The faithful subalgebra

Sarah Reznikoff

joint work with
Jonathan H. Brown, Gabriel Nagy, Aidan Sims, and Dana Williams
funded in part by NSF DMS-1201564

West Coast Operator Algebras Seminar
UCSD, October 11, 2015

Let $\mathscr{G}$ be a graph, k-graph, or groupoid, and $C^*(\mathscr{G})$ the universal C*-algebra defined from it.

Uniqueness Theorems: Under what circumstances is a $*$-homomorphism $\phi : C^*(\mathscr{G}) \to B(H)$ injective?

Classical theorems addressing this question assume either

(a) the existence of intertwining "gauge actions" on the algebras, or

(b) an aperiodicity condition on $\mathscr{G}$ itself.

Theorem (Brown-Nagy-R-Sims-Williams)
There is a canonical subalgebra $\mathscr{M} \subset C^*(\mathscr{G})$ such that a $*$-homomorphism $\phi : C^*(\mathscr{G}) \to B(H)$ is injective iff $\phi|_{\mathscr{M}}$ is injective.

Let E be a graph. Denote by E^n the set of paths in E of length n (E^0 is the set of vertices) and $E^* = \cup E^n$.

A **(nondegenerate*) Cuntz-Krieger E-system** on H is a family $\{T_\lambda, \lambda \in E^*\}$ of partial isometries in $B(H)$ satisfying

(i) $T_v, v \in E^0$, are *nonzero mutually orthogonal projections,

(ii) $T_{\lambda\mu} = T_\lambda T_\mu$ for all λ, $\mu \in E^*$ s.t. $s(\lambda) = r(\mu)$,

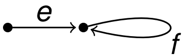
(iii) $T^*_\lambda T_\lambda = T_{s(\lambda)}$ for all $\lambda \in E^*$,

(iv) For $v \in E^0$, $n \in \mathbb{N}$, $\sum\{T_\lambda T^*_\lambda \,|, \lambda \in E^n, r(\lambda) = v\} = T_v$.
(Assuming E is row-finite with no sources.)

Exercise: $C^*\{T_\lambda\} = \overline{\text{span}}\{T_\lambda T^*_\nu \,|\, s(\lambda) = s(\nu)\}$

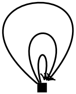
Denote $C^*(E) = C^*\{t_\lambda\}$, where $\{t_\lambda\}$ is a *universal* C-K system.

- **Coburn's Theorem ('67):**



$C^*\{T_\lambda\} \cong \mathcal{T}$

- **Cuntz ('77)**:



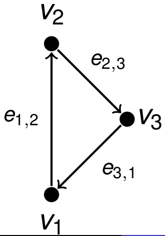
(n loops)

$C^*(T_\lambda) \cong \mathcal{O}_n$
Cuntz algebras

- **Cuntz-Krieger ('80)**: Uniqueness theorem for Cuntz-Krieger algebras $\mathcal{O}_A$.

- **A cycle:**

E



$$\begin{array}{rcl} \phi : C^*(E) & \to & M_3(\mathbb{C}) \\ t_{v_i} & \mapsto & \varepsilon_{ii} \\ t_{e_{i,j}} & \mapsto & \varepsilon_{ji} \end{array}$$

is **not** injective.

Cuntz-Krieger Uniqueness Theorem:
(Kumjian-Pask-Raeburn-Fowler, et. al. ('90's))
If every cycle in E has an entry (L), and ϕ is nondegenerate then ϕ is injective.

Theorem Szymański (2001), Nagy-R (2010):
Condition (L) can be replaced with a condition on the spectrum of $\phi(t_\lambda)$ where the λ are the cycles without entry.

Cycles without entry reveal lack of aperiodicity in the *infinite path space* of the graph. Let us now expand our view to higher rank graphs.

Let $k \in \mathbb{N}^+$. We regard $\mathbb{N}^k$ as a category with a single object, 0, and with composition of morphisms given by addition.

A ***k*-graph** is a countable category Λ along with a "degree" functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *unique factorization property*:

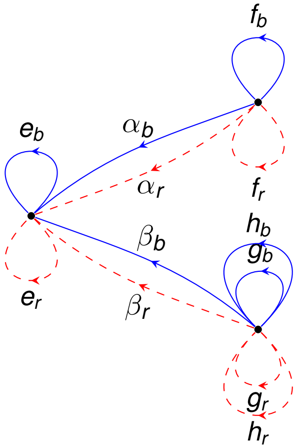
For all $\lambda \in \Lambda$, and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$ then there are unique $\mu \in d^{-1}\{m\}$ and $\nu \in d^{-1}\{n\}$ such that $\lambda = \mu\nu$.

Remarks:

- Think of elements of degree ε_i as edges of color i.
- If $d(\alpha) = \varepsilon_i$, $d(\beta) = \varepsilon_j$ and $s(\alpha) = r(\beta)$, there there are unique α', β' with $d(\alpha') = \varepsilon_j$ and $d(\beta) = \varepsilon_i$ s.t. $\alpha\beta = \alpha'\beta'$.
- These "commuting squares" determine all factorization rules of the k-graph.
- $C^*(\Lambda)$ is defined by associating to each $\lambda \in \Lambda$ a partial isometry t_λ, in accordance with the Cuntz-Krieger relations.

Example

:



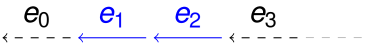
Commutation rules:

$$
\begin{aligned}
e_b\alpha_r &= e_r\alpha_b & e_b\beta_r &= e_r\beta_b \\
\beta_b g_r &= \beta_r g_b & \beta_b h_r &= \beta_r h_b \\
\alpha_b f_r &= \alpha_r f_b & f_b f_r &= f_r f_b \\
g_b g_r &= g_r g_b & g_b h_r &= h_r g_b \\
h_b g_r &= g_r h_b & h_b h_r &= h_r h_b
\end{aligned}
$$

Aperiodicity – defined via the infinite path space $\mathbf{\Lambda}^\infty$

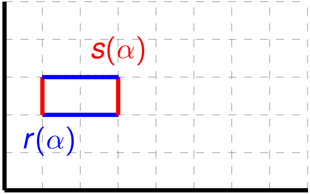
$\Lambda^\infty = \{$degree-preserving covariant functors $\Omega_k \to \Lambda\}$

$k = 1$ picture



$x(1,3) = e_1 e_2$
$d(e_1 e_2) = 2$

$k = 2$ picture



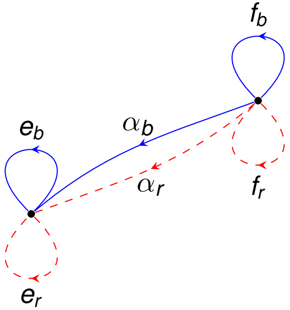
$x((1,2),(3,3)) = \alpha$
$d(\alpha) = (2,1)$

A path $x \in \Lambda^\infty$ is *eventually periodic* if there are $\alpha \neq \beta$ in Λ and $y \in \Lambda^\infty$ such that $x = \alpha y = \beta y$; otherwise x is *aperiodic*.

Λ is **aperiodic** if every vertex is the range of an aperiodic path.

• Uniqueness theorems of Raeburn-Sims-Yeend and Kumjian-Pask assume aperiodicity of the k-graph.

Theorem Nagy-R (2010), Nagy-Brown-R (2013)
A $*$-homomorphism $\phi : C^*(\Lambda) \to \mathcal{A}$ is injective iff it is injective on the subalgebra $\mathscr{M} := C^*(t_\alpha t_\delta^* \mid \forall \gamma \in \Lambda^\infty \; \alpha\gamma = \delta\gamma\}$.
Example:



Commutation rules:

$$e_b\alpha_r = e_r\alpha_b \quad \alpha_b f_r = \alpha_r f_b$$
$$f_b f_r = f_r f_b \quad e_b e_r = e_r e_b$$

Letting $\alpha = e_b\alpha_\beta$, $\delta = e_r\alpha_\beta$, we have $\alpha\gamma = \delta\gamma$ for all $\gamma \in \Lambda^\infty$.

A **groupoid** is a small category $\mathcal{G}$ in which every element has an inverse. A topological groupoid is one in which multiplication and inversion are continuous. It is *étale* if the range and source are local homeomorphisms.

- $C^*(\mathcal{G})$ is defined to be a completion of $C_c(\mathcal{G})$.
- $C^*_r(\mathcal{G})$ is the image of $C^*(\mathcal{G})$ under the direct sum of the left regular representations.
- $\mathcal{G}^{(0)} = \{gg^{-1} \mid g \in \mathcal{G}\}$, the *unit space* of $\mathcal{G}$.
- $\mathrm{Iso}(\mathcal{G}) := \{g \in \mathcal{G} \mid r(g) = s(g)\}$, the *isotropy subgroupoid* of $\mathcal{G}$.

Theorem (Brown-Nagy-R-Sims-Williams, 2014)
Let $\mathcal{G}$ be a locally compact, amenable, Hausdorff, étale groupoid. If $\phi : C^*(\mathcal{G}) \to A$ is a C^*-homomorphism, then the following are equivalent.

(i) ϕ is injective.
(ii) ϕ is injective on $C^*((\mathrm{Iso}(\mathcal{G}))^\circ)$.

Groupoid of a k-graph

To a k-graph Λ, we associate the groupoid

$$\mathcal{G}_\Lambda = \{(\alpha y, d, \beta y) \,|\, y \in \Lambda^\infty,\ \alpha, \beta \in \Lambda,\ d = d_\Lambda(\beta) - d_\Lambda(\alpha)\}$$

$$s(x,d,y) = \ y = r(y,d',z) \qquad (x,d,y)^{-1} = \ (y,-d,x)$$

$$(x,d,y)(y,d',w) \ = \ (x,d+d',w)$$

• The cylinder sets $Z(\alpha,\beta) = \{(\alpha y, d, \beta y)\}$ form a basis for a locally compact, amenable, Hausdorff, étale topology.

• The map $t_\alpha t_\beta^* \mapsto \chi_{Z(\alpha,\beta)}$ implements an iso $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$, which restricts to $C^*(t_\alpha t_\beta^* \,|\, \forall\gamma \in \Lambda^\infty\ \alpha\gamma = \beta\gamma\} \cong C^*(\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ)$.

Properties of the subalgebra

(Renault, '80) A masa C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is **Cartan** if

(i) $\exists$ a faithful conditional expectation $\mathcal{A} \to \mathcal{B}$,
(ii) The normalizer of $\mathcal{B}$ in $\mathcal{A}$ generates $\mathcal{A}$, and
(iii) $\mathcal{B}$ contains an approximate unit of $\mathcal{A}$.

Extension properties for pure states on masa $\mathcal{B} \subset \mathcal{A}$:

(**UEP**) Every pure state extends uniquely to $\mathcal{A}$.

A Cartan subalgebra with the UEP is a Kumjian ***C**-diagonal.**

(**AEP**) Densely many pure states extend uniquely.

Thm (Nagy-R, 2011) When G is a directed graph, $\mathcal{M} \subseteq C^*(G)$ is Cartan and satisfies AEP; i.e, it is a **pseudo-diagonal.**

Thm (BNRSW, 2015) Let $\mathcal{G}$ be a Hausdorff, étale groupoid. Let $\mathcal{M} = C^*(\text{Iso}(\mathcal{G})^\circ)$.

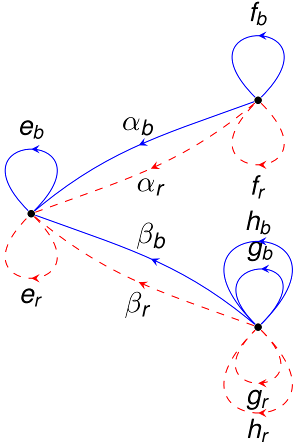
(a) If $(\text{Iso}(\mathcal{G}))^\circ$ is closed and amenable, then the restriction map $f \mapsto f|_{\text{Iso}(\mathcal{G})^\circ}$ extends to a faithful conditional expectation $E : C^*(\mathcal{G}) \to \mathcal{M}$.

(b) If $(\text{Iso}(\mathcal{G}))^\circ$ is not closed, then there is no conditional expectation onto the subalgebra.

(c) If $(\text{Iso}(\mathcal{G}))^\circ$ is abelian and either (i) it is also closed, or (ii) there exists a continuous 1-cocycle $c : \mathcal{G} \to H$ (countable discrete abelian group) s.t. $\forall x \in G^{(0)}$ $c_{G_x^x}$ is injective, then $\mathcal{M}_r$ is masa.

Cor (BNRSW, 2015; Yang, 2014)
$\mathcal{M}$ is always a masa in $C^*(\Lambda)$, Λ a k-graph.

Example of 2-graph C^*-algebras with $(\mathrm{Iso}(\mathcal{G}))^\circ$ not closed, and hence $\mathcal{M}$ not Cartan (no cond. exp.).



Commutation rules:

$$
\begin{aligned}
e_b\alpha_r &= e_r\alpha_b & e_b\beta_r &= e_r\beta_b \\
\beta_b g_r &= \beta_r g_b & \beta_b h_r &= \beta_r h_b, \\
g_b g_r &= g_r g_b & g_b h_r &= h_r g_b, \\
h_b g_r &= g_r h_b & h_b h_r &= h_r h_b \\
\alpha_b f_r &= \alpha_r f_b & f_b f_r &= f_r f_b
\end{aligned}
$$

The element $(e_r(e_b e_r)^\infty, (1, -1), e_b(e_b e_r)^\infty) \in \overline{\mathrm{Iso}(\mathcal{G})^\circ} \setminus \mathrm{Iso}(\mathcal{G})^\circ$.

Abstract Uniqueness Theorem (Brown-Nagy-R)
Let A be a C*-algebra and $M \subset A$ a C^*-subalgebra. Suppose there is a set $\mathcal{S}$ of pure states on M satisfying

(i) each $\psi \in \mathcal{S}$ extends uniquely to a state $\tilde{\psi}$ on A, and
(ii) the direct sum $\oplus_{\psi \in S} \pi_{\tilde{\psi}}$ of the GNS representations associated to the extensions to A of elements in S is faithful on A.

Then a $*$-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective.

Our proof of the main theorem applies the AUT to the set S of pure states of $C_r^*(\mathrm{Iso}(\mathcal{G})^\circ)$ that factor through some $C_r^*(\mathcal{G}_u^u)$ with $\mathcal{G}_u^u = \mathrm{Iso}(\mathcal{G})_u^\circ$ (where $\mathcal{G}_u^u = \mathrm{Iso}(\mathcal{G}) \cap r^{-1}(u)$).

Thank you!

A somewhat arbitrary bibliography follows.

- A. an Huef and I. Raeburn, *The ideal structure of Cuntz-Krieger algebras,* Ergodic Theory Dynam. Systems **17** (1997), 611–624.
- J.H. Brown, G. Nagy, and S. Reznikoff, *A generalized Cuntz-Krieger uniqueness theorem for higher-rank graphs*, J. Funct. Anal. (2013).
- J.H. Brown, G. Nagy, S. Reznikoff, A. Sims, and D. Williams, *Cartan subalgebras of groupoid C^*-algebras*
- K.R. Davidson, S.C. Power, and D. Yang, *Dilation theory for rank 2 graph algebras*, J. Operator Theory.
- D. G. Evans and A. Sims, *When is the Cuntz-Krieger algebra of a higher-rank graph approximately finite-dimensional?*, J. Funct. Anal. **263** (2012), no. 1, 183–215.

- P. Goldstein, *On graph C*-algebras*, J. Austral. Math. Soc. **7**2 (2002), 153–160
- A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. **6** (2000), 1–20.
- A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998) 161–174.
- A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

G. Nagy and S. Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908.

G. Nagy and S. Reznikoff, *Pseudo-diagonals and uniqueness theorems*, (2013), to appear in Proc. AMS.

D. Pask, I. Raeburn, M. Rørdam, A. Sims, *Rank-two graphs whose C*-algebras are direct limits of circle algebras*, J. Functional Anal. **144** (2006), 137–178.

I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C*-algebras*, Proc. Edin. Math. Soc. **46** (2003) 99–115.

N. Phillips, emphCrossed products of the Cantor set by free minimal actions of $\mathbb{Z}^d$, Comm. Math. Phys., **256** (2005), 1–42.

- A. Sims, *Gauge-invariant ideals in the C*-algebras of finitely aligned higher-rank graphs*, Canad. J. Math. **58** (2006), no. 6, 1268–1290.
- J. Spielberg, *Graph-based models for Kirchberg algebras*, J. Operator Theory **57** (2007), 347–374.
- W. Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002) 549–555.
- D. Yang, *Periodic higher rank graphs revisited*, arXiv:1403.6848 (2014).
- T. Yeend, *Groupoid models for the C*-algebras of topological higher-rank graphs*, J. Operator Theory 57:1 (2007), 96–120