HOMEWORK: STATIONARY MEASURES

Stationary measures are quasi-invariant. Consider a continuous group action $G \curvearrowright X$ of a lcsc group G on a lcsc space and take $\mu \in \mathrm{Prob}(G)$. Let $\nu \in \mathrm{Prob}(X)$ be a μ-stationary Borel measure.

(1) Let $A \subset X$ be a Borel subset. Prove that the map $g \in G \mapsto \nu(g^{-1}A) \in [0, 1]$ is continuous.
(2) Assume that $\nu(A) = 0$. Prove that for μ-almost every $g \in G$, $\nu(g^{-1}A) = 0$.
(3) Assume that the support of μ generates G as a semi-group (this means that for any open set $U \subset G$, there exists $n \geq 1$ such that $\mu^{*n}(U) > 0$). Prove that ν is quasi-invariant under G.

Absence of stationary measures. This exercise aims to prove that if G is a non-compact lcsc group then for any probability measure $\mu \in \mathrm{Prob}(G)$ which is equivalent to the Haar measure, there is no μ-stationary Borel probability measure on G for the left translation action $G \curvearrowright G$.

Let us assume by contradiction that there exists such a μ-stationary measure $\nu \in \mathrm{Prob}(G)$.

(1) Using the previous exercise, prove that ν is automatically equivalent to the left Haar measure on G (you may want to use the fact that G is a homogeneous space, and review the exercises about such spaces).
(2) Denote by $f \in L^1(G, \lambda)$ the Radon-Nykodym derivative of ν with respect to the Haar measure. Check that f satisfies the harmonic equation: $f(x) = \int_G f(g^{-1}x)d\mu(g)$ for almost every $x \in G$. Prove that the minimum function $\min(f, 1)$ also satisfies this equation equation.
(3) Prove that a stationary function $F \in L^2(G, \lambda)$ is necessarily constant, and hence equal to 0. *Hint.* Play with the expression $\int_G \langle F, \lambda_g(F)\rangle d\mu(g)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $L^2(G, \lambda)$ and λ_g denotes the left regular representation of G on $L^2(G, \lambda)$.
(4) Derive a contradiction.