Skein theory for subfactors

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Subfactors: inclusions of von Neumann algebras with trivial center.

Theorem ([Jon83])

$$\{index\} = \{4\cos^2\frac{\pi}{n}, n = 3, 4, \cdots\} \cup [4, \infty].$$

Fact (Invariants of subfactors)

Standard Invariant ⇒ Principal Graph ⇒ Index

Theorem ([Pop94])

The standard invariant is a complete invariant of strongly amenable subfactors of the hyperfinite factor of type II_1.

Standard invariant → quantum symmetry

A, D, E classification

Theorem (Jones, Ocneanu,...,90s)

The classification of subfactors with index less 4:

- A_n, *one*
- D_{2n}, *one*
- E_6, *a complex conjugate pair.*
- E_8, *a complex conjugate pair.*

Axiomatizations of the standard invariants

Three axiomatizations:

(1) Ocneanu's paragroup [Ocn88]

(2) Popa's standard λ-lattices [Pop95]

(3) Jones' subfactor planar algebras [Jon98]
A mysterious condition: 360° rotation invariance appeared in (1), (3) but not in (2).

Theorem

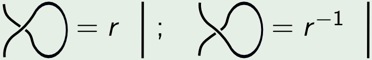
Positivity $+$ *Flatness* $\Longrightarrow$ *Rotation invariance*

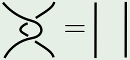
Skein theory

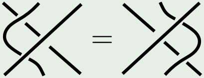
- Skein theory: presenting subfactor planar algebras by generators and (algebraic and topological) relations.
- The Temperley-Lieb-Jones planar algebra has no generators nor relations.
- Three fundamental problems:
  - Evaluation
  - Consistency
  - Positivity

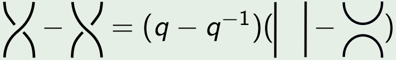
Example (BMW [BW89, Mur87])

The Birman-Murakami-Wenzl (BMW) algebra is a q, r-parameterized (unshaded, unoriented) planar algebra generated by (the universal R matrix)  with the following relations:

Reidemeister move I: 

Reidemeister move II: 

Reidemeister move III: 

BMW relation: 

Universal skein theory

Universal skein theory: a presentation for any subfactor planar algebra (given principal graphs)

Theorem

Evaluation, consistency and positivity can be proved by solving polynomial equations.

Theorem

Connections $\leftrightarrow$ λ *lattices* $\leftrightarrow$ *pre subfactor planar algebras*
Flatness $\leftrightarrow$ *standard* $\leftrightarrow$ *vertical isotopy*

- Pre subfactor planar algebras: subfactor planar algebras without vertical isotopy

Prove the flatness by UST

- step 1: connection is solved in an efficient way
- step 1.5: connection $\rightarrow \lambda$ lattice $\rightarrow$ pre-subfactor planar algebra
- step 2: prove/disprove the flatness

The pre-subfactor planar algebra provides new methods to prove the flatness, such as a good choice of the generator, extra skein relations and the positivity.

Remark

The positivity does NOT rely on the flatness. Instead, it is a powerful tool to prove the flatness and something more interesting.

A, D, E classification by UST

- We have an independent classification and construction of subfactors with index less than 4, i.e. A_n, D_{2n}, E_6, E_8.
- Furthermore, by the universal skein theory, we can construct "D_{2n-1} subfactors".

Another skein theory

The universal skein theory is efficient to construct small index subfactors, but not for large index ones. Moreover, one need to know the principal graph to apply the universal skein theory. We need a different type of skein theory to construct subfactors with large index without knowing the principal graph. That is the *Yang-Baxter relation* motivated by the Yang-Baxter equation.

BMW

Example (BMW)

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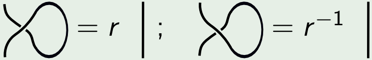
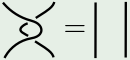
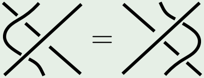
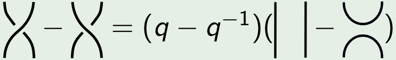
Reidemeister move I: 

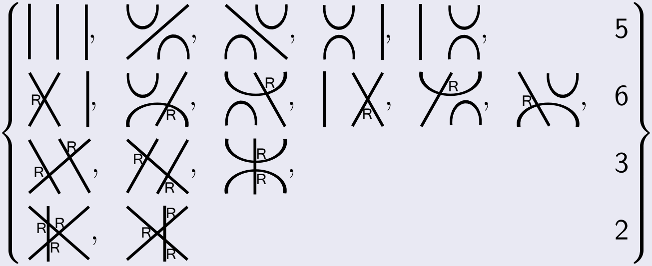
Reidemeister move II: 

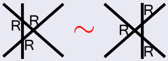
Reidemeister move III: 

BMW relation: 

Proposition (Basis for 2,3-boxes)

$\mathscr{P}_{2,+} = span_{\mathbb{C}}$ 

$\mathscr{P}_{3,+} = span_{\mathbb{C}}$ 



modulo lower terms $15 = 16 - 1$

Based on former work joint with Bisch and Jones, [BJ97, BJ03, BJL]

Theorem ([Liu])

Any unshaded subfactor planar algebra P generated by $\mathscr{P}_2$ and $dim(\mathscr{P}_3) \leq 15$ is one of the following:

(1) *Bisch-Jones;*

(2) *BMW;*

(3) $\mathscr{E}_{N+2}$.

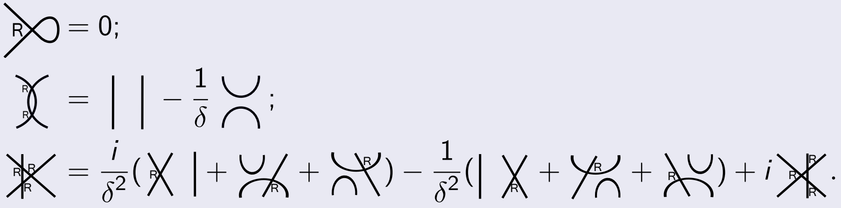
Remark

Case (1) is a limit of case (2).
In case (3), E_{N+2} is a complex conjugate pair.
The generator is self-contragredient in case (1) and (2), but not in case (3).

E Series

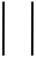
Definition (Generator and relations)

Let $\mathscr{P}$ be the unshaded q-parameterized planar algebra generated by $R =$  which satisfies  $= -i$  and the Yang-Baxter relation:



where $\delta = \dfrac{i(q+q^{-1})}{q-q^{-1}}$.

Algebraic presentation

$\alpha = \frac{q - q^{-1}}{2}$  $- i\frac{q - q^{-1}}{2}$  $+ \frac{q + q^{-1}}{2}$ 

;

$h =$ 

$$\alpha_i - \alpha_i^{-1} = (q - q^{-1})\alpha_i$$
$$\alpha_i\alpha_j = \alpha_j\alpha_i, \ \forall \ |i - j| \geq 2$$
$$\alpha_i\alpha_{i+1}\alpha_i = \alpha_{i+1}\alpha_i\alpha_{i+1}$$
$$h_i^2 = \frac{i(q + q^{-1})}{q - q^{-1}} h_i$$
$$h_i h_j = h_j h_i, \ \forall \ |i - j| \geq 2$$
$$h_i h_{i\pm1} h_i = h_{i\pm1} h_i h_{i\pm1}$$
$$\alpha_i h_i = h_i \alpha_i = q h_i$$
$$\alpha_i h_j = h_j \alpha_i \ \forall \ |i - j| \geq 2$$

$$\alpha_i\alpha_{i+1}h_i = h_{i+1}\alpha_i\alpha_{i+1} = ih_{i+1}h_i$$
$$h_i\alpha_{i+1}\alpha_i = \alpha_{i+1}\alpha_i h_{i+1} = -ih_i h_{i+1}$$
$$\alpha_i h_{i\pm1}\alpha_{i\pm1}^{-1} = \alpha_{i\pm1}^{-1} h_i \alpha_{i\pm1}$$
$$h_i h_{i\pm1}\alpha_i = h_i\alpha_{i\pm1}^{-1}$$
$$\alpha_i h_{i\pm1} h_i = \alpha_{i\pm1} h_i$$
$$h_i\alpha_{i\pm1} h_i = iq^{-1} h_i$$
where $\alpha_i^{-1} = \alpha_i - q - q^{-1}$

Properties

- When $q = e^{\frac{i\pi}{2N+2}}$, we have the subfactor $\mathscr{E}_{N+2}$.
- Principal graph
- Trace formula
- $D_{2(N+1)}$ symmetry (more subfactors are obtained)
- Quotients (more unitary fusion categories are obtained)

Trace formula

Over the field $\mathbb{C}(q)$, the principal graph of $\mathscr{P}$ is Young's lattice.

Theorem ([Liu])

$$< \lambda >= \prod_{c \in \lambda} \frac{i(q^{h(c)} + q^{-h(c)})}{q^{h(c)} - q^{-h(c)}},$$

where $h(c)$ is the hook length of the cell c in the Young diagram λ.

Remark

$$< \lambda >= \prod_{c \in \lambda} \cot(h(c)\theta), \quad when\ q = e^{i\theta},$$

in particular

$$\delta = \cot\theta$$

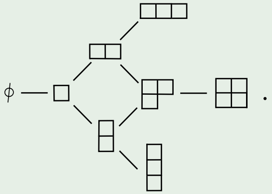
Principal graphs

Theorem ([Liu])

When $q = e^{\frac{i\pi}{2N+2}}$, the principal graph $YL(N)$ of the quotient $\mathscr{E}_{N+2}$ is the sublattice of the Young lattice consisting of Young diagrams whose $(1,1)$ cell has hook length at most N.

Example ([LMP, Liu])

$YL(3) =$



Dihedral symmetry

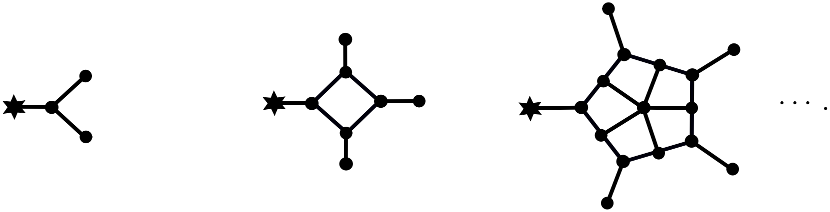
Proposition ([Sut02, Liu])

$$Aut(YL(N)) = D_{2(N+1)}.$$

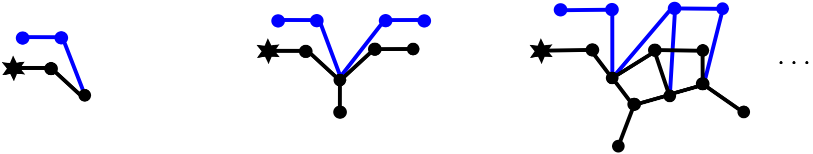
The $\mathbb{Z}_2$ symmetry is from a $\mathbb{Z}_2$ automorphism of $\mathscr{E}_{N+2}$ by mapping R to $-R$. It reflects the Young diagrams by the diagonal.
The $\mathbb{Z}_{N+1}$ symmetry is from invertible objects of $\mathscr{E}_{N+2}$.

$\mathbb{Z}_2$ symmetry

Principal graphs of $\mathscr{E}_{N+2}$, $N = 2, 3, 4 \cdots$:



Principal graphs of the $\mathbb{Z}_2$ fixed point algebras:



$\mathbb{Z}_{N+1}$ symmetry

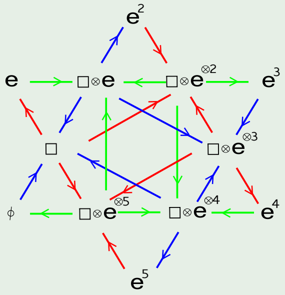
For any odd order subgroup A of $\mathbb{Z}_{N+1}$, there is a A fixed Young diagram λ. Moreover, we have a subfactor with index $\frac{<\lambda>^2}{|A|}$.

Quantum subgroups

There are two *grading operators*
Jones projection: e in $\mathscr{P}_2$
Antisymmetrizer: g in $\mathscr{P}_N$

- Modulo e: $\mathscr{E}_{N+2}$
- Modulo g: $SU(N)_{N+2} \subset SU(N(N+1)/2)_1$
- Modulo $e \otimes g$: $SU(N+2)_N \subset SU((N+2)(N+1)/2)_1$

Example ($SU(3)_5$, [Xu98, Ocn00, Liu] $\cdots$)



$$e^{\otimes 6} = \emptyset$$

$$[1] \otimes e = e \otimes [1]$$

$$[1] \otimes [1] = e \oplus ([1] \otimes e^{\otimes 2}) \oplus ([1] \otimes e^{\otimes 5})$$

3D TQFT

- Turaev-Viro model [TV92]: unitary fusion category $\to$ 3D TQFT
- $\mathscr{P}(q)/e^{\otimes k} \otimes g^{\otimes l} \to ?$
- 

$= \omega$ 

- $O(N)$, $\omega = 1$
- $Sp(2N)$, $\omega = -1$
- $\mathscr{P}(q)$, $\omega = \pm i$

Construction

To construct the sequence of subfactor planar algebras $\mathscr{E}_{N+2}$, we overcome the three fundamental problems in skein theory:

Problem (and Solution)

- *Generator Relations (Classification)*
- *Evaluation (Yang-Baxter relation)*
- *Consistency (Kauffman's argument + HOMFLY-PT invariant)*
- *Positivity (Universal skein theory)*
  - *Constructing Matrix units*
    (Matrix units of Hecke algebras + Basic construction)
  - *Computing the trace formula*
    (q-Murphy operator + calculations)
  - *Taking the quotient*
    (String algebras + Wenzl's formula)

Thank you!

Paper is available on arXiv:
http://arxiv.org/abs/1507.06030

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