A NOTE ON UNIFORMLY BOUNDED COCYCLES INTO FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. We give a short proof of a result of T. Bates and T. Giordano stating that any uniformly bounded Borel cocycle into a finite von Neumann algebra is cohomologous to a unitary cocycle [3]. We also point out a separability issue in their proof. Our approach is based on the existence of a non-positive curvature metric on the positive cone of a finite von Neumann algebra.

1. STATEMENT OF THE MAIN RESULT

Let Γ be a discrete countable group acting on a standard probability space (S, μ) with μ being quasi-invariant and ergodic. Let $\mathcal{M}$ be a von Neumann algebra and denote by $GL(\mathcal{M})$ its invertible group, equipped with strong operator topology. A Borel map $\alpha : \Gamma \times S \to GL(\mathcal{M})$ is called a cocycle if for any $g, h \in \Gamma$, for almost all $s \in S$,

$$\alpha(gh, s) = \alpha(g, hs)\alpha(h, s).$$

A cocycle is said uniformly bounded if there exists $c > 0$ such that for any $g \in \Gamma$, for almost all $s \in S$, $\|\alpha(g, s)\| \leq c$. Two cocyles $\alpha, \beta : \Gamma \times S \to GL(\mathcal{M})$ are cohomologous if there exists a Borel map $\phi : S \to GL(\mathcal{M})$ such that for all $h \in \Gamma$ and almost all $s \in S$,

$$\beta(h, s) = \phi(hs)\alpha(h, s)\phi(s)^{-1}.$$

In this note, we give a new proof of the following result due to T. Bates & T. Giordano, this Theorem generalizes results of both F.-H. Vasilescu & L. Zsidó [6] and R. J. Zimmer [7].

Theorem ([3], Theorem 3.3)**.** *Let Γ be a discrete countable group acting on (S, μ) standard Borel space with probability measure μ which is quasi-invariant and ergodic and $\mathcal{M}$ be a finite von Neumann algebra with separable predual. Let $\alpha : \Gamma \times S \to GL(\mathcal{M})$ be a uniformly bounded Borel cocycle. Then α is cohomologous to a cocycle valued in the unitary group of $\mathcal{M}$.*

Their approach is based on adapting the Ryll-Nardzewski fixed point theorem. However it seems that there is a gap in the argument, and we were not able to determine to what extend this gap was fillable, see the Remark below. Our approach takes a different road, though; it is based on a more geometric property of finite von Neumann algebras, in the spirit of [5].

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2. CIRCUMCENTER AND NON-POSITIVE CURVATURE

Let (X, d) be a metric space and $B \subset X$ a non-empty bounded subset of X. The circumradius of B is the real number

$$r(B) := \inf_{x \in X} \sup_{y \in B} d(x, y).$$

A point $x \in X$ is called a *circumcenter* of B if the closed ball centered at x and with radius $r = r(B)$ contains B. Note that, in general, a circumcenter does not always exist and is not necessarily unique.

A geodesic metric space (X,d) is called a CAT(0)-space if it satisfies the *semi-parallelogram law*: for any $x_1, x_2 \in X$, there exists $z \in X$ such that for all $x \in X$,

$$d(x_1,x_2)^2 + 4d(z,x)^2 \leq 2d(x_1,x)^2 + 2d(x_2,x)^2.$$

In a complete CAT(0)-space, every non-empty bounded subset always admits a unique circumcenter, see for instance [4, Theorem VI.4.2]. Of course it is not always the case that a subset contains its circumcenter, but its closed convex hull does.

The important point for us is that the set of positive elements in a finite von Neumann algebra can be endowed with a metric satisfying the semi-parallelogram law, see [2]. Let $\mathcal{M}$ be a finite von Neumann algebra with finite trace τ. For $x \in \mathcal{M}$, its L_2-norm is denoted by $\|x\|_2 := \tau(x^*x)^{1/2}$. Denote by $GL(\mathcal{M})_+$ the set of positive invertible elements. For $a, b \in GL(\mathcal{M})_+$, set

$$d(a,b) := \| \ln(a^{-1/2}ba^{-1/2})\|_2.$$

This defines a metric on $GL(\mathcal{M})_+$. Here are the main features of this metric; for more details we refer to [5] and the references therein.

(i) For any $g \in GL(\mathcal{M})$, $d(a,b) = d(g^*ag, g^*bg)$;
(ii) The metric d satisfies the semi-parallelogram law;
(iii) For all $c > 1$, the metric d is equivalent to $\|\cdot\|_2$ on the set

$$GL(\mathcal{M})_c := \{x \in GL(\mathcal{M})_+ \,,\ c^{-1} \leq x \leq c\}.$$

In fact, for all $c > 1$, the space $(GL(\mathcal{M})_c, d)$ is a (geodesic) CAT(0)-space, which is bounded, complete and separable (this is not the case of $GL(\mathcal{M})_+$).

Consequently, for all $c > 1$, every non-empty subset $B \subset GL(\mathcal{M})_c$ admits a unique circumcenter, which lies in $GL(\mathcal{M})_c$.

3. Proof of the Theorem

For each $s \in S$, denote $B_s = \{\alpha(g,s)^*\alpha(g,s),\ g \in \Gamma\}$. Since α is a uniformly bounded cocycle, there exists $c > 1$ and a conull Borel set $S_0 \subset S$ such that for all $s \in S_0$ and all $h \in \Gamma$,

$$B_s \subset GL(\mathcal{M})_c \qquad \text{and} \qquad \alpha(h,s)^* B_{hs} \alpha(h,s) = B_s.$$

For every $s \in S_0$, denote by $\gamma(s) \in GL(\mathcal{M})_c$ the unique circumcenter of B_s. By uniqueness, property (i) above implies:

$$\alpha(h,s)^*\gamma(hs)\alpha(h,s) = \gamma(s), \text{ for all } s \in S_0.$$

We claim that the map $s \in S \mapsto \gamma(s)^{1/2} \in \mathcal{M}$ (with γ arbitrarily defined on $S \setminus S_0$) almost surely coincides with a Borel map φ. After we prove this claim, we will get that the Borel map

$$\beta : (h,s) \in \Gamma \times S \longmapsto \varphi(hs)\alpha(h,s)\varphi(s)^{-1} \in GL(\mathcal{M})$$

is a unitary cocycle cohomologous to α, giving the theorem.

To prove the claim we follow the argument in [1, Lemma 3.18]. As the cocycle α is a Borel map and Γ is countable, for all $v \in GL(\mathcal{M})_c$, the map

$$s \in S \longmapsto r(v, B_s) := \sup_{g \in \Gamma} d(v, \alpha(g,s)^*\alpha(g,s))$$

is Borel. By continuity of the maps $v \mapsto r(v, B_s)$, $s \in S$, and separability (iii) of $GL(\mathcal{M})_c$, we deduce that the map

$$s \in S \longmapsto \inf_{v \in GL(\mathcal{M})_c} r(v, B_s) = r(B_s)$$

coincides with an infimum over a countable subset of $GL(\mathcal{M})_c$, and hence is Borel. For $n \geq 1$, the following set D_n is a Borel bundle over S:

$$D_n = \{(s,v) \in S \times GL(\mathcal{M})_c \ :\ r(v, B_s)^2 \leq r(B_s)^2 + n^{-1}\}.$$

By [8, Theorem A.9], there exist Borel maps $\xi_n : S \to GL(\mathcal{M})_c$ such that $(s, \xi_n(s)) \in D_n$, for all s in some conull subset $S_1 \subset S$. For all $s \in S_0 \cap S_1$, the semi-parallelogram law implies that the sequence $(\xi_n(s))_n$ converges to $\gamma(s)$. More precisely, for all fixed n, with $x_1 = \gamma(s)$ and $x_2 = \xi_n(s)$, there exists $z \in GL(\mathcal{M})_c$ such that for all $x \in B_s$,

$$d(x_1, x_2)^2 + 4d(z, x)^2 \leq 2d(x_1, x)^2 + 2d(x_2, x)^2.$$

Taking the supremum over $x \in B_s$ we get,

$$d(\gamma(s), \xi_n(s))^2 + 4r(z, B_s)^2 \leq 2r(\gamma(s), B_s)^2 + 2r(\xi_n(s), B_s)^2 \leq 4r(B_s)^2 + 2n^{-1}.$$

Since $r(z, B_s) \geq r(B_s)$, this readily gives the desired convergence. Therefore γ almost surely coincides with the Borel map $\lim_n \xi_n$, which finishes the proof.

Remark. We point out a gap in the proof of the main result by T. Bates and T. Giordano [3, Theorem 3.3]. With the notations of that proof, at the bottom of p. 747, it is not clear why a countable cover of X should exist. For instance, in the case where $\mathcal{M}$ is the trivial algebra $\mathcal{M} = \mathbb{C}$, then $\widetilde{\mathcal{M}}_C$ and B_ε are simply balls (in the $\| \cdot \|_\infty$-morm) inside $\widetilde{\mathcal{M}} = L^\infty(S)$, of radii C and ε, respectively. Of course, there exists a ultraweakly dense sequence $(\phi_n)_n$ of X, but B_ε has empty interior (for the ultraweak topology) inside $L^\infty(S)$, so $(\phi_n + B_\varepsilon)_n$ has a priori no reason to cover X.

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