HOMEWORK: LINEAR AND PROJECTIVE REPRESENTATIONS

The alternate product. We let V a vector space over any field of characteristic $\neq 2$, with dimension d and we denote by $T(V) = \bigoplus_{k\geq 1} V^{\otimes k}$ its (graded) tensor algebra (without unit). The product on $T(V)$ is simply given by concatenating the tensors. It is associative but not commutative. We denote by $I \subset T(V)$ the two-sided ideal generated by the elements $v \otimes v$, $v \in V$. We denote by $\Lambda(V)$ the quotient $T(V)/I$ and we call it *the exterior algebra*. The product in this algebra is denoted by $\wedge$.

(1) Prove that for every $k \in \mathbb{N}$, $v_1, \dots, v_k \in V$ and every permutation $\sigma \in \mathcal{S}_k$, we have

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = \varepsilon(\sigma) v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} \tag{0.1}$$

(2) Prove that if $e_1, \cdots, e_d$ is a basis of V, then $\Lambda(V)$ is linearly spanned by the family $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$, as $I = \{i_1, \dots, i_k\}$ (always spelled with $i_1 < i_2 < \cdots < i_k$) ranges over the non-empty subsets of $\{1, \dots, d\}$.

(3) Prove that in fact, the family e_I, $I \subset \{1, \dots, d\}$, is a basis of $\Lambda(V)$. There are two steps:
   a) Prove the universal property: If A is an associative algebra with a linear map $\iota : V \to A$ satisfying (0.1) for every $v_1, \dots, v_k \in \iota(V)$, then there is an algebra homomorphism $\Lambda(V) \to A$ whose restriction to V is ι.
   b) Construct an associative algebra $(A, +, \times)$ with a linear basis f_I, $I \subset \{1, \dots, d\}$, such that $f_I = f_{i_1} \times f_{i_2} \cdots \times f_{i_k}$ for every $I = \{i_1, \dots, i_k\}$ (with $i_1 < i_2 < \cdots < i_k$) and in which the subspace spanned by f_I, for I singleton, identifies with V in such a way that (0.1) is satisfied.

In particular, $\Lambda(V)$ is a graded algebra, $\Lambda(V) = \bigoplus_{k=1}^{d} \Lambda^k(V)$, where $\Lambda^k(V)$ is the image of $V^{\otimes k}$.

(4) Prove that any homomorphism $T \in \mathrm{End}(V)$ gives rise to an algebra endomorphism of $\Lambda(V)$, and that in particular, this gives a natural algebra homomorphism $\pi : \mathrm{End}(V) \to \mathrm{End}(\Lambda(V))$, which preserves the grading. For each $k = 1, \dots, d$ we denote by π^k the restriction of π to $\Lambda^k(V)$. Prove that π^k is continuous: if $T_n \in \mathrm{End}(V)$ converges to $T \in \mathrm{End}(V)$ then $\pi^k(T_n)$ converges to $\pi^k(T)$ for any k.

(5) Let $W \subset V$ be a subspace, with dimension $k > 0$. Then $\Lambda^k(W) \subset \Lambda^k(V)$ is a one dimensional subspace. Check that an isomorphism $T \in \mathrm{GL}(V)$ preserves globally W if and only if $\pi^k(T)$ preserves the line $\Lambda^k(W)$.

(6) Prove that if $G \subset \mathrm{GL}(V)$ has proximal rank k, then $\pi^k(G)$ is proximal.

(7) Side question: Can you describe what is the representation π^d on the one dimensional space $\Lambda^d(V)$?

Flag varieties. We denote by V a finite dimensional vector space of dimension d.

(1) Given $0 < k < d$, check that the set of all k-dimensional subspaces W of V can be viewed as a subset of the projective space $\mathbb{P}(\Lambda^k(V))$, and that it is a closed subset. We call it the k-Grassmanian variety of V, and denote it by $\mathrm{Gr}_k(V)$.

(2) The action of $G := \mathrm{PSL}(V)$ on V gives rise to an action on $\mathrm{Gr}_k(V)$. Check that this action is described in terms of the representation π^k from the previous exercise. Prove that this action is transitive, and thus, $\mathrm{Gr}_k(V)$ is identifies (equivariantly) with a homogeneous space G/H, where H is the stabilizer of any given k-dimensional subspace of V.

We elaborate on this example. We define a *flag* of V to be a chain of subspace

$$\mathcal{F} = \{V_1 \subset V_2 \subset \cdots \subset V_n \subset V\}$$

The list $(k_1, \dots, k_n)$ of dimensions $k_1 = \dim(V_1)$, $k_2 = \dim(V_2), \dots, k_n = \dim(V_n)$ is called the *type* of the flag. A *full flag* is a flag whose type is the list of all integers $(1, 2, \dots, d)$.

(3) Prove that the set of all flags in V of a given type $(k_1, \dots, k_n)$ identifies with a closed subset of $\mathrm{Gr}_{k_1}(V) \times \cdots \times \mathrm{Gr}_{k_n}(V)$. We call it the *flag variety* of type $(k_1, \dots, k_n)$.

(4) Prove that $G = \mathrm{PSL}(V)$ acts transitively on any such flag variety.
(5) Identify V with $\mathbb{R}^d$, and denote by $G = \mathrm{PSL}(n, \mathbb{R})$, and by $P < G$ the image of the subgroup of upper triangular matrices. Check that G/P is G-equivariantly homeomorphic with a the flag variety of V.