A NOTE ON UNIFORMLY BOUNDED COCYCLES INTO FINITE VON NEUMANN ALGEBRAS

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Abstract. We give a short proof of a result of T. Bates and T. Giordano stating that any uniformly bounded Borel cocycle into a finite von Neumann algebra is cohomologous to a unitary cocycle [3]. We also point out a separability issue in their proof. Our approach is based on the existence of a non-positive curvature metric on the positive cone of a finite von Neumann algebra.

1. Statement of the main result

Let $\Gamma$ be a discrete countable group acting on a standard probability space $(S, \mu)$ with $\mu$ being quasi-invariant and ergodic. Let $M$ be a von Neumann algebra and denote by $GL(M)$ its invertible group, equipped with strong operator topology. A Borel map $\alpha : \Gamma \times S \to GL(M)$ is called a cocycle if for any $g, h \in \Gamma$, for almost all $s \in S$,

$$\alpha(gh, s) = \alpha(g, hs) \alpha(h, s).$$

A cocycle is said uniformly bounded if there exists $c > 0$ such that for any $g \in \Gamma$, for almost all $s \in S$, $\|\alpha(g, s)\| \leq c$. Two cocycles $\alpha, \beta : \Gamma \times S \to GL(M)$ are cohomologous if there exists a Borel map $\phi : S \to GL(M)$ such that for all $h \in \Gamma$ and almost all $s \in S$,

$$\beta(h, s) = \phi(hs) \alpha(h, s) \phi(s)^{-1}.$$

In this note, we give a new proof of the following result due to T. Bates & T. Giordano, this Theorem generalizes results of both F.-H. Vasilescu & L. Zsidó [6] and R. J. Zimmer [7].

**Theorem** ([3], Theorem 3.3). Let $\Gamma$ be a discrete countable group acting on $(S, \mu)$ standard Borel space with probability measure $\mu$ which is quasi-invariant and ergodic and $M$ be a finite von Neumann algebra with separable predual. Let $\alpha : \Gamma \times S \to GL(M)$ be a uniformly bounded Borel cocycle. Then $\alpha$ is cohomologous to a cocycle valued in the unitary group of $M$.

Their approach is based on adapting the Ryll-Nardzewski fixed point theorem. However it seems that there is a gap in the argument, and we were not able to determine to what extend this gap was fillable, see the Remark below. Our approach takes a different road, though; it is based on a more geometric property of finite von Neumann algebras, in the spirit of [5].

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2. Circumcenter and non-positive curvature

Let $(X, d)$ be a metric space and $B \subset X$ a non-empty bounded subset of $X$. The circumradius of $B$ is the real number

$$r(B) := \inf_{x \in X} \sup_{y \in B} d(x, y).$$

A point $x \in X$ is called a circumcenter of $B$ if the closed ball centered at $x$ and with radius $r = r(B)$ contains $B$. Note that, in general, a circumcenter does not always exist and is not necessarily unique.
A geodesic metric space \((X, d)\) is called a CAT(0)-space if it satisfies the *semi-parallelogram law:* for any \(x_1, x_2 \in X\), there exists \(z \in X\) such that for all \(x \in X\),
\[
d(x_1, x_2)^2 + 4d(z, x)^2 \leq 2d(x_1, x)^2 + 2d(x_2, x)^2.
\]

In a complete CAT(0)-space, every non-empty bounded subset always admits a unique circumcenter, see for instance [1, Theorem VI.4.2]. Of course it is not always the case that a subset contains its circumcenter, but its closed convex hull does.

The important point for us is that the set of positive elements in a finite von Neumann algebra can be endowed with a metric satisfying the semi-parallelogram law, see [2]. Let \(\mathcal{M}\) be a finite von Neumann algebra with finite trace \(\tau\). For \(x \in \mathcal{M}\), its \(L_2\)-norm is denoted by \(\|x\|_2 := \tau(x^*x)^{1/2}\). Denote by \(GL(\mathcal{M})_+\) the set of positive invertible elements. For \(a, b \in GL(\mathcal{M})_+\), set
\[
d(a, b) := \|\ln(a^{-1/2}ba^{-1/2})\|_2.
\]
This defines a metric on \(GL(\mathcal{M})_+\). Here are the main features of this metric; for more details we refer to [3] and the references therein.

(i) For any \(g \in GL(\mathcal{M})\), \(d(a, b) = d(g^*ag, g^*bg)\);
(ii) The metric \(d\) satisfies the semi-parallelogram law;
(iii) For all \(c > 1\), the metric \(d\) is equivalent to \(\|\cdot\|_2\) on the set
\[
GL(\mathcal{M})_c := \{x \in GL(\mathcal{M})_+, \, c^{-1} \leq x \leq c\}.
\]
In fact, for all \(c > 1\), the space \((GL(\mathcal{M})_c, d)\) is a (geodesic) CAT(0)-space, which is bounded, complete and separable (this is not the case of \(GL(\mathcal{M})_+\)).

Consequently, for all \(c > 1\), every non-empty subset \(B \subset GL(\mathcal{M})_c\) admits a unique circumcenter, which lies in \(GL(\mathcal{M})_c\).

3. PROOF OF THE THEOREM

For each \(s \in S\), denote \(B_s = \{\alpha(g, s)^*\alpha(g, s) \, | \, g \in \Gamma\}\). Since \(\alpha\) is a uniformly bounded cocycle, there exists \(c > 1\) and a conull Borel set \(S_0 \subset S\) such that for all \(s \in S_0\) and all \(h \in \Gamma\),
\[
B_s \subset GL(\mathcal{M})_c \quad \text{and} \quad \alpha(h, s)^*B_{hs}\alpha(h, s) = B_s.
\]
For every \(s \in S_0\), denote by \(\gamma(s) \in GL(\mathcal{M})_c\) the unique circumcenter of \(B_s\). By uniqueness, property (i) above implies:
\[
\alpha(h, s)^*\gamma(hs)\alpha(h, s) = \gamma(s), \quad \text{for all } s \in S_0.
\]
We claim that the map \(s \in S \mapsto \gamma(s)^{1/2} \in \mathcal{M}\) (with \(\gamma\) arbitrarily defined on \(S \setminus S_0\)) almost surely coincides with a Borel map \(\varphi\). After we prove this claim, we will get that the Borel map
\[
\beta : (h, s) \in \Gamma \times S \mapsto \varphi(hs)\alpha(h, s)\varphi(s)^{-1} \in GL(\mathcal{M})
\]
is a unitary cocycle cohomologous to \(\alpha\), giving the theorem.

To prove the claim we follow the argument in [1] Lemma 3.18. As the cocycle \(\alpha\) is a Borel map and \(\Gamma\) is countable, for all \(v \in GL(\mathcal{M})_c\), the map
\[
s \in S \mapsto r(v, B_s) := \sup_{g \in \Gamma} d(v, \alpha(g, s)^*\alpha(g, s))
\]
is Borel. By continuity of the maps \(v \mapsto r(v, B_s)\), \(s \in S\), and separability (iii) of \(GL(\mathcal{M})_c\), we deduce that the map
\[
s \in S \mapsto \inf_{v \in GL(\mathcal{M})_c} r(v, B_s) = r(B_s)
\]
coincides with an infimum over a countable subset of \(GL(\mathcal{M})_c\), and hence is Borel. For \(n \geq 1\), the following set \(D_n\) is a Borel bundle over \(S\):
\[
D_n = \{(s, v) \in S \times GL(\mathcal{M})_c : r(v, B_s)^2 \leq r(B_s)^2 + n^{-1}\}.
\]
By [8, Theorem A.9], there exist Borel maps $\xi_n : S \to GL(M)_c$ such that $(s, \xi_n(s)) \in D_n$, for all $s$ in some conull subset $S_1 \subset S$. For all $s \in S_0 \cap S_1$, the semi-parallelogram law implies that the sequence $(\xi_n(s))_n$ converges to $\gamma(s)$. More precisely, for all fixed $n$, with $x_1 = \gamma(s)$ and $x_2 = \xi_n(s)$, there exists $z \in GL(M)_c$ such that for all $x \in B_s$,
\[ d(x_1, x_2)^2 + 4d(z, x)^2 \leq 2d(x_1, x)^2 + 2d(x_2, x)^2. \]
Taking the supremum over $x \in B_s$ we get,
\[ d(\gamma(s), \xi_n(s))^2 + 4r(z, B_s)^2 \leq 2r(\gamma(s), B_s)^2 + 2r(\xi_n(s), B_s)^2 \leq 4r(B_s)^2 + 2n^{-1}. \]
Since $r(z, B_s) \geq r(B_s)$, this readily gives the desired convergence. Therefore $\gamma$ almost surely coincides with the Borel map $\lim_n \xi_n$, which finishes the proof.

**Remark.** We point out a gap in the proof of the main result by T. Bates and T. Giordano [3, Theorem 3.3]. With the notations of that proof, at the bottom of p. 747, it is not clear why a countable cover of $X$ should exist. For instance, in the case where $\mathcal{M}$ is the trivial algebra $\mathcal{M} = C$, then $\mathcal{M}_C$ and $B_\varepsilon$ are simply balls (in the $\| \cdot \|_\infty$-norm) inside $\mathcal{M} = L^\infty(S)$, of radii $C$ and $\varepsilon$, respectively. Of course, there exists a ultraweakly dense sequence $(\phi_n)_n$ of $X$, but $B_\varepsilon$ has empty interior (for the ultraweak topology) inside $L^\infty(S)$, so $(\phi_n + B_\varepsilon)_n$ has a priori no reason to cover $X$.

**References**


