subspace of V admits a complement which is also G-invariant. In particular π is the direct sum of irreducible representations.*

Using the above theorem we may decompose semi-simple groups in terms of simple groups.

DEFINITION 2.25. A connected Lie group G is said to be *simple* if it is non-abelian and has no non-trivial normal connected Lie subgroup.

COROLLARY 2.26. *Any connected semi-simple Lie group with trivial center is the product of finitely many simple groups. Such a product decomposition is unique (up to permutation of the factors).*

SKETCH OF PROOF. We consider the adjoint representation Ad of G on its Lie algebra $\mathfrak{g}$. By Theorem 2.24, this representation can be decomposed as the direct sum of finitely many irreducible G-invariant subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$. Then each $\mathfrak{g}_i$ is invariant under the derivative representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. This means that each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. Note moreover that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$ for all $i \neq j$. This means that the ideals $\mathfrak{g}_i$ commute to each other.

For each i, denote by $G_i < G$ connected component of the subgroup of elements that act trivially on every $\mathfrak{g}_j$, for $j \neq i$. Then using Lemma 2.9, we find that the Lie algebra of G_i is the subalgebra

$$\{X \in \mathfrak{g} \mid \mathrm{ad}(X)(Y) = 0, \text{ for all } Y \in \mathfrak{g}_j, j \neq i\} = \{X \in \mathfrak{g} \mid \mathrm{ad}(X)\mathfrak{g} \subset \mathfrak{g}_i\} = \mathfrak{g}_i.$$

Moreover, one easily checks using Lemma 2.17 that G_i has trivial center. Thus the product $G_1 \times \cdots \times G_n$ is a connected center-free group with Lie algebra $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$, it is isomorphic with G. Note finally that each G_i is a simple Lie group because its Lie algebra has no non-trivial proper ideal.

To prove the uniqueness of such a decomposition, one has to prove the uniqueness of the decomposition of $\mathfrak{g}$ as a direct sum of simple ideals. This reduces to proving that every simple ideal of $\mathfrak{g}$ is one of the $\mathfrak{g}_i$'s. But if $\mathfrak{a}$ is a simple ideal in $\mathfrak{g}$, then $[\mathfrak{a}, \mathfrak{g}_i] \subset \mathfrak{a} \cap \mathfrak{g}_i$. So there must exist some i such that $\mathfrak{a} \cap \mathfrak{g}_i \neq \{0\}$, otherwise $[\mathfrak{a}, \mathfrak{g}] = 0$, which would imply that $\mathfrak{a}$ is an abelian ideal, and would contradict the semi-simplicity of $\mathfrak{g}$. But $\mathfrak{a} \cap \mathfrak{g}_i$ is an ideal of $\mathfrak{a}$ and $\mathfrak{g}_i$ so if it is non-zero, then by simplicity we must have $\mathfrak{a} = \mathfrak{g}_i$. □

Example 2.27. Let us get back to our first examples.

- A solvable or nilpotent Lie group is never semi-simple. In particular the Heisenberg group is not semi-simple.
- The group $\mathrm{SL}_n(\mathbb{R})$ is simple, but it has non-trivial center when n is even. Its quotient $\mathrm{PSL}_n(\mathbb{R})$ is center-free, (connected) and simple.
- The orthogonal group $O(p, q)$ is in general not connected, but its connected component is simple.
- The isometry group of $\mathbb{R}^n$, $\mathrm{SO}(n) \rtimes \mathbb{R}^n$ is of course not semi-simple, since $\mathbb{R}^n$ is a connected normal abelian group.

Another corollary we will use is that every derivation of the Lie algebra $\mathfrak{g}$ of a semi-simple Lie group is inner.

Corollary 2.28. *Let G is a semi-simple Lie group, with Lie algebra $\mathfrak{g}$. Then every derivation of $\mathfrak{g}$ is inner, i.e.* $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Proof. The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ gives rise to another representation $\pi : G \to \mathrm{GL}(\mathrm{End}(\mathfrak{g}))$, given by $\pi(g)T = \mathrm{Ad}(g)T\,\mathrm{Ad}(g^{-1})$. Note that the subspace $\mathrm{Der}(\mathfrak{g})$ is invariant under $\pi(G)$, and so is $\mathrm{ad}(\mathfrak{g})$. By Theorem 2.24, this implies that we may find a supplementary subspace $\mathfrak{a}$ to $\mathrm{ad}(\mathfrak{g})$ inside $\mathrm{Der}(\mathfrak{g})$ which is $\pi(G)$-invariant, i.e. $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g}) \oplus \mathfrak{a}$.

Note that the derivative $d\pi : \mathfrak{g} \to \mathrm{End}(\mathrm{End}(\mathfrak{g}))$ of π is given by $d\pi(X)T = \mathrm{ad}(X)T - T\,\mathrm{ad}(X) = [\mathrm{ad}(X), T]$, for all $X \in \mathfrak{g}$, $T \in \mathrm{End}(\mathfrak{g})$. Since $\mathfrak{a}$ is $\pi(G)$-invariant, it must be invariant under $d\pi$ and therefore $[\mathrm{ad}(X), D] \in \mathfrak{a}$ for every derivation $D \in \mathfrak{a}$. Observe moreover that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}(\mathfrak{g})$. Indeed, if D is a derivation on $\mathfrak{g}$ and $X, Y \in \mathfrak{g}$, then we have

$$\begin{aligned}
[\mathrm{ad}(X), D](Y)] &= \mathrm{ad}(X)DY - D\,\mathrm{ad}(X)Y \\
&= [X, DY] - D([X, Y]) \\
&= -[DX, Y] \\
&= -\mathrm{ad}(DX)(Y).
\end{aligned}$$

Thus for all $X \in \mathfrak{g}$ and $D \in \mathfrak{a}$, $[\mathrm{ad}(X), D] \in \mathrm{ad}(\mathfrak{g}) \cap \mathfrak{a} = 0$.

Let us now conclude that $\mathfrak{a} = 0$. Let us take $D \in \mathfrak{a}$ and $X \in \mathfrak{g}$ and show that $DX = 0$. Since $\mathfrak{g}$ is semi-simple, it has trivial center, so the representation ad is faithful. We only need to prove that $\mathrm{ad}(DX) = 0$. But for Y in $\mathfrak{g}$, the above computation gives $\mathrm{ad}(DX)(Y) = -[\mathrm{ad}(X), D](Y) = 0$. This finishes the proof. □

3. A simplified setting

The main results about lattices in Lie groups that we want to prove involve knowing the structure of semi-simple Lie groups (roots systems, real forms, parabolic groups,...). In order to avoid going through this, we will restrain ourself to the setting where $G = \mathrm{SL}_n(\mathbb{R})$ (or even $\mathrm{PSL}_n(\mathbb{R})$ when we need to assume trivial center) and Γ is an arbitrary lattice in G. The results are already interesting in this case, and the proofs that we will present in these notes are identical for more general Lie groups, for the reader already familiar with semi-simple Lie groups.

We will use the following notation:

- $G = \mathrm{SL}_n(\mathbb{R})$.
- $A < G$ is the subgroup of diagonal matrices. It is a so-called *maximal torus*.
- $P < G$ is the subgroup of all upper triangular matrices. It is a *minimal parabolic subgroup*. In our case P is solvable. In general, P admits a co-compact solvable normal subgroup.
- $K = \mathrm{SO}(n) < G$. It is a maximal compact subgroup.
- $N < P$ the nilpotent subgroup of upper triangular matrices with 1's on the diagonal.

In the trivial center case we will use the same notations to denote the image of any of these groups in $\mathrm{PSL}_n(\mathbb{R})$.

We will need two decompositions of G.

(1) The *Iwasawa decomposition* tells us that any element $g \in G$ can be written as the product $g = kan$ of elements $k \in K$, $a \in A$ and $n \in N$. In our special case $P = AN$, and the decomposition is just saying that the action of K on the homogeneous space G/P is transitive.
(2) The *Cartan decomposition*, or KAK-decomposition tells us that every element of G can be written as a product kak', with $k, k' \in K$, $a \in A$.
(3) The *LU-decomposition*[1] tells us that N^tP is a dense open subset of G. We will give more details when needed. It is a special case of the Bruhat decomposition.

[1]standing for Lower-Upper.

CHAPTER 3

Ergodic theory and unitary representations

In this chapter, we want to study *measurable dynamical systems*, i.e. group actions on measure spaces. We will mostly focus on measure preserving actions and consider the most elementary notions of ergodicity and mixing. We will present the connection with the underlying unitary representation.

1. Mesurable group actions

Definition 3.1. A *measurable group action* is the data of an action σ of a locally compact group G on a measure space $(X, \mathcal{B})$ such that the action map $(g, x) \in G \times X \mapsto g(x) \in X$ is measurable. Given a measure μ on $(X, \mathcal{B})$, we say that the action is

- *non-singular* if $(\sigma_g)_*\mu$ is equivalent to μ for all $g \in G$. This means that for all $g \in G$, and all measurable set $A \subset X$, $\mu(A) = 0$ if and only if $\mu(g^{-1}A) = 0$. We also say that μ is *quasi-invariant*.
- *measure preserving* if $g_*\mu = \mu$ for all $g \in G$. In this case, μ is called *invariant*.

Measurable group actions cover classical measurable dynamical systems (given by a single invertible transformation) and flows (given by actions of $G = \mathbb{R}$).

Example 3.2. In the following examples, the action we consider is not only measurable (with respect to the Borel σ-algebra), it is actually continuous.

(1) If $\alpha \in [0, 1[$, we denote by $R_\alpha : x \in \mathbb{R}/\mathbb{Z} \mapsto x + \alpha \in \mathbb{R}/\mathbb{Z}$. The group generated by this rotation is a copy of $\mathbb{Z}$, which acts on the circle. This action preserves the Lebesgue measure on the circle.
(2) In the same spirit SO(3) acts on the 2-sphere $\mathbb{S}^2$ (and this defines by restriction an action of any subgroup of SO(3)). Again, this action preserves the Lebesgue measure on the sphere.
(3) The projective action of $\mathrm{PSL}_n(\mathbb{R})$ on the projective space $\mathbb{P}^{n-1}$ is also a measurable group action. If $n \geq 2$ this action does not admit an invariant Borel measure.
(4) Any locally compact group G acts on itself by left translation. This action preserves the Haar measure.

In fact, the above examples are special cases of actions on homogeneous spaces.

Example 3.3. Given a lcsc group G and a closed subgroup H, we may consider the natural action $G \curvearrowright G/H$. As is shown in the exercise sheets, there always exists a quasi-invariant Radon measure on G/H, and any two quasi-invariant measures have the same null sets (i.e. they are equivalent). However, there needs not exist an invariant Radon measure.

From now on, we will only consider *non-singular* actions. So a quasi-invariant measure μ will be given and every notion that we will consider will be “up to null sets”. For example, a subset $A \subset X$ we be called G-invariant if $\mu(A \Delta gA) = 0$ for all $g \in G$. Here Δ denotes the symmetric difference. In the case where G is countable, this condition is obviously equivalent to the fact that there exists a truly invariant set $B \subset X$ (i.e. such that $gB = B$ for all $g \in G$), such that $\mu(B \Delta A) = 0$. For example, take $B = \bigcap_g gA$. But this fact is actually true for any lcsc group G.

Definition 3.4. We say that an action $G \curvearrowright (X, \mu)$ is *ergodic* if any G-invariant subset of X is either null or co-null:

$$(\mu(A \Delta gA) = 0, \text{ for all } g \in G) \Rightarrow (\mu(A) = 0 \text{ or } \mu(A^c) = 0)\,.$$

It should be noted that a transitive action is always ergodic. In other words for any lcsc group G and any closed subgroup H, the action $G \curvearrowright G/H$ is ergodic with respect to any quasi-invariant measure on G/H. What is less clear is whether the action of a subgroup G_0 of G on G/H is again ergodic. This kind of question will be crucial for us. In order to answer to such questions, we need to consider the natural unitary representation associated with a measurable group action.

2. Unitary representations associated with actions

If $\mathcal{H}$ is a (complex) Hilbert space, we denote by $\mathcal{U}(\mathcal{H})$ the group of its unitary operators, i.e. the linear operators $u : \mathcal{H} \to \mathcal{H}$ that preserve the inner product and that are surjective. Equivalently u is a unitary operator if it satisfies $u^*u = uu^* = \mathrm{id}_{\mathcal{H}}$.

Definition 3.5. A *unitary representation* of a lcsc group G is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, with the continuity requirement that for all vector $\xi \in \mathcal{H}$, the map $g \in G \mapsto \pi(g)\xi \in \mathcal{H}$ is continuous.

Note that if $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then $\pi_g^* = \pi_{g^{-1}}$, for all $g \in G$. In other words

$$\langle \pi_g(\xi), \eta \rangle = \langle \xi, \pi_{g^{-1}}(\eta) \rangle, \text{ for all } \xi, \eta \in \mathcal{H}. \tag{2.1}$$

The properties that we will study will be relevant only when $\mathcal{H}$ is infinite dimensional.

Definition 3.6. Consider a measure preserving action of a lcsc group G on a space (X, μ). We define the *Koopman representation* π as the unitary representation of G on $L^2(X, \mu)$ defined by the formula

$$\pi_g(f) : x \mapsto f(g^{-1}x), \text{ for all } g \in G, f \in L^2(X, \mu).$$

This definition is relevant for actions on so-called *standard measure spaces*, for which the following lemma holds true. But we restrain our setting to continuous actions, which suffices for our purposes.

Lemma 3.7. *If X is an lcsc space, the action $G \curvearrowright X$ is continuous, and the measure μ is a Radon measure on X, then the Koopman representation π associated with a measure preserving action $G \curvearrowright (X, \mu)$ is indeed a unitary representation.*

Proof. The fact that π_g is a unitary operator comes from the fact that μ is G-invariant. It is also easy to check that $\pi : G \to \mathcal{U}(L^2(X, \mu))$ is a group homomorphism. The only non-trivial fact is that π is continuous. Under the assumptions of the lemma, $C_c(X)$ is dense in $L^2(X, \mu)$. Take $f \in L^2(X, \mu)$ and take a sequence $(g_n)_n$ in G which converges to the identity element $e \in G$. Take $\varepsilon > 0$. By density we may find a function $f_0 \in C_c(X)$ such that $\|f - f_0\|_2 < \varepsilon$. Then for all $n \geq 1$, we have

$$\begin{aligned}\|\pi_{g_n}(f) - f\|_2 &\leq \|\pi_{g_n}(f) - \pi_{g_n}(f_0)\|_2 + \|\pi_{g_n}(f_0) - f_0\|_2 + \|f_0 - f\|_2 \\ &\leq 2\varepsilon + \left(\int_X |f_0(g_n^{-1}x) - f_0(x)|^2 \, \mathrm{d}\mu(x)\right)^{1/2}.\end{aligned}$$

Since f_0 is continuous, we see that if n is large enough, then the later term is less than ε (for instance use the fact that f is uniformly continuous, or use the Lebesgue convergence theorem. In both cases the continuity of the action is crucial). So if n is large enough we get that $\|\pi_{g_n}(f) - f\|_2 < 3\varepsilon$. As ε can be chosen arbitrarily small, we deduce that $\lim_n \|\pi_{g_n}(f) - f\|_2 = 0$. □

In fact one could also define the Koopman representation associated with any nonsingular action, but the formula is more complicated as it involves the Radon-Nikodym derivatives between μ and its translates $g_*\mu$.

Example 3.8. The Koopman representation associated with the left translation action $G \curvearrowright G$ is called the *left regular representation* and is denoted by $\lambda_G : G \to \mathcal{U}(L^2(G))$. Note that we could also define similarly the *right regular representation* ρ_G. One checks that these two representations commute to each other: $(\lambda_G)_g(\rho_G)_h = (\rho_G)_h(\lambda_G)_g$ for all $g, h \in G$. Here we assume that G is unimodular to define these two representations simultaneously.

Example 3.9. If G is a lcsc group and $\Gamma < G$ is a lattice, then we saw in Corollary 1.17 that G/Γ carries a finite G-invariant measure. In this case, we get automatically the corresponding Koopman representation $G \to \mathcal{U}(L^2(G/\Gamma))$.

The Koopman representation can be used to detect ergodicity for measure preserving actions $G \curvearrowright (X, \mu)$, for which the invariant measure μ is finite. By rescaling we may assume that μ is a probability measure, and we say in this case that the action is *pmp* (standing for probability measure preserving).

Lemma 3.10. *Take a pmp action $G \curvearrowright (X, \mu)$. Then the action is ergodic if and only if the only invariant functions $f \in L^2(X, \mu)$ under the Koopman representation are the constant functions.*

Proof. If $A \subset X$ is a G-invariant subset of X, then the indicator function $\mathbf{1}_A$ is invariant under the Koopman representation: $\pi_g(\mathbf{1}_A) = \mathbf{1}_A$ for all $g \in G$. Therefore if the action is not ergodic, there exists a non-constant function which is invariant under the Koopman representation.

Conversely assume that $f \in L^2(X, \mu)$ is G-invariant and non-contant. Then we may find $a \in \mathbb{R}$ such that the level set $\{f > a\}$ is neither null, nor co-null. But this set is obviously G-invariant. □

As a direct application, one can describe exactly which rotations are ergodic.

Example 3.11. The rotation $R_\alpha : x \mapsto x + \alpha$ on the circle $\mathbb{R}/\mathbb{Z}$ is ergodic (viewed as an action of $\mathbb{Z}$) if and only if α is irrational.
Indeed, if α is rational, then the transformation is periodic, and it is easy to find invariant sets that are neither null nor co-null.
Conversely, let us assume that α is irrational, and let us prove that the corresponding

action is ergodic. By the previous lemma, it suffices to prove that the Koopman representation on $L^2(\mathbb{R}/\mathbb{Z})$ has no invariant function other than the constant ones. But we may use the Fourier transform $\mathcal{F}$ to identify $L^2(\mathbb{R}/\mathbb{Z})$ with $\ell^2(\mathbb{Z})$ as follows:

$$\mathcal{F} : f \in L^2(\mathbb{R}/\mathbb{Z}) \mapsto (c_n(f))_n \in \ell^2(\mathbb{Z}),$$

where $c_n(f) = \int_0^1 f(t)\exp(-2i\pi nt)dt$. Now observe that with this identification, the rotation R_α is identified with the transformation

$$T_\alpha : (c_n)_n \in \ell^2(\mathbb{Z}) \mapsto (\exp(-2i\pi n\alpha)c_n)_n \in \ell^2(\mathbb{Z}).$$

More precisely, we have the formula $\mathcal{F}(R_\alpha(f)) = T_\alpha(\mathcal{F}(f))$, for all $f \in L^2(\mathbb{R}/\mathbb{Z})$. So we are left to check that a function which is invariant under T_α is necessarily the Fourier transform of a constant function, i.e. is a multiple of the Dirac sequence to 0, $\delta_0 \in \ell^2(\mathbb{Z})$. Take then such an invariant vector $(c_n) \in \ell^2(\mathbb{Z})$, then for each $n \neq 0$ we must have $c_n = \exp(-2i\pi n\alpha)c_n$, and if α is irrational, this forces $c_n = 0$, as desired.

Example 3.12. Let us give another proof that an irrational rotation is ergodic, not based on the Fourier transform. If α is irrational, then the group it generates has dense image in $\mathbb{R}/\mathbb{Z}$. So if $f \in L^2(\mathbb{R}/\mathbb{Z})$ is R_α-invariant, it must be invariant under the whole of $\mathbb{R}/\mathbb{Z}$, by continuity of the Koopman representation associated with the action $\mathbb{R}/\mathbb{Z} \curvearrowright \mathbb{R}/\mathbb{Z}$. This action being transitive, it is ergodic. So f must be constant.

The above lemma rephrases by saying that there is no G-invariant function in the space $L^2_0(X,\mu)$ of functions whose integral vanishes ($L^2_0(X,\mu)$ is the orthogonal complement of the constant function in $L^2(X,\mu)$).

3. Ergodic theorems

It is hard to imagine a chapter on ergodic theory not discussing ergodic theorems. So even if we won't use this for our purposes, we include von Neumann's and Birkhoff's ergodic theorems (although we won't prove the latter).

Ergodic theorems are about pmp actions of $\mathbb{Z}$, that is, about a single transformation $T : X \to X$ that preserves a probability measure μ and about the average behavior of its iterates. Actually we don't even require T to be invertible, so we are in fact considering pmp actions of the semi-group $\mathbb{N}$. In spirit, ergodic theorems express the fact that the orbits $x, Tx, T^2x, T^3x, \ldots$ become more and more distributed on X according to the mesure μ. That is, when we average a function along such an orbit, we approximate the μ-integral of that function.

We will only prove the weaker result, due to von Neumann, which proves convergence in the L^2-norm, but does not specify the actual convergence at a given point.

Theorem 3.13 (von Neumann). *Let $T : (X,\mu) \to (X,\mu)$ by an ergodic pmp transformation of the probability space (X,μ). Then for every function $f \in L^2(X,\mu)$, we have the following convergence, for the L^2-norm:*

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} f \circ T^k = \int_X f \, \mathrm{d}\mu.$$

This result is based on Hilbert space considerations. As we saw before, an invertible pmp transformation T of a space (X,μ) gives rise to a unitary $u \in \mathcal{U}(L^2(X,\mu))$ given by $u(f) = f \circ T$, for all $f \in L^2(X,\mu)$. When T is not invertible then the same formula makes sense, but $u : L^2(X,\mu) \to L^2(X,\mu)$ defined this way is not invertible anymore. Still, it is an isometry, in the sense that $\|u(f)\| = \|f\|$, which rewrites as $u^*u = \mathrm{id}$.

Let us also emphasize that in the case of a non-invertible pmp transformation T, ergodicity means that any set $A \subset X$ such that $\mu(A \Delta T^{-1}(A)) = 0$ must be null or co-null. The same proof as in Lemma 3.10 shows that T is ergodic if and only if the corresponding isometry u has no non-zero invariant vector in $L^2_0(X, \mu)$.

So von Neumann theorem will follow from the following lemma.

LEMMA 3.14. *Let $\mathcal{H}$ be a Hilbert space and $u : \mathcal{H} \to \mathcal{H}$ be an isometry having no non-zero invariant vector, i.e.* $\ker(u - \mathrm{id}) = \{0\}$. *Then the following facts are true:*

a) The adjoint u^* of u has no invariant vector;
b) The image of $u - \mathrm{id}$ is dense in $\mathcal{H}$;
c) For every $\xi \in \mathcal{H}$, we have $\lim_n \frac{1}{n} \sum_{k=1}^n u^k(\xi) = 0$ in $\mathcal{H}$.

PROOF. *a*) If $\xi \in \mathcal{H}$ satisfies $u^*\xi = \xi$, then we find $\|\xi\|^2 = \langle u^*\xi, \xi\rangle = \langle \xi, u\xi\rangle$. Now the following classical computation shows that this implies that $u\xi = \xi$:

$$\|u\xi - \xi\|^2 = 2\|\xi\|^2 - 2\Re(\langle u\xi, \xi\rangle) = 0. \tag{3.1}$$

b) It is a classical fact about operators on Hilbert spaces that $\overline{\mathrm{Img}(u - \mathrm{id})} = \ker(u^* - \mathrm{id})^\perp$.
c) If ξ is in $\mathrm{Img}(u - \mathrm{id})$ then the sum will be telescopic, so the division by n will ensure convergence to 0. Now take an arbitrary vector $\xi \in \mathcal{H}$, and $\varepsilon > 0$. Then we may find $\xi_0 \in \mathrm{Img}(u - \mathrm{id})$ such that $\|\xi - \xi_0\| < \varepsilon$. For all n, we have

$$\|\frac{1}{n}\sum_{k=1}^n u^k(\xi)\| \leq \|\frac{1}{n}\sum_{k=1}^n u^k(\xi - \xi_0)\| + \|\frac{1}{n}\sum_{k=1}^n u^k(\xi_0)\| \leq \varepsilon + \|\frac{1}{n}\sum_{k=1}^n u^k(\xi_0)\|.$$

So if n is large enough, this can be made less than 2ε, proving that the limit is 0. □

PROOF OF VON NEUMANN'S THEOREM. Take a function $f \in L^2(X, \mu)$, and denote by $f_0 = f - \int_X f d\mu$. For all $n \geq 1$, we have

$$\|\frac{1}{n}\sum_{k=1}^n f \circ T^k - \int_X f \,\mathrm{d}\mu\| = \|\frac{1}{n}\sum_{k=1}^n f_0 \circ T^k\| = \|\frac{1}{n}\sum_{k=1}^n u^k(f_0)\|.$$

Note that f_0 belongs to $L^2_0(X, \mu)$, and since T is assumed to be ergodic, u has no non-zero invariant vector in $L^2_0(X, \mu)$. So the previous lemma shows that the above quantity converges to 0. □

Birkhoff's ergodic theorem actually shows that many orbits tend to become μ-distributed.

THEOREM 3.15 (Birkhoff). *Let $T : (X, \mu) \to (X, \mu)$ by an* ergodic *pmp transformation of the probability space (X, μ). Then for every function $f \in L^1(X, \mu)$, for μ-almost every $x \in X$, we have*

$$\lim_n \frac{1}{n}\sum_{k=1}^n f(T^k x) = \int_X f \,\mathrm{d}\mu.$$

REMARK 3.16. In cases where $L^1(X, \mu)$ is a separable space (for example, if X is a lcsc space and μ is a Borel probability measure on X), we can exchange the quantifiers, and get: for μ-almost every $x \in X$, for every $f \in L^1(X, \mu)$, the convergence holds. This thus ensures that "almost every orbit becomes μ-distributed".

4. Mixing phenomena

The notion of a mixing group action is the most meaningful for probability measure preserving actions $G \curvearrowright (X, \mu)$.

Definition 3.17. A pmp action $G \curvearrowright (X, \mu)$ is said to be *mixing* if for every pair of measurable subsets $A, B \subset X$, and $\varepsilon > 0$, there exists a compact set $K \subset G$ such that gA and B are ε-independent for all $g \in G \setminus K$, in the sense that

$$-\varepsilon < \mu(gA \cap B) - \mu(A)\mu(B) < \varepsilon.$$

In other words, the function $g \mapsto \mu(gA \cap B) - \mu(A)\mu(B)$ belongs to the algebra $C_0(G)$ of functions that converge to 0 at infinity. By definition, $f \in C_0(G)$ if it is continuous and if for every $\varepsilon > 0$, there exists a compact set $K \subset G$ such that $|f(g)| < \varepsilon$ for all $g \in G/K$.

Exercise 3.18. Check that if a pmp action $G \curvearrowright (X, \mu)$ is mixing, then every non-compact subgroup of G acts ergodically on (X, μ).

As for ergodicity, the mixing property can be read on the Koopman representation.

Definition 3.19. A unitary representation $G \to \mathcal{U}(\mathcal{H})$ is called *mixing* and for every $\xi, \eta \in H$, the corresponding coefficient function $g \mapsto \langle \pi_g(\xi), \eta \rangle$ is in $C_0(G)$.

Lemma 3.20. *A pmp action $G \curvearrowright (x, \mu)$ is mixing if and only if the corresponding representation on $L^2_0(X, \mu)$ is mixing.*

Proof. For a set $A \subset X$, we may define a function $f_A \in L^2_0(X, \mu)$ by the formula $\mathbf{1}_A - \mu(A)\mathbf{1}_X$. Then given two subsets $A, B \subset X$, and $g \in G$, we have

$$\begin{aligned} \langle \pi_g(f_A), f_B \rangle &= \int_X (\mathbf{1}_{gA} - \mu(A)\mathbf{1}_X)(\mathbf{1}_B - \mu(B)\mathbf{1}_X)\, \mathrm{d}\mu(x) \\ &= \mu(gA \cap B) - \mu(A)\mu(B). \end{aligned}$$

So if the representation of G on $L^2_0(X, \mu)$ is mixing, then the action is mixing. To prove the converse, observe that f_A is the orthogonal projection of $\mathbf{1}_A$ on $L^2_0(X, \mu)$. Since the linear span of simple functions $\mathbf{1}_A$ is dense in $L^2(X, \mu)$, we see that the linear span E of the functions f_A is dense in $L^2_0(X, \mu)$. If the action is mixing then we know that the coefficient function $c : g \mapsto \langle \pi_g(\xi), \eta \rangle$ is in $C_0(G)$ if ξ and η are of the form $\mathbf{1}_A$, and by linearity this remains true if they are of the linear combinations of such functions. Assume now that $\xi, \eta \in L^2_0(X, \mu)$ are arbitrary functions. Then we may find sequences $\xi_n, \eta_n \in E$ such that $\|\xi_n - \xi\|_2$ and $\|\eta_n - \eta\|_2$ converge to 0. Then we find from Cauchy-Schwarz inequality that the coefficient functions $c_n : g \mapsto \langle \pi_g(\xi_n), \eta_n \rangle$ converge uniformly to c. Since each c_n lies in $C_0(G)$, this must also be the case of c (exercise). □

Exercise 3.21. Prove that a rotation on the circle is never mixing.

We are now going to discuss the following very useful theorem.

Theorem 3.22 (Howe-Moore). *If G is a connected semi-simple Lie group with finite center, then any unitary representation of G with no (non-zero) invariant vector is mixing.*

In other words, the presence of invariant vectors is the only obstruction to be mixing. In order to avoid going through the structure of (semi-)simple Lie groups, we will only prove this result for $G = \mathrm{SL}_n(\mathbb{R})$. Before getting into the proof, let us provide some examples.

EXAMPLE 3.23. Let G be as in the theorem and let Γ be a lattice in G. Then the action $G \curvearrowright G/\Gamma$ is mixing. In particular, for every non-compact closed subgroup $H < G$, H acts ergodically on G/Γ. Note that this later fact is equivalent to saying that every subset $X \subset G$ which is left H-invariant and right Γ-invariant is either null or co-null. Equivalently, Γ acts ergodically on G/H. So we also deduce ergodicity results for actions that need not admit an invariant measure.

EXERCISE 3.24. Apply the above example to prove that any lattice of $\mathrm{SL}_n(\mathbb{R})$ acts ergodically on the projective space, or on $\mathbb{R}^n$;

Let us now discuss the proof of Howe-Moore theorem. Our main ingredient is the so-called Mautner phenomenon described in the following lemma.

LEMMA 3.25 (Mautner 1). *Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a lcsc group, and take a sequence $(g_n)_n \in G$ which fixes a vector $\xi \in \mathcal{H}$, i.e. $\pi_{g_n}(\xi) = \xi$ for all n. Take an element $u \in G$ such that $\lim_n g_n^{-1} u g_n = e$. Then $\pi_u(\xi) = \xi$.*

We directly prove the following refinement which involves the weak topology on $\mathcal{H}$. Recall that by self duality of Hilbert spaces, a sequence $\xi_n \in \mathcal{H}$ converges weakly to a vector $\eta \in \mathcal{H}$ if and only if for every vector ξ', we have $\lim_n \langle \xi_n, \xi' \rangle = \langle \eta, \xi' \rangle$.

LEMMA 3.26 (Mautner 2). *Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a lcsc group, and take a sequence $(g_n)_n \in G$ and $u \in G$ such that $\lim_n g_n^{-1} u g_n = e$. Take a vector $\xi \in \mathcal{H}$ such that $\pi_{g_n}(\xi)$ converges weakly to a vector $\xi_\infty \in \mathcal{H}$. Then $\pi_u(\xi_\infty) = \xi_\infty$.*

PROOF. We first compute
$$\lim_n \|\pi_u \pi_{g_n}(\xi) - \pi_{g_n}(\xi)\| = \lim_n \|\pi_{g_n^{-1} u g_n}(\xi) - \xi\| = 0.$$
So the two sequences $\pi_u \pi_{g_n}(\xi)$ and $\pi_{g_n}(\xi)$ must have the same weak limits. Since these weak limits are $\pi_u(\xi_\infty)$ and ξ_∞ respectively, we are done. □

4.1. The case of $\mathrm{SL}_2(\mathbb{R})$. In this paragraph, we denote by $G = \mathrm{SL}_2(\mathbb{R})$. For $a \in \mathbb{R}^*$, $x \in \mathbb{R}$, we define
$$s_a := \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad u_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$
We denote by $A < G$ the subgroup consisting of all diagonal matrices s_a, $a \in \mathbb{R}^*$ and by $N < G$ the upper unipotent subgroup, consisting of all u_x, $x \in \mathbb{R}$. We also denote by N^{op} the lower unipotent subgroup, whose elements are the transpose of u_x.

Observe that A normalizes N, with the action given by $s_a u_x s_a^{-1} = u_{a^2 x}$. In particular, if (a_n) is a sequence of real numbers that goes to infinity as $n \to \infty$, then for any $x \in \mathbb{R}$, we have $\lim_n s_{a_n}^{-1} u_x s_{a_n} = e$. This is precisely the situation where we can apply Mautner phenomenon.

LEMMA 3.27. *If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then*

a) any vector in $\mathcal{H}$ which is invariant under A, N and N^{op} is G-invariant;
b) any A-invariant vector in $\mathcal{H}$ is invariant under N and N^{op};
c) any N-invariant vector in $\mathcal{H}$ is A-invariant, hence G-invariant.

PROOF. Since A, N and N^{op} generate G, item a) is trivial.
b) is a consequence of Mautner lemma 1. We leave it as an exercise.
c) Assume that $\xi \in H$ is a vector invariant under $\pi(N)$. Consider the function
$$f : g \in G \mapsto \langle \pi_g(\xi), \xi \rangle.$$

This function is continuous. It is obviously right N-invariant, and equation (2.1) tells us that it is also left N-invariant. The right invariance allows us to view f as a continuous function on G/N. And observe that there is a G-equivariant homeomorphism $G/N \to \mathbb{R}^2 \setminus \{(0,0)\}$. More precisely, consider the natural action of G on $\mathbb{R}^2$, denote by $e_1 = (1,0) \in \mathbb{R}^2$ and consider the orbit map

$$g \in G \mapsto g(e_1) \in \mathbb{R}^2.$$

Then the range of this map is $\mathbb{R}^2 \setminus \{(0,0)\}$ and the stabilizer of e_1 is N. So this map induces a G-equivariant homeomorphism $G/N \to \mathbb{R}^2 \setminus \{(0,0)\}$, which we may use to view f as a continuous function on $\mathbb{R}^2 \setminus \{(0,0)\}$. Now the left N-invariance of f implies that

$$f(a,b) = f(u_x(a,b)) = f(a+bx,b), \text{ for all } x \in \mathbb{R}, (a,b) \in \mathbb{R}^2.$$

So f is constant along any horizontal line $L_b := \{(a,b)\,,\, a \in \mathbb{R}\}$, with $b \neq 0$. But since f is continuous, this implies that f is also constant along the horizontal axis $L_0 \setminus \{(0,0)\}$. This means that $f(1,0) = f(a,0) = f(s_a(1,0))$ for all $a \in \mathbb{R}^*$. Therefore, going back to the description of f as a right-N-invariant function on G, we conclude that $f(s_a) = f(e)$, for all $a \in \mathbb{R}^*$. In other words $\langle \pi_{s_a}(\xi), \xi\rangle = \|\xi\|^2$, for all $a \in A$. This means that ξ is A invariant. □

PROPOSITION 3.28. *If $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation without (non-zero) invariant vectors then its restriction to A is mixing.*

PROOF. We prove the contraposite. Assume that the representation of A is not mixing.

Claim 1. There exist two vectors $\xi, \eta \in \mathcal{H}$ and a sequence of real numbers $(a_n)_n$ that goes to infinity as $n \to \infty$, and such that $\langle \pi_{s_{a_n}}(\xi), \eta\rangle$ converges to a non-zero scalar number.

Since the representation of A is not mixing, then there exists two vectors $\xi, \eta \in \mathcal{H}$ such that the corresponding coefficient function $g \in A \mapsto \langle \pi_g(\xi), \eta\rangle$ is not in $C_0(A)$. This means that we can find a sequence $(g_n)_n$ of elements in A that goes to infinity in A and such that $\langle \pi_{g_n}(\xi), \eta\rangle$ converges to a non-zero complex number $c \in C$.

For each n, we may write $g_n = s_{a_n}$ for some $a_n \in \mathbb{R}$. Saying that the sequence $(g_n)_n$ goes to infinity in A implies that, up to taking a subsequence, $(a_n)_n$ goes to 0 or to ∞. In the latter case we are done. In the case where $(a_n)_n$ converges to 0, then the sequence $(a_n^{-1})_n$ goes to infinity and satisfies

$$\lim_n \langle \pi_{s_{a_n^{-1}}}(\eta), \xi\rangle = \lim_n \langle \eta, \pi_{g_n}(\xi)\rangle = \bar{c} \neq 0.$$

So the sequence $(a_n^{-1})_n$ satisfies the claim, by interchanging the vectors ξ and η.

Claim 2. With the notation of Claim 1, we may assume that the sequence $\xi_n := \pi_{s_{a_n}}(\xi)$ converges weakly to a non-zero vector ξ_∞, which is G-invariant.

This simply follows from the fact that the sequence $(\xi_n)_n$ is bounded in H, contained in the closed ball B with center 0 and radius $\|\xi\|$, and that this ball is compact for the weak topology (Banach-Alaoglu theorem). So we may find a subsequence of ξ_n that converges weakly to some vector ξ_∞[1]. Now by Claim 1, we must have

$$\langle \xi_\infty, \eta\rangle = \lim_n \langle \pi_{s_{a_n}}(\xi), \eta\rangle \neq 0.$$

[1]Here we are implicitly using the fact that B is sequentially compact, which holds only when H is a separable Hilbert space. But in order to prove Howe-Moore theorem, we can always reduce to the case where H is separable (why?).

So ξ_∞ is non-zero. Mautner lemma 2 tells us that ξ_∞ is N-invariant. So Lemma 3.27.c implies that it is G-invariant. □

Caution. The above lemma is *not* saying that every representation of A without invariant vectors is mixing!

Now Howe-Moore theorem follows from the KAK-decomposition, which we more generally state for representations of $\mathrm{SL}_n(\mathbb{R})$, $n \geq 2$.

LEMMA 3.29. *Let now $G = \mathrm{SL}_n(\mathbb{R})$, and denote by A the subgroup of diagonal matrices in G. Consider a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, whose restriction to A is mixing. Then π is mixing.*

PROOF. Denote by $K = \mathrm{SO}(n) < G$ the compact subgroup of orthogonal matrices. Then we have the KAK decomposition. In other words every element $g \in G$ can be written as a product $g = kak'$, with $k, k' \in K$, $a \in A$.

Take $\xi, \eta \in \mathcal{H}$ and $\varepsilon > 0$. The sets $\pi(K)(\xi)$ and $\pi(K)(\eta)$ are compact subsets of $\mathcal{H}$. So we may find finitely many vectors $\xi_1, \ldots, \xi_n \in \pi(K)(\xi)$ and $\eta_1, \ldots, \eta_m \in \pi(K)(\eta)$, such that

- for any vector $\xi' \in \pi(K)(\xi)$, there exists i such that $\|\xi_i - \xi'\| < \varepsilon$.
- for any vector $\eta' \in \pi(K)(\eta)$, there exists j such that $\|\eta_j - \eta'\| < \varepsilon$.

Since the representation of A on $\mathcal{H}$ is mixing, we may find a compact set $C \subset A$ such that

$$|\langle \pi_a(\xi_i), \eta_j \rangle| < \varepsilon \text{ for all } a \in A \setminus C, i \leq n, j \leq m.$$

Now denote by $C' \subset G$ the compact set KCK. Then for any $g \in G \setminus C'$, we may write $g = kak'$, with $a \in A \setminus C$, and we may find $i \leq n$, $j \leq m$ such that $\|\pi_{k'}(\xi) - \xi_i\| < \varepsilon$ and $\|\pi_{k^{-1}}(\eta) - \eta_j\| < \varepsilon$. And we get

$$\begin{aligned}
|\langle \pi_g(\xi), \eta \rangle &= |\langle \pi_a \pi_{k'}(\xi), \pi_{k^{-1}}(\eta) \rangle| \\
&< |\langle \pi_a(\xi_i), \pi_{k^{-1}}(\eta) \rangle| + \|\pi_a \pi_{k'}(\xi) - \pi_a(\xi_i)\| \|\eta\| \\
&< |\langle \pi_a(\xi_i), \eta_j \rangle| + \|\xi\| \|\pi_{k^{-1}}(\eta) - \eta_j\| + \varepsilon \|\eta\| \\
&< \varepsilon(1 + \|\xi\| + \|\eta\|).
\end{aligned}$$

This proves that the coefficient function $g \in G \mapsto \langle \pi_g(\xi), \eta \rangle$ is in $C_0(G)$ □

4.2. The case of $\mathrm{SL}_d(\mathbb{R})$, $d \geq 3$. in order to prove Howe Moore theorem for general semi-simple Lie groups one needs to use roots systems and $\mathfrak{sl}_2$-triples. We illustrate the proof for $G = \mathrm{SL}_d(\mathbb{R})$, $d \geq 3$. Thanks to Lemma 3.29, it suffices to prove the following proposition.

PROPOSITION 3.30. *Consider a unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ with no non-zero invariant vector. Then the restriction of π to the diagonal subgroup A is mixing.*

PROOF. The proof reuses all the tools we used for $\mathrm{SL}_2(\mathbb{R})$, and thus makes a good exercise.

- Prove that if a sequence $a_n = \mathrm{diag}(a_{n,1}, \ldots, a_{n,d})$ goes to infinity in A then, taking a subsequence if necessary, we may find indices i, j such that $\lim_n a_{n,i} = +\infty$ and $\lim_n a_{n,j} = 0$. Deduce that $a_n^{-1} u_{i,j}(x) a_n$ converges to e, where $u_{i,j}(x) \in G$ is the element with 1's one the diagonal, x in position (i, j) and 0 elsewhere.
- Prove that if A is not mixing then there exist $1 \leq i \neq j \leq d$ and a non-zero vector $\xi \in H$ which is invariant under $u_{i,j}(x)$, for all $x \in \mathbb{R}$.

- Show that ξ is invariant under the subgroup $A_{i,j} < A$ consisting of diagonal matrices with 1's on the diagonal entries other than the ith and jth.
- Prove then that ξ is invariant under all $u_{k,l}(x)$, $x \in \mathbb{R}$ whenever $\{k,l\} \cap \{i,j\} \neq \emptyset$.
- Conclude that ξ is G-invariant. □

5. Stationary dynamical systems

5.1. Definitions and examples.

Definition 3.31. Consider a measurable group action $G \curvearrowright X$ on a measure space and take probability measures $\mu \in \text{Prob}(G)$ and $\nu \in \text{Prob}(X)$ (we don't require that ν is quasi-invariant under G). Then we define the convolution measure $\mu * \nu \in \text{Prob}(X)$ to be the push-forward measure of the product measure $\mu \otimes \nu \in \text{Prob}(G \times X)$ under the action map $(g,x) \in G \times X \mapsto gx \in X$.

Example 3.32. If we consider G acting on itself by left multiplication, we thus obtain a product law on $\text{Prob}(G)$, which is associative, and whose neutral element is δ_e. In particular we may define the convolution powers μ^{*n}, $n \geq 1$, of a measure $\mu \in \text{Prob}(G)$. Then μ^{*n} describes the distribution of an element $g_1 \cdots g_n$ obtained as the product of n independent random elements $g_1, \dots, g_n$, each of them having law μ.

If G acts on a measurable space X, and if $\nu \in \text{Prob}(X)$ then the measures $\mu^{*n} * \nu$, $n \geq 1$, describe the random walk on X whose starting point is a random point with law ν, and where each step of the walk is obtained by applying a random element of G with law μ. Each new convolution by μ corresponds to an extra step in the walk.

Definition 3.33. Consider a measurable group action $G \curvearrowright X$ and take a Borel probability measure $\mu \in \text{Prob}(G)$. We say that a probability measure $\nu \in \text{Prob}(X)$ is *μ-stationary* if $\mu * \nu = \nu$.

Roughly speaking, a μ-stationary is a measure which is "invariant under the μ-random walk": if the starting point is distributed according to a μ-stationary measure ν, then the position of the random walk at any time $n \geq 1$ is also distributed according to the measure ν.

Of course any G-invariant measure is μ-stationary, for any $\mu \in \text{Prob}(G)$, but the gap between invariance and stationarity is huge: the following lemma says that stationary measures (on compact spaces) always exist, while this is far from true for invariant measures.

Lemma 3.34. *Consider a continuous action of a topological group G on a* compact *space X. Then for any Borel measure $\mu \in \text{Prob}(G)$, there always exists a μ-stationary Borel measure $\nu \in \text{Prob}(X)$.*

Proof. This is a fixed point argument. If X is compact, then $\text{Prob}(X)$ is a compact convex subset of the dual of $C(X)$. Consider the convolution map $T : \nu \in \text{Prob}(X) \mapsto \mu * \nu \in \text{Prob}(X)$. This map is affine and continuous on $\text{Prob}(X)$ and we want to prove that it admits a fixed point. Take an arbitrary initial measure $\nu_0 \in \text{Prob}(X)$, and consider for all $n \geq 1$ the measure

$$\nu_n := \frac{1}{n} \sum_{k=1}^{n} T^k(\nu_0) \in \text{Prob}(X).$$

Observe that $T(\nu_n) - \nu_n = (\mu^{*n+1} * \nu_0 - \nu_0)/n$ converges *-weakly to 0. So any weak-* cluster point of the sequence ν_n is fixed under T. Since $\text{Prob}(X)$ is compact, such cluster points do exist. □

In the light of a subsequent chapter on amenability, the above proof shows in fact that the semi-group $\mathbb{N}$ is amenable. The compactness of X is crucial, for example if G is any non-compact group and $\mu \in \mathrm{Prob}(G)$ is any probability measure equivalent to the Haar measure, there is no μ-stationary measure on G for the left translation action $G \curvearrowright G$.

Definition 3.35. A (G, μ)-*space* will be the data (X, ν) of a lcsc space X on which G acts continuously together with a μ-stationary Borel measure $\nu \in \mathrm{Prob}(X)$.

Example 3.36. Here are two classes of examples:

- Any continuous finite dimensional representation $\pi : G \to \mathrm{GL}(V)$ of G induces a projective action $G \curvearrowright \mathbb{P}(V)$ which is continuous. By Lemma 3.34, we may find a μ-stationary measure on $\mathbb{P}(V)$ for any Borel probability measure μ on G. This is a typical example of a (G, μ)-space that we will use in the next chapter.
- For any co-compact subgroup $Q < G$, the action of G on G/Q is continuous, so as in the previous example, such a space G/Q can be turned into a (G, μ)-space for any choice of $\mu \in \mathrm{Prob}(G)$. The typical example will be that where $Q = P$ is a minimal parabolic subgroup, or where Q contains such a group P, see Chapter 2, Section 3 for the definitions of P in our special setting).

5.2. Conditional measures. As we explained, μ-stationary measures may be interpreted as invariant measures under the μ-random walk. This probabilistic point of view will be useful to study μ-stationary measures.

We take a measure $\mu \in \mathrm{Prob}(G)$ and we consider the infinite product space $(\Omega, P) = \prod_{n \geq 1}(G, \mu)$. As a set this is nothing but $G^{\mathbb{N}^*}$, endowed with the σ-algebra generated by the finite cylinders $\mathcal{C} := A_1 \times \cdots \times A_n \times G \times G \times \cdots$, where each A_i is a Borel subset of G. The measure of such a cylinder is given by $P(\mathcal{C}) = \mu(A_1)\mu(A_2) \cdots \mu(A_n)$. By Caratheodory extension theorem, such a measure P exists, and it is unique.

The letter ω will stand for an element of Ω, and $g_n(\omega)$ will denote its nth coordinate, for $n \geq 1$.

Lemma 3.37. *Consider a (G, μ)-space (X, ν). Then for P-almost every $\omega \in \Omega$, the sequence of probability measures $g_1(\omega)_* \cdots g_n(\omega)_*\nu$ is weak-* convergent. We denote by $\nu_\omega \in \mathrm{Prob}(X)$ its limit, and we call it the* conditional measure *at ω. We have*

$$\nu = \int_\Omega \nu_\omega \, \mathrm{d}P(\omega).$$

Proof. Let us fix $f \in C_0(X)$, and consider the sequence of functions

$$f_n(\omega) := \int_X f(g_1(\omega) \cdots g_n(\omega)x) \, \mathrm{d}\nu(x).$$

Then f_n are bounded functions on Ω, with $\|f_n\|_\infty \leq \|f\|_\infty$. To prove that this sequence is almost surely convergent, we will use the martingale convergence theorem.

Denote by $\mathcal{B}$ the product σ-algebra on Ω, on which P is naturally defined. For all $n \in \mathbb{N}$, denote by $\mathcal{B}_n \subset \mathcal{B}$ the σ-subalgebra generated by the cylinders of the form $\mathcal{C} := A_1 \times \cdots \times A_n \times G \times G \times \cdots$ (with the same n). So the $\mathcal{B}_n$-measurable functions are the ones that only depend on the coordinates $g_1, \ldots, g_n$. It is then clear that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, for all $n \in \mathbb{N}$. We denote by $E_n : L^\infty(\Omega, \mathcal{B}) \to L^\infty(\Omega, \mathcal{B}_n)$ the conditional expectation.

For all $\omega \in \Omega$, we have

$$E_n(f_{n+1})(\omega) = \int_G \int_X f(g_1(\omega)\cdots g_n(\omega)gx)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(g)$$
$$= \int_X f(g_1(\omega)\cdots g_n(\omega)x)\,\mathrm{d}(\mu * \nu)(x) = f_n(\omega).$$

So (f_n) is a uniformly bounded martingale with respect to the filtration $(\mathcal{B}_n)_n$. By the martingale convergence theorem, it is almost surely convergent. So we find that for every fixed function $f \in C_0(X)$, for almost every $\omega \in \Omega$, $f_n(\omega)$ converges to a limit, denoted by $\nu_\omega(f)$. Since X is a lcsc space, the function algebra $C_0(X)$ is separable: it admits a dense sequence $(f_k)_{k\geq 1}$ (with respect to the uniform norm).

Since the countable intersection of conull sets is conull, we may find a conull set $\Omega_0 \subset \Omega$ such that for all $\omega \in \Omega_0$, for all $k \geq 1$, we have

$$\lim_n \int_X f_k(g_1(\omega)\cdots g_n(\omega)x)\,\mathrm{d}\nu(x) = \nu_\omega(f_k).$$

By density of the set $\{f_k\,,\, k \geq 1\}$, one checks that for every $\omega \in \Omega_0$, for every $f \in C_0(X)$, the sequence $f_n(\omega) = \int_X f(g_1(\omega)\cdots g_n(\omega)x)d\nu(x)$ is a Cauchy sequence so it is convergent to a limit denoted by $\nu_\omega(f)$.

Then by uniqueness of the limit $\nu_\omega(f)$ associated to each function $f \in C_0(X)$, we see that the map $f \in C_0(X) \mapsto \nu_\omega(f) \in \mathbb{R}$ is a positive, unital, linear functional, so it corresponds to a probability measure ν_ω, by Riesz representation theorem. By definition, ν_ω is the weak-* limit of the sequence $g_1(\omega)_* \cdots g_n(\omega)_*\nu$.

Moreover, for every $f \in C(X)$, the Lebesgue convergence theorem implies that

$$\int_\Omega \nu_\omega(f)\ \mathrm{d}P(\omega) = \lim_n \int_\Omega \int_X f(g_1(\omega)\cdots g_n(\omega)x)\,\mathrm{d}\nu(x)\ \mathrm{d}P(\omega)$$
$$= \lim_n \int_X f(x)\ \mathrm{d}(\mu^{*n} * \nu)(x) = \nu(f). \qquad \square$$

LEMMA 3.38. *For every $k \geq 1$, for μ^{*k}-almost every $g \in G$, and for P-almost every $\omega \in \Omega$, the sequence of measures $g_1(\omega)_* \cdots g_n(\omega)_* g_*\nu$ weak-* converges to ν_ω.*

PROOF. Fix $k \geq 1$, $f \in C_0(X)$, and define a function $\Phi : G \to \mathbb{R}$ by the formula

$$\Phi(f)(g) = \int_X f(gx)\,\mathrm{d}\nu(x),\ \text{for all } g \in G.$$

This function is right-μ-harmonic, in the sense that $\Phi(g) = \int_G \Phi(gh)d\mu(h)$, for all $g \in G$. This is because ν is μ-harmonic. As in the previous proof, we denote by f_n the function on Ω given by

$$f_n(\omega) := \Phi(g_1(\omega)\cdots g_n(\omega)), \omega \in \Omega.$$

and we also introduce $f_n^g : \omega \in \Omega \mapsto \Phi(g_1(\omega)\cdots g_n(\omega)g)$. We compute

$$I_n := \int_\Omega \int_G |f_n(\omega) - f_n^g(\omega)|^2\,\mathrm{d}P(\omega)\,\mathrm{d}\mu^{*k}(g)$$
$$= \int_G \int_G |\Phi(h) - \Phi(hg)|^2\,\mathrm{d}\mu^{*n}(h)\,\mathrm{d}\mu^{*k}(g)$$
$$= \int_\Omega |f_{n+k}(\omega) - f_n(\omega)|^2\,\mathrm{d}\omega$$
$$= \|f_{n+k} - f_n\|_2^2 = \|f_{n+k}\|_2^2 - \|f_n\|_2^2$$

We implicitly used the fact that f_n is a martingale, to get the last equality, through the equality $\langle f_{n+k}, f_n\rangle = \|f_n\|_2^2$.

This computation shows that I_n is a summable sequence:

$$\sum_{n\in\mathbb{N}} I_n = \sum_{i=0}^{k-1}\sum_{n\in\mathbb{N}} I_{nk+i} = \sum_{i=0}^{k}\sum_{n\in\mathbb{N}} \|f_{(n+1)k+i}\|_2^2 - \|f_{nk+i}\|_2^2 \leq 2k\|f\|_\infty^2.$$

By Fubini-Tonelli's theorem, we find that

$$\int_\Omega \int_G (\sum_n |f_n(\omega) - f_n^g(\omega)|^2)\, \mathrm{d}P(\omega)\, \mathrm{d}\mu^{*k}(g) < \infty.$$

This shows that for P-almost every $\omega \in \Omega$ and for μ^{*k}-almost every $g \in G$, the sum $\sum_n |f_n(\omega) - f_n^g(\omega)|^2$ is finite. In particular, for such ω and g, $f_n^g(\omega)$ converges to the same limit as $f_n(\omega)$, which is $\nu_\omega(f)$.

In order to conclude, we need to show that the convergence holds almost surely, independently on f, i.e. the conull sets of ω's and g's on which convergence holds must be independent on $f \in C_0(X)$. But this is ok thanks to the separability of $C_0(X)$, by proceeding as in the proof of the previous lemma. □

CHAPTER 4

Projective dynamics

In this chapter we present some special features of dynamics on projective spaces. We will be given a representation of a group G on a finite dimensional vector space V and we will study invariant and stationary measures on the projective space $\mathbb{P}(V)$. The invariant setting will be used to prove the so-called Borel density theorem, asserting that lattices in semi-simple Lie groups are Zariski dense. The stationary setting will be used later, in our proof of Margulis superrigidity result.

For simplicity, we will restrain ourselves to vector spaces over $K = \mathbb{R}$ or $\mathbb{C}$, but everything in this chapter applies in fact to any local field (meaning: any valued field whose valuation gives a locally compact topology). For instance it could be applied to $K = \mathbb{Q}_p$ for any prime p.

1. A general lemma

In this chapter we will study the support of certain invariant or stationary measures on projective spaces.

DEFINITION 4.1. The *support* of a Borel measure μ on a second countable topological space X, denoted by $\mathrm{supp}(\mu)$, is the complementary of the largest open subset $U \subset X$ such that $\mu(U) = 0$. Such an open set exists by second countability: just take for U the union of all open sets with measure 0, and observe that this U can be realized as a countable union, and thus has measure 0.

In particular, when we say that a probability measure μ on a lcsc space X is supported on a closed subset $F \subset X$, we simply mean that $\mathrm{supp}(\mu) \subset F$ or equivalently, that $\mu(F) = 1$.

Let V be a finite dimensional vector space. For any $g \in \mathrm{End}(V)$ and $x \in \mathbb{P}(V)$, simply denote by $g(x) \in \mathbb{P}(V)$ the image of x by the projective transformation $\bar{g} \in \mathrm{PGL}(V)$.

All our results about projective dynamics are based on generalizations of the following observation: if a_n, b_n, $n \in \mathbb{N}$ are real numbers such that $\lim_n |a_n/b_n| = \infty$, then the sequence of matrices $g_n = \mathrm{diag}(a_n, b_n)$ acts on the projective line $\mathbb{P}^1 = \mathbb{P}(\mathbb{R}^2)$ by "pushing almost every point towards the point $\mathbb{R}e_1 \in \mathbb{P}(\mathbb{R}^2)$". More precisely, for every $x \in \mathbb{P}^1$, with $x \neq \mathbb{R}e_2$, we have $\lim_n g_n(x) = \mathbb{R}e_1$. In particular, every probability measure on $\mathbb{P}^1$ which is invariant under the elements g_n is supported on $\mathbb{R}e_1 \cup \mathbb{R}e_2$.

This situation somewhat specific to diagonal matrices. A more general situation would be that of a sequence $g_n = k_n \,\mathrm{diag}(a_n, b_n)$, with k_n a sequence of rotations: $k_n \in \mathrm{SO}(2)$. Then this sequence k_n must admit a convergent subsequence so in the end, the projective action of g_n is also easy to describe. This example should be the one to have in mind in the following lemma.

LEMMA 4.2. *Consider a finite dimensional vector space V and a sequence of transformations $g_n \in \mathrm{GL}(V)$ for which there exists a sequence of scalar numbers $(\lambda_n)_n$ such that*

$\lambda_n g_n$ converges to a non-zero endomorphism $A \in \mathrm{End}(V)$. Take a Borel probability measure $\nu \in \mathrm{Prob}(\mathbb{P}(V))$, and assume that the sequence of measures $(g_n)_*\nu \in \mathrm{Prob}(\mathbb{P}(V))$ converges to a measure $\nu_\infty \in \mathrm{Prob}(\mathbb{P}(V))$. Then the following two facts hold true:

- *ν_∞ is supported on a union $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$, where $V_1 = \mathrm{Img}(A)$ and $\dim(V_2) = \dim(\mathrm{Ker}(A))$ (but V_2 needs not be equal to $\mathrm{Ker}(A)$).*
- *If $\nu(\mathbb{P}(\mathrm{Ker}(A))) = 0$, then ν_∞ is supported on $\mathbb{P}(\mathrm{Img}(A))$, and in fact $\nu_\infty = A_*\nu$, in the sense that $\nu_\infty(Y) = \nu(A^{-1}(Y))$, for every Borel subset $Y \subset \mathbb{P}(V)$.*

PROOF. Denote by $X := \mathbb{P}(V)$ and set $k := \dim(\mathrm{Ker}(A))$. For $v \in V$, we denote by $\bar{v} \in X$ the corresponding line. If $v \in V \setminus \mathrm{Ker}(A)$, then $\lambda_n g_n(v)$ converges to $A(v)$, and hence $\lim_n g_n(\bar{v}) \in \mathbb{P}(\mathrm{Img}(A))$. Moreover, by Exercise 4.3 below, up to taking a subsequence of $(g_n)_n$, there exists a k-dimensional subspace $\mathbb{P}(V_2) \subset X$ such that any limit point of the sequence $(g_n(\bar{v}))_n$ belongs to $\mathbb{P}(V_2)$, for all $v \in \mathrm{Ker}(A)$.

In this case, if we take a continuous function $f \in C(X)$ such that $0 \leq f \leq 1$ and $f = 1$ on $\mathbb{P}(\mathrm{Img}(A)) \cup \mathbb{P}(V_2)$, we see that $\lim_n f(g_n(x)) = 1$ for all $x \in X$. For any neighborhood U of $\mathbb{P}(\mathrm{Img}(A)) \cup \mathbb{P}(V_2)$, we may find such a function f which is supported on U. Then, applying Lebesgue convergence theorem, we get

$$\nu_\infty(U) \geq \int_X f \, \mathrm{d}\nu_\infty = \lim_n \int_X f(g_n(x)) \, \mathrm{d}\nu(x) = \int_X \lim_n f(g_n(x)) \, \mathrm{d}\nu(x) = 1.$$

So $\nu_\infty(U) = 1$. Since we may find a countable family of such open sets U whose intersection is $\mathbb{P}(\mathrm{Img}(A)) \cup \mathbb{P}(V_2)$, this gives the first assertion of the lemma:

$$\nu_\infty(\mathbb{P}(\mathrm{Img}(A)) \cup \mathbb{P}(V_2)) = 1.$$

In the special case where $\nu(\mathbb{P}(\mathrm{Ker}(A))) = 0$, then the transformation $A : \bar{v} \in X \mapsto \overline{Av} \in X$ is well defined ν-almost everywhere on X. And for ν-almost every $x \in X$, we have $\lim_n g_n(x) = A(x)$. As before we may apply Lebesgue convergence theorem to deduce that for every continuous function $f \in C(X)$, we have

$$\int_{\mathbb{P}(V)} f \, \mathrm{d}\nu_\infty = \lim_n \int_{\mathbb{P}(V)} f(g_n(x)) \, \mathrm{d}\nu(x) = \int_{\mathbb{P}(V)} f(Ax) \, \mathrm{d}\nu(x).$$

This means that $\nu_\infty = A_*\nu$, which is exactly the formula given in the second item. In particular, ν_∞ is supported on $\mathrm{Img}(A)$. ☐

EXERCISE 4.3. Consider a vector space V, an integer $k \leq \dim(V)$ and a sequence $(V_n)_{n\geq 1}$ of k-dimensional subspaces of V. Prove that, up to replacing $(V_n)_n$ by a subsequence, there exists a k-dimensional subspace V_∞ of V such that any limit point of a sequence $(\overline{v_n})_{n\geq 1}$ in $\mathbb{P}(V)$ such that $v_n \in V_n$ for each n belongs to $\mathbb{P}(V_\infty)$.

2. Invariant measures and the Borel density theorem

The next lemma is due to Furstenberg, and follows easily from Lemma 4.2.

LEMMA 4.4 (Furstenberg). *Let $G \subset \mathrm{GL}(V)$ be a subgroup such that the projective action $G \curvearrowright \mathbb{P}(V)$ admits an invariant probability measure ν. Then either the image of G in $\mathrm{PGL}(V)$ has compact closure, or there exists two proper subspaces $V_1, V_2 \subset V$ such that $\dim(V) = \dim(V_1) + \dim(V_2)$ and ν is supported on $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$.*

PROOF. Assume that the image of G in $\mathrm{PGL}(V)$ does not have compact closure. Then the following claim and Lemma 4.2 show that ν is supported on a union $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$ with $V_1, V_2 \subset V$ two proper subspaces satisfying $\dim(V) = \dim(V_1) + \dim(V_2)$.

Claim. There exists a sequence $(g_n)_n$ in G and a sequence of scalar numbers $(\lambda_n)_n$ such that $\lambda_n g_n$ converges to a non-zero, non-invertible endomorphism $A \in \mathrm{End}(V)$.

Denote by $\|\cdot\|$ an arbitrary nom on the vector space $\mathrm{End}(V)$. If there is no such sequence (g_n), then the closure $\mathcal{C}$ of the set $\{g/\|g\|\,,\, g \in G\}$ inside $\mathrm{End}(V)$ is contained in $\mathrm{GL}(V)$. Moreover, $\mathcal{C}$ is a bounded closed subset of $\mathrm{End}(V)$ so it is compact. Therefore, the image of $\mathcal{C}$ in $\mathrm{PGL}(V)$ is a compact subset which contains the image of G. Thus the image of G in $\mathrm{PGL}(V)$ has compact closure. □

PROPOSITION 4.5. *Let G be a connected semi-simple Lie group with trivial center and no compact factor. Then for any continuous representation $\pi : G \to \mathrm{GL}(V)$ of G on a finite dimensional vector space V, any G-invariant Borel probability measure on $\mathbb{P}(V)$ is supported on the set of G-fixed points.*

PROOF. Write G as a direct product of simple factors, $G = G_1 \times \cdots \times G_n$. Then every quotient of G by a continuous map is a product of some of these factors. Denote by $\nu \in \mathrm{Prob}(\mathbb{P}(V))$ a G-invariant Borel probability measure.

Claim Let $W \subset V$ be a subspace such that $\nu(\mathbb{P}(W)) > 0$ and which is minimal for this property. Then W is globally G-invariant and G acts trivially on W.

Denote by $\alpha := \nu(\mathbb{P}(W))$. For every $g \in G$, we have $\nu(\mathbb{P}(\pi(g)W)) = \alpha$. Moreover since W is minimal, so is $\pi(g)W$. If g, h are such that $\pi(g)W \neq \pi(h)W$, then by minimality, $\nu(\mathbb{P}(\pi(g)W) \cap \mathbb{P}(\pi(h)W)) = 0$. Let us now see G as acting on the set of all subspaces of V and denote by $\mathcal{O} := \{\pi(g)W\,,\, g \in G\}$ the orbit of W under G. From the hayes rule, we have

$$\nu(\bigcup_{E\in\mathcal{O}} \mathbb{P}(E)) = \sum_{E\in\mathcal{O}} \alpha = \alpha|\mathcal{O}|.$$

Since ν is a finite measure, we deduce that $\mathcal{O}$ must be finite, and the stabilizer of W in G is a finite index closed subgroup H. As such, it must also be open and since G is connected, we conclude that $H = G$: W is indeed G-invariant.

By assumption, the action $G \curvearrowright \mathbb{P}(W)$ has an invariant finite measure (namely the restriction of ν to $\mathbb{P}(W)$). So we may apply Lemma 4.4 to the representation $G \to \mathrm{GL}(W)$. By minimality of W, we deduce that the image of G in $\mathrm{PGL}(W)$ is compact. But since every compact quotient of G is trivial, we deduce that the morphism $G \to \mathrm{PGL}(W)$ is trivial. Thus G acts on W by dilations. This means that there exists a continuous morphism $\phi : G \to \mathbb{R}$ such that an element g of G acts on W by multiplication by $\phi(g)$. But since any abelian quotient of G is trivial, we deduce that ϕ is trivial, i.e. G acts trivially on W. This prove the claim.

To conclude, denote by $V_0 \subset V$ the largest subspace of V on which G acts trivially. Denote by ν_0 the restriction of ν to $\mathbb{P}(V_0)$. In order to prove the lemma, it suffices to show that $\nu_0 = \nu$. If not, then $\nu - \nu_0$ is a non-zero G-invariant finite Borel measure on $\mathbb{P}(V)$. We may renormalize it to assume that it is a probability measure ν_1. It satisfies $\nu_1(\mathbb{P}(V_0)) = 0$. Then denote by $W \subset V$ a minimal subset of V such that $\nu_1(\mathbb{P}(W)) > 0$. Then the claim applied to ν_1 implies that W is globally G-invariant and $W \subset V_0$. This contradicts the fact the $\nu_1(\mathbb{P}(V_0)) = 0$. □

The following corollary is the so-called Borel density lemma, which asserts that lattices in semi-simple Lie groups are Zariski dense.

COROLLARY 4.6. *Let Γ be a lattice in a connected semi-simple Lie group G with trivial center and no compact factor. Let $\pi : G \to \mathrm{GL}(V)$ be a continuous representation of*

G on a finite dimensional vector space V. Then any subspace of V which is globally Γ-invariant is globally G-invariant.

PROOF. Let $W \subset V$ be a Γ-invariant subspace and denote by $k := \dim(W)$.

STEP 1. We use an exterior product trick to reduce to the case where $k = 1$.

We define a linear representation $\widetilde{\pi}$ on the exterior product $\Lambda^k(V)$ by the formula

$$\widetilde{\pi}(g)(v_1 \wedge \cdots \wedge v_k) = (\pi(g)v_1) \wedge \cdots \wedge (\pi(g)v_k).$$

This is well defined by the universal property of exterior products, and it is still continuous. Moreover, the subspace $\Lambda^k(W) \subset \Lambda^k(V)$ is a one dimensional subspace which is invariant under $\widetilde{\pi}(\Gamma)$. If the corollary holds for one dimensional subspaces, then we deduce that $\Lambda^k(W) \subset \Lambda^k(V)$ is $\widetilde{\pi}(G)$-invariant. This implies that W is $\pi(G)$-invariant (exercise).

STEP 2. We prove the corollary in the case where $k = 1$.

In this case W defines a point $x \in \mathbb{P}(V)$, which is Γ-invariant. Then the orbit map $g \in G \mapsto \pi(g)(x) \in \mathbb{P}(V)$ factors to a G-equivariant continuous map $\theta : G/\Gamma \to \mathbb{P}(V)$. Since Γ is a lattice in G, G/Γ admits a G-invariant Borel probability measure ν_0, whose push-forward $\nu := \theta_* \nu_0$ is a G-invariant probability measure on $\mathbb{P}(V)$, supported on the orbit of x. Proposition 4.5 then implies that this orbit consists of G-invariant points, i.e. x is G-invariant. This proves that W is G-invariant. □

3. Stationary measures on projective spaces

Proposition 4.5 tells us that invariant probability measures on projective spaces can only occur for obvious reasons: they are supported on the set of fixed points; in particular there must exist fixed points. In particular for irreducible non-trivial representations of semi-simple Lie group there is no invariant probability measure at all. But as we observed in Lemma 3.34, there always exist stationary measures with respect any probability measure μ on the acting group G. In this section, we give conditions ensuring that such a measure is unique: an irreducibility condition and a proximality condition.

Consider a continuous representation $\pi : G \to \mathrm{GL}(V)$ of a lcsc group G on a finite dimensional vector space V.

DEFINITION 4.7. The representation π is called *irreducible* if the only G-invariant subspaces of V are $\{0\}$ and V. It is called *strongly irreducible* if its restriction to any finite index subgroup of G is irreducible.

EXERCISE 4.8. Check that π is strongly irreducible if and only if there is no finite family $\mathcal{F}$ of non-trivial proper subspaces of V which is G-invariant (meaning that $\pi(g)W \in \mathcal{F}$ for all $W \in \mathcal{F}$).

DEFINITION 4.9. The *proximal rank* of π is the minimal rank of a matrix $A \in \mathrm{End}(V)$ which is the limit of a sequence $\lambda_n \pi(g_n)$, with $g_n \in G$ and λ_n scalar numbers. We say that π is *proximal* if its proximal rank is 1.

We also need a condition on the choice of the measure μ (for instance the Dirac measure at the trivial element $\mu = \delta_e$ is not expected to generate interesting random walks...).

DEFINITION 4.10. A measure $\mu \in \mathrm{Prob}(G)$ is called *generating* if the semi-group generated by its support is dense in G.

THEOREM 4.11. *Let $\mu \in \mathrm{Prob}(G)$ be a generating measure. If π is strongly irreducible and proximal then there is a unique μ-stationary measure $\nu \in \mathrm{Prob}(\mathbb{P}(V))$. Better, there is a unique μ-stationary measure in* $\mathrm{Prob}(\mathrm{Prob}(\mathbb{P}(V)))$*; it is supported on the closed subset of Dirac measures on $\mathbb{P}(V)$.*

The proof of the theorem is based on a series of lemmas. Let $\mu \in \mathrm{Prob}(G)$ be a generating measure and assume that π is strongly irreducible and proximal. We denote by ν a μ-stationary measure on $\mathbb{P}(V)$. We will use the notation (Ω, P) and $\{\nu_\omega\}_{\omega\in\Omega}$ introduced in Section 5.

LEMMA 4.12. *The measure $\nu \in$ $\mathrm{Prob}(\mathbb{P}(V))$ does not charge proper subspaces of V: $\nu(\mathbb{P}(W)) = 0$, for every proper subspace $W \subset V$.*

PROOF. Otherwise, we may consider the minimal dimension $k < \dim(V)$ of a subspace $W \subset V$ such that $\nu(\mathbb{P}(W)) \neq 0$. For $\varepsilon > 0$, denote by

$$\mathcal{F}_\varepsilon := \{W \subset V \mid \dim(W) = k, \nu(\mathbb{P}(W)) \geq \varepsilon\}.$$

By minimality of k, for any two distinct subspaces $W, W' \in \mathcal{F}_\varepsilon$, we have $\nu(\mathbb{P}(W) \cap \mathbb{P}(W')) = 0$. So for any finite subset $\mathcal{F}_0 \subset \mathcal{F}_\varepsilon$, we get

$$1 \geq \nu\Big(\bigcup_{W\in\mathcal{F}_0} \mathbb{P}(W)\Big) = \sum_{W\in\mathcal{F}_0} \nu(\mathbb{P}(W)) \geq \varepsilon|\mathcal{F}_0|.$$

This shows that $\mathcal{F}_0$ has cardinality at most $1/\varepsilon$, and thus $\mathcal{F}_\varepsilon$ is finite. So if we define

$$\alpha := \sup(\{\nu(\mathbb{P}(W)) \mid W \subset V\ :\ \dim(W) = k\}),$$

we find that α is attained, and $\mathcal{F}_\alpha$ is finite.

Now, since ν is μ-stationary, we have, for all $W \in \mathcal{F}_\alpha$,

$$\alpha = \nu(\mathbb{P}(W)) = \int_G \nu(\mathbb{P}(\pi(g^{-1})W))\, \mathrm{d}\mu(g).$$

By maximality of α we have $\nu(\mathbb{P}(\pi(g^{-1})W)) \leq \alpha$ for all $g \in G$ and since μ is a probability measure, the above equality actually implies that $\nu(\mathbb{P}(\pi(g^{-1})W)) = \alpha$ for μ-almost every $g \in G$. In particular, $\pi(g^{-1})W \in \mathcal{F}_\alpha$ for almost every $g \in G$ and every $W \in \mathcal{F}_\alpha$. But observe that the set of elements $g \in G$ such that $\pi(g^{-1})W \in \mathcal{F}_\alpha$ for every $W \in \mathcal{F}_\alpha$ is a closed subgroup of G. Since it has measure 1, it must contain the support of μ, and also the closure of the semi-group generated by $\mathrm{supp}(\mu)$. since μ is a generating measure, we conclude that G preserves the finite family $\mathcal{F}_\alpha$, which contradicts our assumption that π is strongly irreducible. □

LEMMA 4.13. *For P-almost every $\omega \in \Omega$, the conditional measure ν_ω is a Dirac measure $\nu_\omega = \delta_{p(\omega)}$. The point $p(\omega)$ depends almost surely only on ω, not on the choice of the stationary measure ν.*

PROOF. By Lemma 3.37, there exists a co-null set $\Omega_0 \subset \Omega$ such that $\pi(g_1(\omega)\cdots g_n(\omega))_*\nu$ converges weakly to ν_ω for every $\omega \in \Omega_0$. For $\omega \in \Omega_0$, we denote by $V_\omega \subset V$ the smallest subspace $W \subset V$ such that $\mathbb{P}(W)$ contains the support of ν_ω.

Fix $\omega \in \Omega_0$ and choose a norm $\|\cdot\|$ on $\mathrm{End}(V)$. Denote by A a limit point of the sequence $(\pi(g_1(\omega)\cdots g_n(\omega))/\|\pi(g_1(\omega)\cdots g_n(\omega))\|)_n$. By Lemma 4.12, $\nu(\mathbb{P}(\mathrm{Ker}(A))) = 0$ and thus Lemma 4.2 implies that $\nu_\omega = A_*\nu$ is supported on the image of A. Moreover, using again Lemma 4.12, we see that $\nu_\omega(\mathbb{P}(W)) = \nu(A^{-1}(W)) = 0$, for any proper subspace $W \subset \mathrm{Img}(A)$. So $\mathrm{Img}(A) = V_\omega$.

In particular, for P-almost every $\omega \in \Omega$, V_ω does not depend on the stationary measure ν, but only on ω.

Claim. V_ω is one dimensional for P-almost every $\omega \in \Omega$.

This is where we use the fact that π is proximal. By Lemma 3.38, we may replace Ω_0 by a co-null subset if necessary to ensure that for every $\omega \in \Omega_0$, every $k \geq 1$, and μ^{*k}-almost every $g \in G$, we have $\nu_\omega = \lim_n \pi(g_1(\omega)\cdots g_n(\omega)g)_*\nu$. Fix such $\omega \in \Omega_0$, $k \geq 1$, and $g \in G$. Then we know that $A\pi(g) \in \mathrm{End}(V)$ is a limit point of the sequence $(\pi(g_1(\omega)\cdots g_n(\omega)g)/\|\pi(g_1(\omega)\cdots g_n(\omega))\|)_n$. Thanks to Lemma 4.2, we have $\nu_\omega = (A\pi(g))_*\nu$. By continuity, this equation holds for every $g \in \mathrm{supp}(\mu^{*k})$, for every k. Since μ is generating, this holds for every $g \in G$.

Since π is proximal, we may find a sequence $(h_k)_{k\in\mathbb{N}}$ in G and a sequence of scalars (λ_k) such that $\lambda_k\pi(h_k)$ converges to a rank one endomorphism $B \in \mathrm{End}(V)$ as $k \to \infty$. By irreducibility of π, we may assume that $\mathrm{Img}(B) \nsubseteq \mathrm{Ker}(A)$, so that $AB \neq 0$. Recall that $A : \mathbb{P}(V) \to \mathbb{P}(V)$ is in fact only defined on the co-null open subset $\mathbb{P}(V) \setminus \mathbb{P}(\mathrm{Ker}(A))$, on which it is continuous. Thus, for every $v \in V \setminus \mathrm{Ker}(AB)$, we have $Bv \notin \mathrm{Ker}(A)$, and so $\pi(h_k)v \notin \mathrm{Ker}(A)$ for k large enough. In particular $A\pi(h_k)\bar{v} \in \mathbb{P}(V)$ converges to $AB\bar{v}$. By Lemma 4.12, we have $\nu(\mathrm{Ker}(AB)) = 0$, and thus for ν-almost every $x \in \mathbb{P}(V)$, $\lim_k A\pi(h_k)x = ABx$. Lebesgue convergence theorem then implies that $(AB)_*\nu = \lim_k(A\pi(h_k))_*\nu = \nu_\omega$. In particular, ν_ω is supported on $\mathrm{Img}(AB) \subset \mathrm{Img}(A) = V_\omega$. By minimality, we must have $\mathrm{Img}(AB) = \mathrm{Img}(A)$. Since AB has rank one, so does A. This proves the claim.

The claim now implies that ν_ω is the Dirac measure at the point $p(\omega) \in \mathbb{P}(V)$ corresponding to the line $V_\omega = \mathrm{Img}(A)$. Thus it only depends on ω, and not on the choice of the measure ν. $\square$

Proof of Theorem 4.11. Denote by $X := \mathbb{P}(V)$. The uniqueness of a μ-stationary measure on X follows from the previous lemma and the fact that a stationary measure ν can be reconstructed as the integral of its conditional measures. Denote by $\nu \in \mathrm{Prob}(X)$ this unique μ-stationary measure and let $\widetilde{\nu} \in \mathrm{Prob}(\mathrm{Prob}(X))$ be an arbitrary μ-stationary measure.

We will use the barycenter map $\mathrm{Bar} : \mathrm{Prob}(\mathrm{Prob}(X)) \to \mathrm{Prob}(X)$, given by

$$\mathrm{Bar}(\sigma) = \int_{\mathrm{Prob}(X)} \zeta \, \mathrm{d}\sigma(\zeta).$$

This formula means that for every $f \in C(X)$, we define $\mathrm{Bar}(\sigma)(f)$ to be $\int_{\mathrm{Prob}(X)} \zeta(f)\,\mathrm{d}\sigma(\zeta)$, and that this defines a unital, positive linear functional $\mathrm{Bar}(\sigma)$ on $C(X)$, i.e. a probability measure on X. This barycenter map is well behaved: it is continuous with respect to weak-* topologies, and it is G-equivariant with respect to the natural G-actions on X and $\mathrm{Prob}(X)$. It is moreover an affine map, if we view $\mathrm{Prob}(\mathrm{Prob}(X))$ and $\mathrm{Prob}(X)$ as convex subsets of the dual spaces of $C(\mathrm{Prob}(X))$ and $C(X)$ respectively.

Thanks to these properties, we see that the barycenter of $\widetilde{\nu}$ must be μ-stationary. So by uniqueness, we must have $\mathrm{Bar}(\widetilde{\nu}) = \nu$. Now for almost every $\omega \in \Omega$, we have simultaneous convergence to conditional measures: $\lim_n(g_1(\omega)\cdots g_n(\omega))_*\widetilde{\nu} = \widetilde{\nu}_\omega$ and $\lim_n(g_1(\omega)\cdots g_n(\omega))_*\nu = \nu_\omega$. By continuity and equivariance of the barycenter map, this gives:

$$\nu_\omega = \mathrm{Bar}(\widetilde{\nu}_\omega).$$

By Lemma 4.13, we know that almost every ν_ω is a Dirac measure, and thus is extremal in $\mathrm{Prob}(X)$. This forces $\widetilde{\nu}_\omega$ to be the Dirac measure at ν_ω. So $\widetilde{\nu}_\omega$ does not depend on

$\widetilde{\nu}$ for almost every ω, thus $\widetilde{\nu}$ is unique. Moreover, $\widetilde{\nu}_\omega$ is supported on the set of Dirac measures on X, and thus, so is $\widetilde{\nu}$. □

The following corollary will play a crucial role in our proof of Margulis superrigidity theorem.

COROLLARY 4.14. *Let G be an lcsc group, with a generating measure $\mu \in \mathrm{Prob}(G)$. Let $\pi : G \to \mathrm{GL}(V)$ be a strongly irreducible and proximal representation of G. Let (X_0, ν_0) be a (G,μ)-space. Then*

a) Any G-equivariant measurable map $\theta : X_0 \to \mathrm{Prob}(\mathbb{P}(V))$ almost surely ranges into the set of Dirac measures.
b) There exists at most one such map, up to ν_0-almost everywhere equality.

PROOF. The push-forward $\theta_* \nu_0$ is a μ-stationary measure on $\mathrm{Prob}(\mathbb{P}(V))$. By Theorem 4.11, it must be supported on the subset of Dirac measures on $\mathbb{P}(V)$. This easily implies *a*). Now given two such G-maps $\theta_1, \theta_2 : X_0 \to \mathrm{Prob}(\mathbb{P}(V))$, we can form the middle map:

$$\theta : x \in X_0 \mapsto \frac{1}{2}(\theta_1(x) + \theta_2(x)) \in \mathrm{Prob}(\mathbb{P}(V)).$$

Then by item *a*), almost every value $\theta(x)$ is a Dirac measure, which implies by extremality that $\theta_1(x) = \theta_2(x)$, almost surely. □

In order to apply the above corollary, it will be convenient to be able to construct proximal, irreducible representations.

PROPOSITION 4.15. *Let G be a connected semi-simple Lie group with trivial center, and let $\Delta \subset G$ be a subgroup with non-compact closure. Then there exists a continuous linear representation $\pi : G \to \mathrm{GL}(V)$ which is irreducible (hence strongly irreducible), and whose restriction to Δ is proximal.*

PROOF. Since G is semi-simple with trivial center, the adjoint representation π_0 of G on its Lie algebra is a homeomorphism onto its image. In particular $\pi_0(\Delta)$ has a non-compact closure inside $\mathrm{GL}(\mathfrak{g})$. Note moreover that since G is semi-simple it is perfect (i.e. it has no non-trivial abelian quotient). Therefore $\pi_0(G)$ is contained in $\mathrm{SL}(\mathfrak{g})$. So as a non-compact subgroup of $\mathrm{SL}(\mathfrak{g})$, $\pi_0(\Delta)$ contains a sequence (g_n) such that $g_n/\|g_n\|$ converges to a non-invertible matrix (here we pick for $\|\cdot\|$ an arbitrary norm on $\mathrm{End}(V)$). This shows that the proximal rank of the restriction of π_0 to Δ is an integer k strictly smaller than $\dim(\mathfrak{g})$.

Therefore the representation π_0^k of G on the exterior product $\Lambda^k(\mathfrak{g})$ has its restriction to Δ which is proximal. Since G is semi-simple, we may decompose this representation of G as a direct sum of irreducible ones. Then one of them, at least, has a restriction to Δ which is proximal. This is the desired representation.

Note that since G is connected, it has no proper finite index closed subgroup. So any irreducible representation of G is strongly irreducible. □

CHAPTER 5

Extra topics

The proof of Margulis' superrigidity theorem will require two more lemmas that we isolate in this chapter. The first lemma is the existence of boundary maps, Lemma 5.8, which is based on the classical notion of an amenable group. The second lemma is Lemma 5.9 in which we construct a nice measure on a lattice Γ in a semi-simple Lie group, which will be our reference measure for some dynamical systems.

1. Amenable groups

There are many equivalent formulations of amenability for topological groups. The one that we will use is based on a fixed point property for affine actions on compact convex sets.

Definition 5.1. A *compact convex space* is a non-empty compact convex subset $\mathcal{C}$ of a locally convex topological vector space space E. We denote by $\mathrm{Aff}(\mathcal{C})$ the vector space of all affine functions on $\mathcal{C}$, i.e., of all restrictions to $\mathcal{C}$ of continuous affine functions $E \to \mathbb{R}$. An *affine action* of a group G on a compact convex space is by definition a group action $G \curvearrowright \mathcal{C}$ which extends to a *continuous* action on E by affine transformations.

Example 5.2. The typical example is that of probability spaces. If X is a compact set, then the space $\mathrm{Prob}(X)$ of all Borel probability measures on X is a compact convex space, when viewed as a subset of $E = C(X)'$, endowed with the weak-* topology. This is due to Banach-Alaoglu theorem. Any continuous action σ of a group G on X induces a norm continuous action of G on $C(X)$ by the formula $g \cdot f := f \circ \sigma^{-1}$, for all $f \in C(X)$, $g \in G$. By duality, this gives a weak-* continuous action of G on E (by linear transformations). When restricted to $\mathrm{Prob}(X)$, this action is exactly the action given by the push forward of measures. Indeed, we have for all $\mu \in \mathrm{Prob}(X)$, $f \in C(X)$, $g \in G$:

$$(g \cdot \mu)(f) = \mu(g^{-1} \cdot f) = \mu(f \circ \sigma_g) = \int_X f(gx)\, \mathrm{d}\mu(x) = (g_*\mu)(f).$$

We get that $G \curvearrowright \mathrm{Prob}(X)$ is an affine action.

Definition 5.3. A topological group G is said to be *amenable* if every affine action of G on a compact convex space admits a fixed point.

Alternatively, amenability can be expressed in terms of invariant measures.

Lemma 5.4. *A topological group G is amenable if and only if for every continuous action of G on a compact space X (not necessarily convex), there exists a G-invariant probability measure on X.*

Proof. If G is amenable, and if $G \curvearrowright X$ is a continuous action on a compact space X, then the push forward action $G \curvearrowright \mathrm{Prob}(X)$ is an affine action on a compact convex space. So it must have a fixed point, i.e. a G-invariant probability measure on X.

The converse is based on the barycenter map Bar : $\mathrm{Prob}(\mathcal{C}) \to \mathcal{C}$, where $\mathrm{Bar}(\mu)$ is the unique point $x \in \mathcal{C}$ such that $f(x) = \int_{\mathcal{C}} f(y)\,\mathrm{d}\mu(y)$, for all affine map $f \in \mathrm{Aff}(\mathcal{C})$. Take an affine action of G on the compact convex set $\mathcal{C}$. If there is a G-invariant probability measure on $\mathcal{C}$, then its barycenter must be a G-invariant point in $\mathcal{C}$. □

The class of amenable groups is rather large, as the following lemma shows. But the class of non-amenable groups is very large as well. Typically, connected semi-simple Lie groups will never be amenable, unless they are compact.

Lemma 5.5. *The following facts are true.*

(1) *Compact groups are amenable.*
(2) *Abelian groups are amenable.*
(3) *Amenability is stable under taking (continuous) quotients, extensions.*
(4) *A connected semi-simple Lie group is amenable if and only if it is compact.*

Proof. *(1)* If G is a compact group acting on a compact set X, then we may fix a point $x \in X$, and consider the orbit map $g \in G \mapsto gx \in X$. Then the normalized Haar measure on G is left G-invariant, and its push forward under the orbit map becomes a G-invariant probability measure on X.
(2) Assume that G is any abelian topological group. Consider an affine action of G on a compact convex space $\mathcal{C}$. For any subgroup $H < G$, we denote by $\mathcal{C}^H \subset \mathcal{C}$ the closed subset of fixed points under H. Since G is abelian, this subset is globally G-invariant.

We first claim that if $H = \langle g \rangle$ is generated by a single element g, then $\mathcal{C}^H$ is non-empty. Indeed, take an arbitrary point $x \in \mathcal{C}$ and consider the sequence $(x_n)_{n\geq 1}$ defined by

$$x_n = \frac{1}{n}\sum_{k=0}^{n-1} g^k(x) \in \mathcal{C}.$$

Denote by $y \in \mathcal{C}$ any limit point of this sequence. Then for all $n \geq 1$, we have $gx_n - x_n = (g^n(x) - x)/n$, which tends to 0 in E as n goes to infinity. This shows that $gy = y$, and hence $\mathcal{C}^H$ is not empty.

More generally, if H is a subgroup of G such that $\mathcal{C}^H$ is not empty and $g \in G$ is any element, then the above paragraph applied to the action $G \curvearrowright \mathcal{C}^H$ shows that $\mathcal{C}^{\langle H,g\rangle} = (\mathcal{C}^H)^g$ is not empty. We may apply this observation inductively to prove that $\mathcal{C}^H$ is non-empty for any finitely generated subgroup H of G.

Therefore, the family $\mathcal{C}^{\langle g\rangle}$, $g \in G$, has the finite intersection property: any finite intersection of such sets is non-empty. This proves that

$$\mathcal{C}^G = \bigcap_{g\in G} \mathcal{C}^{\langle g\rangle} \neq \emptyset.$$

(3) Any action of a quotient of G is in particular an action of G, so it is straightforward from the definition that amenability passes to quotients. Consider now an extension $1 \to H \to G \to Q \to 1$ with H and Q both amenable. Take an affine action on a compact convex space $G \curvearrowright \mathcal{C}$. Then by assumption the set $\mathcal{C}^H$ is non-empty. It is moreover globally G-invariant since H is normal inside G. The action $G \curvearrowright \mathcal{C}^H$ then factors to an affine action of Q, and thus $(\mathcal{C}^H)^Q$ is non-empty. But this set coincides with $\mathcal{C}^G$.
(4) We only treat the case where G has finite center. The proof of the general case is the same but one needs to know that the universal cover of a compact semi-simple group is compact.

If G is a non-compact connected semi-simple Lie group with finite center, then it admits a simple quotient with trivial center which is non-compact. Let us show that such a quotient G_0 is non-amenable. By simplicity, in the adjoint representation of G_0 there is no invariant vector. Since G_0 is non-compact Theorem 4.5 tells us that there is no invariant probability measure under the projective action of G_0 on $\mathbb{P}(\mathfrak{g}_0)$. So G_0 is non-amenable. □

REMARK 5.6. Although we will not use this, let us mention that amenability passes to *closed* subgroups. This relies on the axiom of choice. Be careful however, amenability does not pass to arbitrary subgroups! For example, one may embed a non-abelian free group (densely) into the compact group SO(3). But non-abelian free groups, viewed as discrete groups, are not amenable.

COROLLARY 5.7. *Solvable groups are amenable. In particular the minimal parabolic subgroup P of a connected semi-simple Lie group is amenable.*

PROOF. The fat that solvable groups are amenable is checked by induction thanks to the previous lemma. Now it is known from Iwasawa decomposition that a minimal parabolic subgroup P as in the statement is a compact extension of a solvable group. So P must be amenable. □

Of course, in our special case $G = \mathrm{SL}_d(\mathbb{R})$, P is nothing but the (solvable) subgroup of upper triangular matrices. So it is clearly amenable.

2. A boundary map

PROPOSITION 5.8. *Let Γ be a lattice in a connected semi-simple Lie group G and denote by P a minimal parabolic subgroup. Consider a continuous action of Γ on a compact metrizable space X. Then there exists a Γ-equivariant map $\theta : G/P \to \mathrm{Prob}(X)$.*

Here we mean that the map $\theta : G/P \to \mathrm{Prob}(X)$ is such that $\theta(\gamma x) = \gamma\theta(x)$ for every $\gamma \in \Gamma$ and almost every $x \in G/P$. The only property of P used in the proof is amenability.

PROOF. As usual, we view $\mathrm{Prob}(X)$ as a compact convex subset of the dual space $C(X)'$, endowed with the weak-* topology.

We denote by $F := L^1_\Gamma(G, C(X))$ the Banach space of all measurable Γ-equivariant functions $f : G \to C(X)$ such that

$$\|f\| := \int_{\Gamma\backslash G} \|f(g)\| \,\mathrm{d}g < \infty.$$

In the above integral, we use the fact that f is Γ-equivariant, and thus, $g \in G \mapsto \|f(g)\|$ is left Γ-invariant to view the latter as a function on $\Gamma\backslash G$, which we integrate with respect to the unique right-G-invariant probability measure on $\Gamma\backslash G$. We'll use this again implicitly a few lines below.

On the other hand we denote by $E := L^\infty_\Gamma(G, C(X)')$ the vector space of all measurable, bounded, Γ-equivariant maps $\phi : g \in G \mapsto \phi_g \in C(X)'$.

In the definition of E and F, $C(X)$ (resp. $C(X)'$) is endowed with the Borel σ-algebra of its norm topology (resp. its weak-* topology). Measurability of functions is thus considered with respect to these σ-algebras. Of course we abuse with notation here, and in fact elements of E and F are only considered modulo almost everywhere equality.

Since X is separable, one can prove that E identifies with the dual of the Banach space F thanks to the pairing

$$(m, f) \mapsto \int_{\Gamma\backslash G} \phi_g(f(g))\,\mathrm{d}g.$$

So we may endow E with the weak-* topology. Inside E we have the convex subset $\mathcal{C} := L^\infty_\Gamma(G, \mathrm{Prob}(X))$. This set is easily seen to be bounded in E with respect to the dual norm on E (coming from the norm on F), moreover it is a weak-* closed subset of E, so it must be weak-* compact, thanks to the Banach-Alaoglu theorem.

Now observe that the right action of G on itself gives by pre-composition a norm continuous linear action of G on F, and hence a weak-* continuous action on E. Observe that since the measure on $\Gamma\backslash G$ is right G-invariant, this dual action on E is also given by pre-composition, i.e.

$$(g \cdot \phi)_h = \phi_{hg}, \text{ for all } \phi \in E, g, h \in G.$$

Moreover $\mathcal{C}$ is clearly globally invariant under this action. By amenability of P, we may find a fixed point $\theta \in \mathcal{C}$ under the action of P. This fixed point is then a map $G \to \mathrm{Prob}(X)$ which is left Γ-equivariant and right P-invariant. We may thus factorize it to a Γ equivariant map $G/P \to \mathrm{Prob}(X)$. □

3. Existence of nice measures on lattices

Let G be a connected semi-simple Lie group P a minimal parabolic subgroup and K a maximal compact subgroup of G. Then it follows from the Iwasawa decomposition K acts transitively on G/P. Moreover, since K is amenable, there exists a K-invariant Borel probability measure ν_0 on G/P. By transitivity, this measure is unique, and any K-quasi-invariant measure on G/P is equivalent to ν_0. In particular any G-quasi-invariant measure on G/P is equivalent to ν_0, which implies that ν_0 is in fact quasi-invariant under G.

Moreover from uniqueness, it follows that if $\mu_0 \in \mathrm{Prob}(G)$ is any left K-invariant probability measure on G, then any μ_0-stationary measure on G/P must be K-invariant, and hence, ν_0 is the unique μ_0-stationary measure on G/P. The following proposition shows that even though a lattice in G does not carry a K-invariant probability measure, there still exist measures μ on it such that ν_0 is μ-stationary.

Lemma 5.9. *Let Γ be a lattice in a connected semi-simple Lie group G with trivial center. Denote by ν_0 the unique K-invariant measure on G/P. Then there exists a probability measure $\mu \in \mathrm{Prob}(\Gamma)$ whose support is all of Γ and such that ν_0 is μ-stationary.*

Proof. For $t \in [0, 1]$, we define a set

$$\mathcal{P}_t := \{\mu \in \mathrm{Prob}(G) \mid \mu(\Gamma) \geq 1 - t \text{ and } \mu(\{h\}) > 0, \text{ for all } h \in \Gamma\}.$$

Consider the function $\tau : G \to [0, 1]$ given by

$$\tau(g) = \inf\{t \in [0, 1] \mid \exists \mu \in \mathcal{P}_t : g_*\nu_0 = \mu * \nu_0\}.$$

We admit that this is a measurable function on G, with respect to the σ-algebra of Haar measurable subsets. This can be proved using [**Arv76**, Section 3], but this would require us introducing too much formalism in view of the time we have.

We now prove a series of claims that will eventually lead to the fact that τ is uniformly bounded away from 1.

Claim 1. For every $g \in G$, $\gamma \in \Gamma$, there exists $\varepsilon_{g,\gamma} > 0$ and a finite measure $\mu_{g,\gamma}$ on G such that $g_*\nu_0 = \varepsilon_{g,\gamma}\gamma_*\nu_0 + \mu_{g,\gamma} * \nu_0$.

We first introduce an auxiliary measure $\mu_0 \in \mathrm{Prob}(G)$. We let $U \subset G$ be an open neighborhood of the identity with compact closure and which is left K-invariant. We denote by $\mu_0 \in \mathrm{Prob}(G)$ the restriction to U of an appropriately normalized Haar measure on G (to ensure that μ_0 is indeed a probability measure on G). Observe that $\bigcup_{n\geq 1} U^n$ contains an open subgroup of G, hence is equal to G by connectedness. Then we know that ν_0 is a μ_0-stationary measure on G/P.

Observe that any convolution power μ_0^{*n}, $n \geq 2$, of μ_0 is absolutely continuous with respect to the Haar measure, and that the corresponding Radon-Nykodim derivative is continuous and supported on U^n. Since U is compact and $\bigcup_{n\geq 1} U^n = G$, we may find n such that gU^n contains $\gamma(U) = \mathrm{supp}(\gamma_*\mu_0)$. Therefore there exists $\varepsilon_{g,\gamma} > 0$ such that $g_*\mu_0^{*n} > \varepsilon_{g,\gamma}\gamma_*\mu_0$. In particular, the measure $\mu_{g,\gamma} := g_*\mu_0^{*n} - \varepsilon_{g,\gamma}\gamma_*\mu_0$ is a positive measure. And we have

$$g_*\nu_0 = g_*(\mu_0^{*n} * \nu_0) = (\varepsilon_{g,\gamma}\gamma_*\mu_0 + \mu_{g,\gamma}) * \nu_0 = \varepsilon_{g,\gamma}\gamma_*\nu_0 + \mu_{g,\gamma} * \nu_0.$$

This proves Claim 1.

Claim 2. For every $g \in G$, $\tau(g) < 1$.

Choose a family $(a_\gamma)_{\gamma\in\Gamma}$ of (strictly) positive numbers such that $\sum_\gamma a_\gamma = 1$, and sum up the equalities from Claim 1 as γ varies:

$$g_*\nu_0 = \sum_{\gamma\in\Gamma} a_\gamma g_*\nu_0 = \sum_{\gamma\in\Gamma} a_\gamma(\varepsilon_{g,\gamma}\gamma_*\nu_0 + \mu_{g,\gamma} * \nu_0) = (\sum_{\gamma\in\Gamma} a_\gamma\varepsilon_{g,\gamma}\delta_\gamma + a_\gamma\mu_{g,\gamma}) * \nu_0.$$

Note that the measure $\mu := \sum_{\gamma\in\Gamma} a_\gamma\varepsilon_{g,\gamma}\delta_\gamma + a_\gamma\mu_{g,\gamma}$ is in $\mathcal{P}_t$, where $t = 1 - \sum_{\gamma\in\Gamma} a_\gamma\varepsilon_{g,\gamma} < 1$. This proves that $\tau(g) < 1$.

Claim 3. We may find $\ell < 1$ such that $\tau(g) < \ell$ for every $g \in G$.

We will use the auxiliary measure $\mu_0 \in \mathrm{Prob}(G)$ again. First observe that τ is left Γ-invariant: $\tau(\gamma g) = \tau(g)$ for every $g \in G$, $\gamma \in \Gamma$. Moreover, since ν_0 is μ_0-stationary, we have

$$\tau(g) \leq \int_G \tau(gh)\,\mathrm{d}\mu_0(h). \tag{3.1}$$

This is based on von Neumann's measurable selection theorem, see for instance [**Arv76**, Theorem 3.4.3]. Although we omit the proof of the theorem, let us briefly sketch how to use it to prove the inequality. Fix $\varepsilon > 0$ and consider the Borel subset of $G \times \mathrm{Prob}(G)$

$$X := \{(g,\mu) \in G \times \mathrm{Prob}(G) \mid \mu \in \mathcal{P}_{\tau(g)+\varepsilon}, g_*\nu_0 = \mu * \nu_0\}$$

The Borel map $\theta : (g,\mu) \in X \mapsto g \in G$ is surjective. Then the measurable selection theorem tells us that we may find a measurable section $\sigma : G \to X$ such that $\theta \circ \sigma = \mathrm{id}$. Then $\sigma(g)$ is of the form (g, μ_g), and $g \in G \mapsto \mu_g \in \mathrm{Prob}(G)$ is measurable. For every $g \in G$, we then have

$$(\int_G \mu_{gh}\,\mathrm{d}\mu_0(h)) * \nu_0 = \int_G \mu_{gh} * \nu_0\,\mathrm{d}\mu_0(h) = \int_G (gh)_*\nu_0\,\mathrm{d}\mu_0(h) = g_*\nu_0.$$

Moreover the measure $\mu = \int_G \mu_{gh}\,\mathrm{d}\mu_0(h)$ has atoms at every point of Γ and satisfies

$$\mu(\Gamma) = \int_G \mu_{gh}(\Gamma) d\mu_0(h) \geq 1 - \int_G \tau(gh) d\mu_0(h) - \varepsilon.$$

This shows that $\tau(g) \leq \int_G \tau(gh) d\mu_0(h) + \varepsilon$. Since ε can be arbitrarily small, we get (3.1).

Now we view τ as a function on $X := \Gamma\backslash G$. Denote by λ_X the right G-invariant probability measure on X. Equation (3.1) and Cauchy-Schwarz inequality give

$$\begin{aligned}\int_X \tau(x)^2 \,\mathrm{d}\lambda_X(x) &\leq \int_X \Big(\int_G \tau(xh)\,\mathrm{d}\mu_0(h)\Big)^2 \,\mathrm{d}\lambda_X(x)\\ &\leq \int_X \int_G \tau(xh)^2 \,\mathrm{d}\mu_0(h))\,\mathrm{d}\lambda_X(x)\\ &= \int_G \int_X \tau(xh)^2 \,\mathrm{d}\lambda_X(x)\,\mathrm{d}\mu_0(h)\\ &= \int_X \tau(x)^2 \,\mathrm{d}\lambda_X(x).\end{aligned}$$

So all the implied inequalities are actually equalities. In particular the equality in the Cauchy-Schwarz inequality tells us that $h \mapsto \tau(xh)$ is μ_0-essentially constant for almost every $x \in G$. We further deduce that τ is essentially constant on G. This does not give on the nose that τ is constant on G, since it is a priori not continuous, but since (3.1) is true for every $g \in G$, this gives the claim.

Applying von Neumann's measurable selection theorem once again, and using an averaging argument, we conclude that for every $\mu_1 \in \mathrm{Prob}(G)$ we may find a measure $\mu \in \mathcal{P}_\ell$ such that $\mu_1 * \nu_0 = \mu * \nu_0$. We may decompose μ as a convex combination $\mu = (1-\ell)\mu_1' + \ell\mu_2$ where $\mu_1', \mu_2 \in \mathrm{Prob}(G)$, $\mathrm{supp}(\mu_1') = \Gamma$. So we can inductively construct measures $\mu_n, \mu_n' \in \mathrm{Prob}(G)$, $n \geq 1$, such that $\mathrm{supp}(\mu_n') = \Gamma$, $\mu_1 = \delta_e$, and

$$\begin{aligned}\nu_0 = \mu_1 * \nu_0 &= (1-\ell)\mu_1' * \nu_0 + \ell\mu_2 * \nu_0\\ &= (1-\ell)\mu_1' * \nu_0 + \ell((1-\ell)\mu_2' * \nu_0 + \ell\mu_3 * \nu_0)\\ &= (1-\ell)\mu_1' * \nu_0 + (1-\ell)\ell\mu_2' * \nu_0 + (1-\ell)\ell^2\mu_3' * \nu_0 + \dots.\end{aligned}$$

Thus, we find that $\nu_0 = \mu_\infty * \nu_0$, where $\mu_\infty = (1-\ell)\sum_{k\geq 0}\ell^k\mu_{k+1}'$. This measure has support equal to Γ. $\square$

CHAPTER 6

Margulis superrigidity theorem

1. Algebraic preliminaries

The general statement of Margulis superrigidity theorem requires the language of algebraic groups. We will try to keep it to a minimum, and rather state the result in the setting of Lie groups. Nevertheless some unavoidable language needs to be introduced.

We will use a naive approach to algebraic geometry, focusing on the points of rather than the function rings.

1.1. The Zariski topology. In this section k denotes an arbitrary field.

Definition 6.1. A subset $V \subset k^n$ is said to be *Zariski closed* if there exists a family of polynomials $I \subset k[X_1, \ldots, X_n]$ such that V is exactly the set $Z(I)$ of common zeroes of all elements of I.

One checks that the Zariski closed sets form the family of closed sets of a topology, called the *Zariski topology.* If V is Zariski closed, then in fact $V = Z(I(V))$, where $I(V) \subset k[X_1, \ldots, X_n]$ is the ideal of all polynomials vanishing on V. More generally, if $V \subset k^n$ is any subset, then $Z(I(V))$ is a closed set containing V, it turns out to be equal to the Zariski closure of V, denoted by $\overline{V}^Z$ (or $\overline{V}$ when there is no ambiguity).

Example 6.2. Let us give some comments and examples in the setting of groups.

- The set $\mathrm{SL}(n,k)$ is Zariski closed inside $M_n(k) \simeq k^{n^2}$, being the zero set of $\det -1$.
- The set $\mathrm{GL}(n,k)$ is not Zariski closed inside $M_n(k)$, but it can still be seen as a Zariski closed subset of $M_n(k) \times k \simeq k^{n^2+1}$, as the zero set of the polynomial $P(A,x) = 1 - x\det(A)$, $A \in M_n(k)$, $x \in k$. Concretely, this corresponds to identify the group $GL(n,k)$ with its image under the map
$$A \in \mathrm{GL}(n,k) \mapsto (A, 1/\det(A)) \in M_n(k) \times k.$$
Exercise: check that the induced topology on $\mathrm{GL}(n,k)$ of the Zariski topology in any of these two embeddings is the same.

Let us record an easy fact.

Lemma 6.3. *If $G \subset \mathrm{GL}(n,k)$ is a subgroup, then its Zariski closure $\overline{G}$ inside $GL(n,k)$ is again a subgroup.*

Proof. By the comments above, $\overline{G}$ is equal to $Z(I(G)) \cap \mathrm{GL}(n,k)$, where $I(G)$ is the set of polynomials in n^2 variables that vanish on $G \subset M_n(k) \simeq k^{n^2}$. Assume that $a, b \in \overline{G}$ and take $P \in I(G)$. Since G is a group, we know that for all $h \in G$, the polynomial $R_h : x \mapsto P(xh)$ vanishes on G, i.e. $R_h \in I(G)$. Since $a \in \overline{G}$, we deduce that $P(ah) = R_h(a) = 0$ for all $h \in G$. Thus the polynomial $L_a : y \mapsto P(ay)$ vanishes on G. Since $b \in \overline{G}$, we conclude that $P(ab) = L_a(b) = 0$. Hence $ab \in \overline{G}$.

We want to proceed similarly for the inverse, but unfortunately, the inverse map $x \in \mathrm{GL}(n,k) \mapsto x^{-1} \in \mathrm{GL}_n(k)$ does not extend to a nice polynomial map on $M_n(k)$. Instead, we use the other embedding $\mathrm{GL}(n,k) \subset M_n(k) \times k$ described above. With this embedding, the inverse map extends to the map $\theta : (A,x) \in M_n(k) \times k \mapsto (xA', \det(A)) \in M_n(k) \times k$, where A' is the transpose of the commatrix of A.

Again, $\overline{G}^Z$ may be expressed as $Z(I(G))$ where this time $I(G)$ is the set of polynomials in n^2+1 variables that vanish on $G \subset M_n(k) \times k$. If $a \in \overline{G}$, and $P \in I(G)$, then the map $x \mapsto P(\theta(x))$ is a polynomial that vanishes on G, thus it vanishes on a: $P(\theta(a)) = P(a^{-1}) = 0$. This shows that $a^{-1} \in \overline{G}$. □

The above proof illustrates that the embedding $\mathrm{GL}(n,k) \subset M_n(k) \times k$ is more relevant to consider functions on G. In this respect, we will say that a function $\mathrm{GL}(n,k) \to k$ is *rational* if it is the restriction to $\mathrm{GL}(n,k)$ of a polynomial map on $M_n(k) \times k$. In particular, the map $g \mapsto \det(g)^{-1}$ is rational.

DEFINITION 6.4. An (affine) *algebraic group* is a Zariski closed subgroup of some $\mathrm{GL}(n,k)$. A function $G \to k$ on such a group is called rational if it is the restriction to G of a rational function on $\mathrm{GL}(n,k)$.

Note that in the above definition the ambient linear group $\mathrm{GL}(n,k)$ is part of the data. This is because we didn't take the time to describe the relevant category associated with the Zariski topology and rational maps. This may seem annoying but it will be enough for our purposes.

DEFINITION 6.5. An *algebraic representation* of an algebraic group on a k-vector space V is a group homomorphism $\pi : G \to \mathrm{GL}(V)$ such that for any rational function $f : \mathrm{GL}(V) \to \mathbb{C}$, the function $f \circ \pi$ is rational on G.

EXERCISE 6.6. Check that π is algebraic if and only if for every $v \in V$ and $\chi \in V^*$, the coefficient function $g \mapsto \chi(\pi(g)v)$ is rational on G.

LEMMA 6.7. *If $G \subset \mathrm{GL}(n,k)$ is an algebraic group and $\Gamma \subset G$ is a Zariski dense subset, then for any algebraic representation $\pi : G \to \mathrm{GL}(V)$, $\pi(G)$ is Zariski closed and $\pi(\Gamma)$ is Zariski dense in $\pi(G)$.*

PROOF. We admit the first fact. It is based on commutative algebra. The second fact is trivial: in any topological space the image of a dense set by a continuous map is dense. □

1.2. Algebraic Lie groups. In this section, we specify to the case where $k = \mathbb{R}$ or $k = \mathbb{C}$. In this case, k and $M_n(k)$, $\mathrm{GL}(n,k)$,... all carry both a Zariski topology and a locally compact topology. We will add the term Zariski to any property that refers to the Zariski topology.

REMARK 6.8. An algebraic group $G \subset \mathrm{GL}(n,k)$ is closed inside $\mathrm{GL}(n,k)$ (for both topologies), so G is naturally endowed with a Lie group structure over k.

The following result is a special feature of algebraic Lie groups.

THEOREM 6.9. *An algebraic group $G \subset \mathrm{GL}(n,k)$ has only finitely many connected components, when seen as a Lie group over k.*

Elements of proof. The proof splits into two parts: First one shows that G has only finitely many Zariski-connected components. Then one shows that if G is Zariski connected, then it has only finitely many connected component, when viewed as a Lie group.
The first part follows from the Noetherian property of the ring $k[X_1, \dots, X_{n^2+1}]$: any non-decreasing chain of ideals is eventually constant. So topologically, any non-increasing chain of Zariski closed subsets of G is eventually constant. Exercise: fill the blanks!
The second part is more involved. A good reference is [**Sha13**, Chapter 7.2]. □

As for Lie groups, the adjoint representation plays a pivotal role in the study of algebraic groups.

Lemma 6.10. *Let $G \subset \mathrm{GL}(n,k)$ be an algebraic group. Then the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is algebraic.*

Proof. The result is obvious for $G = \mathrm{GL}(n,k)$ itself as the adjoint representation is just given by $\mathrm{Ad}(g)(A) = gAg^{-1}$, $g \in G$, $A \in M_n(k)$.

In the general case we use a restriction trick. Since $G \subset \mathrm{GL}(n,k)$, we have an embedding of Lie algebras $\mathfrak{g} \subset M_n(k)$. Denote by $\widetilde{G} \subset \mathrm{GL}(M_n(k))$ the algebraic subgroup of elements that leave globally invariant the subspace $\mathfrak{g} \subset M_n(k)$. Then the adjoint representation $\mathrm{GL}(n,k) \to \mathrm{GL}(M_n(k))$ restricts to an algebraic representation $\beta : G \to \mathrm{GL}(M_n(k))$, with range inside $\widetilde{G}$. Moreover, the restriction map $\alpha : \widetilde{G} \to \mathrm{GL}(\mathfrak{g})$ is clearly algebraic. So the result follows from the observation that $\alpha \circ \beta : G \to \mathrm{GL}(\mathfrak{g})$ is exactly the adjoint representation of G. □

Given a connected semi-simple Lie group G with trivial center, we will freely speak about the Zariski topology on G to refer to the Zariski topology in its adjoint representation, i.e. in the algebraic group $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$.

2. The statement

Theorem 6.11 (Margulis superrigidity theorem). *Let G and H be connected semi-simple Lie groups with trivial center. Assume that G has rank at least equal to 2 and that H has no compact factor. Let Γ be an irreducible lattice in G and $\pi : \Gamma \to H$ a group homomorphism such that $\pi(\Gamma)$ is Zariski dense in H.*

Then π extends to a Lie group homomorphism $G \to H$.

Let us discuss some of the assumptions of this theorem.

Remark 6.12 (Assumptions on G). The requirement that G is connected and has trivial center is a natural assumption and will not be discussed. The rank condition on G is necessary. In our simplified setting where $G = \mathrm{PSL}_d(\mathbb{R})$, it means that $d \geq 3$. In fact, we may observe that the theorem fails when $d = 2$. Indeed, $G = \mathrm{PSL}_2(\mathbb{R})$ contains a lattice which is isomorphic with a free group on two generators $\mathbb{F}_2$. This free group itself has a finite index subgroup Γ isomorphic with a free group on three generators $\mathbb{F}_3$. Then Γ is still a lattice in G, while the surjective morphism $\Gamma \to \mathbb{F}_2$ given by killing one of the generators does not extend to a continuous morphism $G \to G$, because G is simple, while the morphism $\Gamma \to \mathbb{F}_2$ has infinite kernel.

REMARK 6.13 (Assumptions on Γ). The irreducibility assumption is necessary, otherwise we could produce a counterexample by considering a product lattice $\mathbb{F}_3 \times \mathbb{F}_3$ inside $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ as before.
Besides, note that the definition of irreducibility we gave prevents G to admit compact factors. A softer definition could allow this situation, but we would then have to assume that G has no compact factors.

REMARK 6.14 (Assumptions on H). In fact in the conclusion of the theorem, one can prove that the extension of π is a surjective morphism. Thus several assumptions on H are necessary: trivial center, semi-simplicity, no compact factors. Let us mention about this last assumption that the morphisms from Γ into compact simple Lie groups are well understood. We provide below one example of such a morphism.

EXAMPLE 6.15. Consider the quadratic form $Q : \mathbb{R}^5 \to \mathbb{R}$ given by

$$Q(x) = x_1^2 + x_2^2 + x_3^2 - \sqrt{2}x_4^2 - \sqrt{2}x_5^2, \text{ for all } x \in \mathbb{R}^5.$$

Then it follows from Borel and Harish-Chandra's theorem that $\mathrm{SO}(Q, \mathbb{Z}[\sqrt{2}])$ is a lattice inside $G := \mathrm{SO}(Q, \mathbb{R})$. But the non-trivial Galois automorphism σ of $\mathbb{Z}[\sqrt{2}]$ over $\mathbb{Z}$ changes the quadratic form Q to a positive definite quadratic form Q^σ. This Galois automorphism σ induces an embedding of rings $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R}$ and further a group homomorphism $\pi : \Gamma \hookrightarrow \mathrm{GL}(5, \mathbb{R})$ such that $\pi(\Gamma) \subset H := \mathrm{SO}(Q^\sigma, \mathbb{R})$. But since G is simple and non-compact there is no non-trivial continuous morphism $G \to H$ into the compact group H.

About the conclusion of the theorem, we make the following observation.

PROPOSITION 6.16. *The extension morphism given by Margulis' superrigidity theorem is unique.*

PROOF. Representing H on a finite dimensional vector space V, it suffices to show that two continuous representations $\pi_1, \pi_2 : G \to \mathrm{GL}(V)$ which coincide on a lattice Γ are equal. To that aim, define a finite dimensional representation $\widetilde{\pi} : G \to \mathrm{GL}(\mathrm{End}(V))$ by the formula

$$\widetilde{\pi}(g) : T \in \mathrm{End}(V) \mapsto \pi_1(g)T\pi_2(g)^{-1} \in \mathrm{End}(V), \text{ for all } g \in G.$$

Then by assumption $\widetilde{\pi}(\Gamma)$ fixes the line $\mathbb{R}\,\mathrm{id} \subset \mathrm{End}(V)$. So by Borel-density theorem (Corollary 4.6), $\widetilde{\pi}(G)$ must fix this line as well. Since G is semi-simple, it has no character, so in fact $\widetilde{\pi}(G)$ must fix the element $\mathrm{id} \in \mathrm{End}(V)$. This implies that $\pi_1(g) = \pi_2(g)$ for all $g \in G$. □

3. The proof

3.1. First reduction. We observe that it suffices to prove Margulis' theorem in the case where H is simple. Indeed, if this is true in this case, then if H is an arbitrary connected semi-simple Lie group with trivial center, we may write H as a product $H_1 \times \cdots \times H_n$ of simple Lie groups with trivial center. Then composing the morphism $\pi : \Gamma \to H$ with each projection on the factors H_i, we obtain morphisms $\pi_i : \Gamma \to H_i$, which must extend to continuous morphisms $G \to H_i$, still denoted by π_i. Then the morphism $g \in G \mapsto (\pi_1(g), \ldots, \pi_n(g)) \in H$, is a continuous extension of π.

3.2. What do we aim for? It will be useful for us to assume that H is represented (non-trivially) on a vector space V. Since H is simple, this amounts to having an embedding $\sigma : H \subset \mathrm{GL}(V)$. But we also want to keep track of the Zariski topology, so we will assume that this representation σ is algebraic. We then claim that $\sigma(H) \subset \mathrm{GL}(V)$ is equal to the connected component of its Zariski closure. Indeed, first recall that since H is simple, it is equal to the connected component of the algebraic group $\mathrm{Aut}(\mathfrak{h})$. In this context, an algebraic representation of H is by definition the restriction to H of an algebraic representation σ of the algebraic group $\mathrm{Aut}(\mathfrak{h})$ on V. By Lemma 6.7, we deduce that $\sigma(\mathrm{Aut}(\mathfrak{h}))$ is Zariski closed and contains $\sigma(H)$. Which easily implies the claim.

From now on we omit the letter σ and really see $H \subset \mathrm{GL}(V)$ as a subset. In this context, we may view π as a representation of Γ on V.

A classical construction when we are given a representation of Γ is to induce it to a representation of G. So let us denote by $L^0_\Gamma(G, V)$ the space of all measurable maps f from G into V that are Γ-equivariant, in the sense that $f(\gamma g) = \pi_\gamma(f(g))$ for all $\gamma \in \Gamma$, and almost every $g \in G$. Consider the induced representation $\widetilde{\pi}$ of G on $L^0_\Gamma(G, V)$ given by right translation:

$$(\widetilde{\pi}_g(f))(h) = f(hg), \quad \text{for all } f \in L^0_\Gamma(G, V), g, h \in G.$$

Since the initial representation π is not unitary, we need not find a nice G-invariant norm on $L^0_\Gamma(G, V)$. However, this representation is continuous for the topology of convergence in measure[1]. All what we need to know about this is that if $E \subset L^0_\Gamma(G, V)$ is a finite dimensional G-invariant subspace then the restricted representation $G \to \mathrm{GL}(E)$ is continuous. This essentially follows from Lemma 6.19.

Exercise 6.17. Assume that the initial representation π does extend to a representation of G (still denoted by π). Check that $\widetilde{\pi}$ is then conjugate with the representation of G on $L^0(G/\Gamma) \otimes V$ defined by the formula $\lambda_{G/\Gamma}(g) \otimes \pi(g)$, $g \in G$. (*Hint.* Study the map $\theta : L^0(G/\Gamma) \otimes V \to L^0(G, V)$ defined by $\theta(f \otimes v) : g \in G \mapsto f(g^{-1}\Gamma)\pi_g(v) \in V$.)
Deduce that the induced representation $\widetilde{\pi}$ contains a finite dimensional subspace which is globally invariant, and on which the representation $\widetilde{\pi}$ is conjugate with the initial representation π.

From the above exercise, it appears that a good step towards proving the theorem is to try to find a finite dimensional G-invariant subspace inside $L^0_\Gamma(G, V)$. Let us prove that this is indeed sufficient.

Proposition 6.18. *If there exists a finite dimensional subspace of $L^0_\Gamma(G, V)$ which is globally G-invariant and which contains a non-constant function, then the homomorphism $\pi : \Gamma \to H$ extends to a Lie group homomorphism $G \to H$.*

Proof. Let $E \subset L^0_\Gamma(G, V)$ be such a G-invariant finite dimensional subspace. Let us first prove a very useful claim.

Claim 1. Any element $f \in E$, viewed as a measurable function defined almost surely, admits a continuous representative.

The restriction of the induced representation $\widetilde{\pi}$ to E gives a representation $\pi_0 : G \to \mathrm{GL}(E)$, which can be checked to be a measurable group homomorphism. Thanks to Lemma 6.19 below, we deduce that π_0 is in fact continuous. Take an element $f \in E$ and

[1] We will not use this but it may be reassuring to know that this is a so-called Polish topology: it is metrizable, separable and complete.

fix a basis $f_1, \dots, f_\ell$ of E. Then we may find continuous functions $c_1, \dots, c_\ell : G \to \mathbb{R}$ such that for all $g \in G$, we have

$$\pi_0(g)(f) = \sum_{i=1}^{\ell} c_i(g) f_i.$$

By Fubini's theorem, we deduce that for almost every $h \in G$, for almost every $g \in G$, we have

$$f(hg) = \sum_{i=1}^{\ell} c_i(g) f_i(h).$$

In particular, we may find an element $h \in G$ such that for almost every $g \in G$, $f(g) = \sum_{i=1}^{\ell} c_i(h^{-1}g) f_i(h)$. This last expression is continuous in the variable $g \in G$ (we emphasize that $h \in G$ is fixed). This proves Claim 1.

Observe that since E contains a non-constant function, π_0 is a non trivial representation of G. By Borel density theorem, it is a non-trivial representation of Γ.

Thanks to the claim, we may select a (unique) continuous representative of each element in E. From uniqueness it actually follows that this representative is everywhere Γ-equivariant. Then the following evaluation map is well defined, and linear:

$$\Psi : f \in E \mapsto f(e) \in V.$$

Moreover, the Γ-equivariance property of each element $f \in E \subset L^0_\Gamma(G, V)$ implies that for all $\gamma \in \Gamma$, we have $\Psi((\pi_0)_\gamma(f)) = f(\gamma) = \pi_\gamma(f(e)) = \pi_\gamma(\Psi(f))$. In other words, Ψ intertwines the representations π_0 and π of Γ.

Claim 2. Ψ is injective and $\Psi(E) \subset V$ is globally H-invariant.

Note that $\mathrm{Ker}(\Psi)$ is $\pi_0(\Gamma)$-globally invariant. By Borel density theorem, we deduce that $\mathrm{Ker}(\Psi)$ is globally $\pi_0(G)$-invariant. So if $f \in \mathrm{Ker}(\Psi)$, then $(\pi_0)_g(f) \in \mathrm{Ker}(\Psi)$ for all $g \in G$, and hence $f(g) = 0$ for all $g \in G$. This proves that Ψ is injective. For the second part, observe that $\Psi(E)$ is globally $\pi(\Gamma)$-invariant. Since π is an algebraic representation of H, and Γ is Zariski dense in H, Lemma 6.7 shows that $\pi(\Gamma)$ is Zariski dense in $\pi(H)$. This implies that $\pi(H)$ is contained in the algebraic group of elements of $\mathrm{GL}(V)$ that leave $\Psi(E)$ globally invariant, as desired.

Since $W := \Psi(E)$ is globally H-invariant, we may consider the restricted representation $H \to \mathrm{GL}(W)$. This is a non-trivial representation of H. Indeed, we observed that Γ acts non-trivially on E, so Claim 2 and the Γ-equivariance property of Ψ show that Γ acts non-trivially on W. Since H is simple, the representation $H \to \mathrm{GL}(W)$ is faithful. Now, observe that the group homomorphism

$$\rho : g \in G \mapsto \Psi(\pi_0)_g \Psi^{-1} \in \mathrm{GL}(W)$$

is continuous and satisfies $\rho(\gamma) = \pi(\gamma)$ for all $\gamma \in \Gamma$. In particular, $\rho(\Gamma) \subset H$. Since G is semi-simple and connected, Borel density theorem implies that $\rho(G)$ is contained in the Zariski closure of H. Since $\rho(G)$ is connected, it must be contained in the connected component of this Zariski closure, i.e. in H. So ρ is the desired extension. $\square$

We used the following general lemma.

Lemma 6.19. *A measurable morphism ϕ between two lcsc groups G and H is continuous.*

Proof. Let $O \subset H$ be an open neighborhood of the identity in H and U an open set such that $UU^{-1} \subset O$. Then using countably many translates of U one can cover the whole H. Better: $\phi(G) \subset \bigcup_n \phi(h_n)U$ for countably many $h_n \in G$. This implies that

$G = \cup_n h_n \phi^{-1}(U)$. Hence the measurable set $\phi^{-1}(U)$ has positive Haar measure. This means that $\phi^{-1}(UU^{-1})$ contains an open neighborhood of e_G, because the convolution function $1_{\varphi^{-1}(U)} * 1_{\phi^{-1}(U^{-1})}$ is continuous, supported inside $\phi^{-1}(UU^{-1})$, and non-zero at e_G. Hence $\phi^{-1}(O)$ contains a neighborhood of e_G, which shows that ϕ is continuous. □

3.3. Finding a nice representation. To apply the previous strategy, we will need to find a nice representation of H.

PROPOSITION 6.20. *Let H be a connected non-compact simple Lie group and let $\Delta < H$ be a Zariski-dense subgroup. Then there exists an algebraic representation $H \to \mathrm{GL}(V)$ which is irreducible and such that Δ acts proximally on V. In fact this representation is strongly irreducible for Δ.*

PROOF. This is based on Proposition 4.15. In order to apply it, we need to check that Δ does not have compact closure. View H as a subgroup of $\mathrm{GL}(\mathfrak{h})$. If Δ had compact closure, then it would preserve some scalar product on $\mathfrak{h}$, hence it would be contained in an orthogonal group (which is algebraic). The intersection of this orthogonal group with H would be an algebraic group containing Δ, and strictly contained in H,because H is non-compact. This would contradict the fact that Δ is Zariski dense in H. So Proposition 4.15 applies, and it is clear from its proof that the representation that we get is algebraic.

Moreover if $W \subset V$ is a subspace invariant under a finite index subgroup Δ_0 of Δ, then it is invariant under its Zariski closure. Denote by H_0 this Zariski closure. Since Δ_0 has finite index inside Δ, we may find $g_1, \ldots, g_m \in \Delta$ such that

$$\Delta = \bigsqcup_{i=1}^{m} g_i \Delta_0.$$

Since the left multiplication by each g_i is algebraic, we find that the Zariski closure of $g_i \Delta_0$ is $g_i H_0$. So taking the Zariski closure in the above equality gives $H = \bigcup_{i=1}^{m} g_i H_0$. So H_0 has finite index inside H. Since H is connected, it has no-proper closed finite index subgroup. So $H = H_0$ and W is in fact H-invariant. This forces $W = V$. □

3.4. Finding a finite dimensional subspace in the induced representation. Represent the group H on some vector space V as in Proposition 6.20, applied to $\Delta = \pi(\Gamma)$. We denote by ρ this representation. In fact the representation to which we will apply Proposition 6.18 is the representation $\sigma : H \to \mathrm{GL}(\mathrm{End}(V))$, given by $\sigma_h(T) = \rho_h T \rho_h^{-1}$, for all $T \in \mathrm{End}(V)$, $h \in H$.

A first step will be to find an element f_0 in the induced space $L^0_\Gamma(G, \mathrm{End}(V))$ with good invariance properties under the G-action. Then we will make sure that the G-orbit of this function spans a finite dimensional subspace.

The composition of $\pi : \Gamma \to H$ with the representation $\rho : H \to \mathrm{GL}(V)$ gives a linear representation of Γ on V, and therefore, an action of Γ on the projective space $\mathbb{P}(V)$.

From now on, we assume that $G = \mathrm{PSL}_d(\mathbb{R})$, and we use the notation A, P, N given in Chapter 2, Section 3.

LEMMA 6.21. *There exists a Γ-equivariant measurable map $G/P \to \mathbb{P}(V)$.*

PROOF. By Proposition 5.8, there exist a Γ-equivariant measurable map $\theta : G/P \to \mathrm{Prob}(\mathbb{P}(V))$. Denote by ν_P the unique K-invariant probability measure on G/P. Thanks to Lemma 5.9, we may find a probability measure $\mu \in \mathrm{Prob}(\Gamma)$ whose support is the whole of Γ and such that $\mu * \nu_P = \nu_P$. Since the representation $\rho \circ \pi : \Gamma \to V$ is proximal and strongly irreducible, the result follows from Corollary 4.14. $\square$

In the above proof, it may not be so obvious why we need to use Lemma 5.9 at all. Indeed one could a priori take for μ any probability measure on Γ, and denote by ν_0 a μ-stationary measure on G/P, and apply Corollary 4.14 in the same way. However, doing so would only tell us that the map $\theta(x)$ is a Dirac measure on $\mathbb{P}(V)$ for ν_0-almost every $x \in G/P$. If we don't have any control on the measure ν_0, this condition could only happen for a very small set of x's (for instance it might only happen on a countable set of x's). So Lemma 5.9 allows us to make sure that this happens for almost every x, with respect to ν_P (which is a nice G-quasi-invariant measure).

Lemma 6.22. *There exists a non-constant* Γ*-equivariant measurable map* $f_0 : G/A \to \mathrm{End}(V)$.

Proof. The map $G/P \to \mathbb{P}(V)$ given by Lemma 6.21 also exists for the dual representation $\rho^* : H \to \mathrm{GL}(V^*)$, given by $\rho^*_h(\chi) = \chi \circ \rho_h^{-1} \in V^*$, for all $\chi \in V^*$, $h \in H$.
Exercise. Check that the representation $\rho^* \circ \pi : \Gamma \to \mathrm{GL}(V^*)$ is also proximal and strongly irreducible.
So we also have a Γ-equivariant map $G/P \to \mathbb{P}(V^*)$.

Therefore, we may form the product map, $G/P \times G/P \to \mathbb{P}(V) \times \mathbb{P}(V^*)$, which is Γ-equivariant with respect to the diagonal actions $\Gamma \curvearrowright G/P \times G/P$ and $\Gamma \curvearrowright \mathbb{P}(V) \times \mathbb{P}(V^*)$. This map is measurable, and maps the product measure $\nu_P \times \nu_P$ to the product measure $\nu \times \nu^*$, where ν (resp. ν^*) is the unique μ-stationary measure on $\mathbb{P}(V)$ (resp. $\mathbb{P}(V^*)$).

Claim 1. There is one G-orbit in $G/P \times G/P$ (for the diagonal action) which has full measure. It is isomorphic with G/A.

It is well known that any matrix having no vanishing principal minor can be written as the product LU of a lower triangular matrix L with an upper triangular matrix U. This is the so-called LU-decomposition, and can be checked by induction on the size of the matrix. In particular, since the condition of having vanishing minors is of measure 0 inside G, we find that almost every matrix $g \in G$ decomposes as a product LU.

We deduce that the group $\overline{P}$ of lower triangular matrices has a co-null orbit inside G/P, namely the orbit of the point P. Denote by $\mathcal{O} = \overline{P}P$ this orbit. Now observe that $\overline{P}$ is conjugate to P via the permutation matrix w_0 associated with the permutation $\sigma_0 : i \mapsto n+1-i$. In conclusion, the G-orbit of the point (P, w_0P) contains its P-orbit, which is equal to $(P, Pw_0P) = (P, w_0\overline{P}P) = (P, w_0\mathcal{O})$. So $G \cdot (P, w_0P)$ contains all fibers $(gP, gw_0\mathcal{O})$ for all $g \in G$. As $gw_0\mathcal{O}$ has full measure inside G/P, the first part of the claim follows[2].

Note moreover that the stabilizer of (P, w_0P) is $P \cap w_0Pw_0^{-1} = P \cap \overline{P} = A$. So the G-orbit of (P, w_0P) is isomorphic (as a G-space), with G/A.

Claim 2. In $\mathbb{P}(V) \times \mathbb{P}(V^*)$, the set $Z := \{(\bar{x}, \bar{\chi}) \mid \chi(x) = 0\}$ has measure 0. It is also invariant under the diagonal Γ-action.

Since we are dealing with a product measure it suffices to check that for every $\bar{\chi} \in \mathbb{P}(V^*)$, the fiber $Z_\chi := \{\bar{x} \in \mathbb{P}(V) \mid \chi(x) = 0\}$ has measure 0: $\nu(Z_\chi) = 0$. But each such fiber Z_χ is equal to a (projective) hyperplane. So the claim follows from Lemma 4.12.

[2]When G is a general semi-simple Lie group, the decomposition LU that we used is superseded by the so-called *Bruhat decomposition*. This decomposition gives a complete description of all the orbits of the action of G on $G/P \times G/P$

So, we get a measurable Γ-equivariant map $G/A \to Z^c$ (by perturbing our initial map on a subset of measure 0 if needed). We may compose it with the map

$$(\bar{x}, \bar{\chi}) \in Z^c \mapsto \{v \in V \mapsto \frac{\chi(v)}{\chi(x)} x \in V\} \in \mathrm{End}(V).$$

One checks that the composed map $f_0 : G/A \to \mathrm{End}(V)$ is Γ-equivariant when Γ acts on $\mathrm{End}(V)$ by the formula $\gamma(T) = \rho_{\pi(\gamma)} T \rho_{\pi(\gamma)}^{-1}$, for all $T \in \mathrm{End}(V)$, $\gamma \in \Gamma$. One easily checks that f_0 is non-constant. □

We may view the function f_0 from Lemma 6.22 as an element $f_0 \in L^0_\Gamma(G, \mathrm{End}(V))$ which is $\widetilde{\pi}(A)$-invariant, and non-constant. Here, we recall that $\widetilde{\pi}$ is defined by the formula $\widetilde{\pi}_g(f)(h) = f(hg)$ for all $g, h \in G$, $f \in L^0_\Gamma(G, \mathrm{End}(V))$.

We want to check that $\mathrm{span}(\widetilde{\pi}(G)f_0)$ is finite dimensional. This is where we will use the higher rank condition on G (in our case, the $d \geq 3$ condition for $G = \mathrm{PSL}_d(\mathbb{R})$). More precisely, we will need the following property.

Lemma 6.23. *There exist infinite subgroups $A_1, \dots, A_k \subset A$ such that every element of g can be written as a product $g_1 \dots g_k$ of elements in the centralizers $g_i \in Z_G(A_i)$. Formally, $G = Z_G(A_1) \cdots Z_G(A_k)$.*

Proof. For $1 \leq i \leq n-1$, define $A_i = \{\mathrm{diag}(a_1, \dots, a_n) \mid a_i = a_{i+1}, \prod a_j = 1\}$. *Exercise.* Check that P is contained in a product $Z(A_{i_1}) \cdots Z(A_{i_\ell})$ for appropriate indices $i_1, \dots, i_\ell$. Taking the transpose this is also the case of $\overline{P}$, and hence of $Y := \overline{P}P$. By the LU-decomposition, this set is co-null in G. Hence $G = YY$, which implies the lemma. □

Proposition 6.24. *Let $E \subset L^0_\Gamma(G, \mathrm{End}(V))$ be a finite dimensional subspace which is globally A-invariant. Let $A_0 \subset A$ be an infinite subgroup and denote by $C := Z_G(A_0)$. Then the linear span E' of $\widetilde{\pi}(C)E$ is still finite dimensional and globally A-invariant.*

The proposition relies on the following consequence of Howe-Moore theorem.

Lemma 6.25. *Let $A_0 \subset G$ be a non-compact subgroup and consider a linear, finite dimensional $\Gamma \times A_0$-module W. Note that $\Gamma \times A_0$ acts on G by left-right action. Then the vector space of all measurable $\Gamma \times A_0$-equivariant functions from G to W is finite dimensional.*

Proof. Take finitely many such measurable functions $f_1, \dots, f_n : G \to W$. For all $g \in G$, we denote by $d(g)$ the dimension of $\mathrm{span}(\{f_1(g), \dots, f_n(g)\})$. Then $d : G \to \mathbb{R}$ is a bounded measurable function, which is $\Gamma \times A_0$ invariant. By Howe-Moore theorem, A_0 acts ergodically on G/Γ, so $\Gamma \times A_0$ acts ergodically on G. This shows that d is a constant function, bounded by $\dim(W)$. We may now choose the family $f_1, \dots, f_n$ such that the corresponding pointwise dimension d is maximal. We may also assume that $n = d$.

Take now any other $\Gamma \times A_0$-equivariant measurable function $f : G \to W$. Then for almost every $g \in G$, we may write $f(g)$ as a (unique) linear combination $f(g) = \sum_{i=1}^d c_i(g) f_i(g)$. By uniqueness, the coefficient functions c_i must be measurable. Since f and the f_i's are equivariant, the functions c_i must also be $\Gamma \times A_0$-invariant (here we use the fact that W is a *linear* $\Gamma \times A_0$-module). Applying Howe-Moore theorem again, these functions must be constant. Then we deduce that

$$f = \sum_{i=}^{d} c_i f_i.$$

So $f_1, \dots, f_d$ is a generating family, proving that our space has finite dimension. □

Proof of Proposition 6.24. Note that A normalizes A_0, so it normalizes $C = Z_G(A_0)$ as well. Therefore, $\widetilde{\pi}(a)\widetilde{\pi}(c)f = \widetilde{\pi}(aca^{-1})(\widetilde{\pi}(f)) \in E'$, for all $a \in A$, $c \in C$, $f \in E$. This implies that E' is globally A-invariant.

We now need to prove that E' has finite dimension. For this, note that every element $c \in C$ acts as an A_0-equivariant linear map from E into $L^0_\Gamma(G, \mathrm{End}(V))$.

Claim. The vector space $X := \mathrm{Hom}_{A_0}(E, L^0_\Gamma(G, \mathrm{End}(V)))$ is finite dimensional.

This claim implies the proposition, because E' is contained in the image of the bilinear map

$$(L, f) \in X \times E \mapsto L(f) \in L^0_\Gamma(G, \mathrm{End}(V)).$$

To prove the claim, denote by $W := \mathrm{Hom}(E, \mathrm{End}(V))$. This finite dimensional vector space is endowed with a linear action of $\Gamma \times A_0$, coming from the action of Γ on $\mathrm{End}(V)$, and the action of A_0 on E. The proposition follows from Lemma 6.25 and the following exercise.

Exercise. Prove that X identifies as a vector space with $L^0_{\Gamma\times A_0}(G, W)$. □

Using this proposition, we can prove by induction that for every finite family of infinite subgroups $A_1, \ldots, A_n \subset A$, the vector space $\mathrm{span}(\{Z_G(A_1) \cdots Z_G(A_n) f_0\})$ is finite dimensional. So Lemma 6.23 implies that indeed the G-orbit of f_0 spans a finite dimensional subspace. This concludes the proof of Margulis superrigidity theorem.

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