HOMEWORK: LOCALLY COMPACT GROUPS AND LATTICES

1. Topology of the basic examples

1.1. Connectedness.

(1) Prove that the unipotent group U of upper triangular matrices in with 1's the diagonal is connected and simply connected. Prove that this is also the case of the groups A of diagonal matrices, and the group P of all triangular matrices. Does this change something if we instead look at the intersection of these groups with $\mathrm{SL}_n(\mathbb{R})$?
(2) Prove that for $n \geq 2$, the group $\mathrm{SO}(n)$ is connected (see the first exercise on homogeneous spaces).
(3) Using the Iwasawa decomposition mentioned in the notes (and discussed below) prove that $\mathrm{SL}_n(\mathbb{R})$ is connected for every n.

1.2. Fundamental groups.

(1) $(n = 2)$. Compute the fundamental groups of $\mathrm{SO}(2)$ and $\mathrm{SL}_2(\mathbb{R})$.
(2) $(n \geq 3)$. Prove that the fundamental group of $\mathrm{SO}(3)$ is $\mathbb{Z}^2$.
Hint. One possibility is to use long exact sequences for homotopy groups associated with the fibrations $\mathrm{SO}(n) \subset \mathrm{SO}(n+1) \to \mathbb{S}^n$. Or you can check by hand that any loop in $\mathrm{SO}(n)$ can be contracted to a loop in $\mathrm{SO}(2)$. Then prove that if γ is the generator of $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$, then $\gamma^2 = 0$ in $\mathrm{SO}(n)$ whenever $n \geq 3$. It remains to prove that γ is a non-trivial loop in $\mathrm{SO}(n)$, which is harder to check by hand, but doable.

2. Classical decompositions

Iwasawa decomposition. Prove the Iwasawa decomposition in the special setting $G = \mathrm{SL}_n(\mathbb{R})$.
Hint. Use the Gramm-Schmidt orthonormalization procedure.

KAK-decomposition. Prove the KAK decomposition.
Hint. Use the polar decomposition.

LU decomposition. Prove that any matrix having no vanishing principal minor can be written as the product LU of a lower triangular matrix L with an upper triangular matrix U. *Hint.* Proceed by induction.