C*-simplicity for discrete groups

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We have inclusions

$$\mathbb{C}[G] \subset \mathrm{C}_r^*(G) \subset L(G).$$

Group von Neumann algebras

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The canonical trace on $L(G)$ extends the canonical trace on $\mathbb{C}[G]$,

$$\tau(\textstyle\sum_{s\in G} a_s\lambda_s) = a_e.$$

(Reduced) group C*-algebras

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Theorem (Powers 1975)

The reduced C-algebra* $\mathrm{C}^*_r(\mathbb{F}_2)$ *is simple and has a unique trace.*

We say that $\mathbb{F}_2$ is **C*-simple** and has the **unique trace property**.

Variants of Powers' proof became the main method for establishing these properties.

Definition

A group G has *Powers' averaging property* if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

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Proof.

For C*-simplicity, let I be a non-trivial closed two-sided ideal of $\mathrm{C}^*_r(G)$. By faithfulness there is $a \in \mathrm{C}^*_r(G)$ with $\tau(a) = 1$. Applying Powers' averaging property implies $1 \in I$. The unique trace property is similarly straightforward.

□

Theorem (Powers 1975)

The free group $\mathbb{F}_2$ has Powers' averaging property. Hence it is C-simple and has the unique trace property.*

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C*-simple and unique trace property (if $R_a(G)$ trivial)	**Authors**
Free groups $\mathbb{F}_n$ for $n \geq 2$	Powers (1975)
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Linear groups	T. Poznansky (unpublished, 2008)
Groups with zero first ℓ^2-Betti number	J. Peterson and A. Thom (2010)
Acylindrically hyperbolic groups	F. Dahmani, V. Guirardel, and D. Osin (2011)
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See e.g. de la Harpe's survey from 2007.

Necessary and sufficient conditions

Theorem (KK 2014)

A discrete group is C-simple if and only if it has a (topologically) free boundary action.*

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Theorem (BKKO 2014)

A discrete group has the unique trace property if and only if it has a trivial amenable radical. Hence every C-simple group has the unique trace property.*

G-Boundaries

Definition (Furstenberg 1973)

A compact G-space X is a G-**boundary** if for every probability measure $\mu \in \mathcal{P}(X)$, the weak* closure of the orbit $G\mu$ contains the point masses $\{\delta_x \mid x \in X\}$.

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1. Boundaries have nice rigidity properties.
2. Topological "boundaries" arising in nature are often boundaries in this sense, e.g. Gromov boundaries of non-elementary hyperbolic groups.
3. Boundary actions encode interesting properties of G. For example, G is amenable if and only if there are no non-trivial G-boundaries.

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3. *The reduced crossed product $C(X) \rtimes_r G$ is an essential extension of $\mathrm{C}^*_r(G)$ in the category of operator systems equipped with G-equivariant unital completely positive maps.*

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Definition

Fix a category $\mathcal{C}$ of objects and morphisms with a notion of embedding. For $A, B, E \in \mathcal{C}$ with $A \subset E$, we say E is an **essential extension** of A if whenever $\phi : E \to B$ is a morphism such that $\phi|_A$ is an embedding, then ϕ is an embedding.

Main results

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"Half" of the result follows from results about injectivity.

The other half is more difficult. For an ideal J of $\mathrm{C}^*_r(G)$, key idea is to find an approximate G-equivariant embedding of $C(\partial_F G)$ as a subalgebra of $\mathrm{C}^*_r(G)/J$.

Some Applications

Definition

A subgroup $H < G$ is said to be *normalish* if for every $t_1, \ldots, t_n \in G$, the intersection

$$t_1 H t_1^{-1} \cap \cdots \cap t_n H t_n^{-1}$$

is infinite.

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Theorem (BKKO 2014)

A group with no non-trivial finite normal subgroups and no amenable normalish subgroups is C-simple.*

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Theorem (BKKO 2014)

A group with no non-trivial finite normal subgroups and no amenable normalish subgroups is C-simple.*

We recover essentially all previously known examples of C*-simple groups. Plus some new ones.

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This implies the C*-simplicity of (torsion and torsion-free) Tarski monster groups. Plus the result of Olshanskii-Osin about the C*-simplicity of free Burnside groups $B(m, n)$ for $m \geq 2$ and n odd and large.

Uniqueness of the trace

Theorem (BKKO 2014)

Every tracial state on $\mathrm{C}^*_r(G)$ is concentrated on the amenable radical $R_a(G)$ (the largest amenable normal subgroup).

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This means that for every tracial state τ on $\mathrm{C}^*_r(G)$, $\tau(\lambda_s) = 0$ for $s \in G \backslash R_a(G)$.

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Corollary

Every C-simple group has the unique trace property.*

Question

Does the unique trace property imply C*-simplicity?

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Answer: No!

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Example (Le Boudec 2015)

There are groups with the unique trace property that are not C*-simple. Examples are enlargements of groups acting on their Bass-Serre tree.

Characterizations of C*-simplicity

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Where do traces on $\mathrm{C}^*_r(G)$ "come from?"

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Partial Answer: Amenable invariant subgroups of G give rise to traces on $\mathrm{C}^*_r(G)$.

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Definition (Abèrt-Glasner-Viràg 2014)

An *invariant random subgroup* μ of G is a G-invariant probability measure on the space of subgroups $\mathcal{S}(G)$ of G. If μ is supported on amenable subgroups it is said to be *amenable*.

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Theorem (Tucker-Drob 2012)

If μ is an amenable invariant random subgroup on G, then

$$\tau_\mu(\lambda_g) = \mu\{H \in \mathcal{S}(G) \mid g \in H\}$$

extends to a trace on $\mathrm{C}^*_r(G)$.

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Connection to IRSs: The support of any non-trivial (i.e. $\mu \neq \delta_{\{e\}}$) ergodic invariant random subgroup μ contains a non-trivial uniformly recurrent subgroup.

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Key idea is that amenable uniformly recurrent subgroups correspond to boundaries in the state space of $\mathrm{C}_r^*(G)$. Special case is that amenable invariant random subgroups correspond to traces on $\mathrm{C}_r^*(G)$.

Theorem (Haagerup[1] 2015, K 2015)

A group G is C-simple if and only if it has Powers' averaging property, i.e. if and only if for every $a \in \mathrm{C}^*_r(G)$ and $\epsilon > 0$ there are $g_1, \ldots, g_n \in G$ such that*

$$\left\| \frac{1}{n} \sum \lambda_{g_i} a \lambda_{g_i^{-1}} - \tau(a) 1 \right\| < \epsilon.$$

[1] We recently learned from Mikael Rørdam that Haagerup independently obtained this result earlier this year.

Characterization in terms of uniformly recurrent subgroups leads to an intrinsic characterization of C*-simplicity.

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A subgroup $H < G$ is *recurrent* if for every sequence (g_n) in G there is a subsequence (g_{n_k}) such that

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Leads to an easy proof that Le Boudec's examples have the unique trace property but are not C*-simple.

An application to Thompson's group F

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Big Open Question

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It is easy to see that F is recurrent as a subgroup of T. By the previous characterization of C*-simplicity, this implies the following result:

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Theorem (Haagerup-Olesen 2014)

If T is C-simple, then F is non-amenable.*

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If this conjecture holds, then based on work of Brin, it should be the case that every recurrent subgroup of T contains a copy of F.

From the characterization of C*-simplicity, this would imply the converse of Haagerup-Olesen, i.e. that the non-amenability of F is equivalent to the C*-simplicity of T.

Thanks!