HOMEWORK: ERGODIC THEORY

Invariant sets. Consider a non-singular action of a lcsc group $G \curvearrowright (X, \mu)$, and take a subset $A \subset X$ with the property that $\mu(A \Delta gA) = 0$ for all $g \in G$. We want to prove that there exists a set $B \subset X$ which is truly G-invariant and such that $\mu(A \Delta B) = 0$.

(1) Denote by $f : G \times X \to \mathbb{R}$ the Borel function such that $f(g, x) = |\mathbf{1}_{gA}(x) - \mathbf{1}_A(x)|$. Prove that $\int_G \int_X f(g, x) d\mu(x) dg = 0$.
(2) Deduce that the set $Y := \{x \mid \int_G f(g, x) dg = 0\}$ is a measurable subset of X, which is co-null and G-invariant.
(3) Prove that $B := \{x \in Y \mid \int_G \mathbf{1}_{gA}(x) dg \neq 0\}$ satisfies the desired conclusion.

Poincaré recurrence theorem. Let (X, μ) be a probability space and $T : X \to X$ be a μ-preserving map (we do not necessarily require that T is invertible). Given $A \subset X$, the Poincaré recurrence theorem asserts that for μ-almost every $x \in A$, the orbit $x, Tx, T^2x, \dots$ intersects A infinitely often.

(1) Prove first that the orbit of almost every $x \in A$ goes back to x at least once. That is, prove that the following set B has measure 0:
$$B := \{x \in A \mid T^n(x) \notin A \text{ for all } n \geq 1\}.$$
Hint. Compute the intersection $T^{-n}(B) \cap T^{-m}(B)$ for $n \neq m$.
(2) Complete the proof of Poincaré recurrence theorem.

Ergodicity and Koopman representation. Find an example of a measure preserving action which is not ergodic, but whose Koopmann representation has no non-constant invariant functions.

Weakly mixing actions. A pmp action $G \curvearrowright (X, \mu)$ is said to be *weakly mixing* if the diagonal action $G \curvearrowright X \times X$ is ergodic.

(1) Check that a weakly mixing action is ergodic. Prove that an irrational rotation on the circle is not weakly mixing (although it is ergodic).
(2) Prove that a mixing action is always weakly mixing. *Hint.* Prove that the linear span of product functions $(x, y) \mapsto f_1(x)f_2(y)$ with $f_1, f_2 \in L^2(X)$ is dense in $L^2(X \times X)$.
(3) Give an example of a weakly mixing action which is not mixing. *Hint.* Use a quotient $G \to H$ with a non-compact kernel, and a suitable action of H.
(4) We aim to prove that if $G \curvearrowright (X, \mu)$ is weakly mixing and if $G \curvearrowright (Y, \nu)$ is another ergodic pmp action, the diagonal action $G \curvearrowright X \times Y$ is ergodic. Here are some steps.
   a) Assume that the action $G \curvearrowright X \times Y$ is not ergodic. Then there exists an L^2-function $f : X \times Y \to \mathbb{R}$ which is G-invariant, has integral 0 but is non-zero. Check that for almost every $y \in Y$, the partial integral $\int_X f(x, y) d\mu(x)$ vanishes.
   b) Prove that there exists $a \in L^2(X)$ such that the function $y \in Y \mapsto \int_X a(x) f(x, y) dx$ is non-zero. Here one needs to use that $L^2(X)$ is separable.
   c) Prove that the function $F : (x, x') \in X \times X \mapsto \int_Y f(x, y) f(x', y) d\nu(y) \in \mathbb{R}$ is a G-invariant in $L^2(X \times X)$ with integral 0, but which is non-zero. Conclude.

Howe-Moore theorem and lattices. Let Γ be a discrete countable group. Given a subgroup $\Lambda < \Gamma$, we may consider Bernoulli shift action $\sigma : \Gamma \curvearrowright \{0, 1\}^{\Gamma/\Lambda}$, defined by $\sigma_g(x) = y$, where $y \in \{0, 1\}^{\Gamma/\Lambda}$ is the element such that $y_{h\Lambda} = x_{g^{-1}h\Lambda}$.

(1) Check that this indeed defines a measurable action (even continuous) which preserves the product measure $\mu_0^{\otimes \Gamma/\Lambda}$, where μ_0 is any probability measure on $\{0, 1\}$.
(2) Prove that if $\Lambda < \Gamma$ has infinite index (i.e. if Γ/Λ is an infinite set), then this action is ergodic.
(3) Prove that if Λ is infinite, then this action is not mixing.

We conclude from these facts that many discrete groups don't have the Howe-Moore property. For instance one can check (in many ways) that a lattices in non-compact connected semi-simple Lie group never has the Howe-Moore property, even though the ambient Lie group does.

Induction. Take a lattice Γ in a general lcsc group G. We explain how to induce an action/a unitary representation of Γ to one of G. Let us choose a fundamental domain $\mathcal{D} \subset G$ for the right action of Γ on G.

Part 0. Some notations.

(1) Check that choosing such a domain Ω amounts to choosing a section to the projection map $p : G \to G/\Gamma$, i.e. a measurable map $s : G/\Gamma \to G$ such that $p \circ \sigma = \mathrm{id}$.

Unfortunately, in general, there is no reason for such a section s to be G-equivariant, even though the projection map p is. Define thus a map $c : G \times G/\Gamma \to G$ by the formula $c(g, a) := s(ga)^{-1}gs(a)$.

(2) Check that the map c has its image in Γ, and that it satisfies the so-called *cocycle* identity

$$c(gh, a) = c(g, ha)c(h, a), \quad \text{for all } g, h \in G, a \in G/\Gamma.$$

Part 1. Induced action.

(3) Given a measure preserving action $\sigma : \Gamma \curvearrowright (X, \mu)$, prove that the following formula defines an action $\widetilde{\sigma}$ of G on $\widetilde{X} := G/\Gamma \times X$, which preserves the product measure $\widetilde{\mu} := \lambda_{G/\Gamma} \otimes \mu$:

$$\widetilde{\sigma}_g(a, x) := (ga, \sigma_{c(g,a)}(x)), \quad \text{for all } g \in G, a \in G/\Gamma, x \in X.$$

We call this G-action the *induced action* associated with σ.

(4) Prove that if $\mathcal{D}$ and $\mathcal{D}'$ are two distinct fundamental domains (or equivalently if s and s' are two distinct sections to the map p), then the corresponding induced G-actions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ are conjugate, in the sense that there exists a G-equivariant measurable isomorphism $\theta : \widetilde{X} \to \widetilde{X}$ which preserves the measure $\widetilde{\mu}$ and such that $\theta \circ \widetilde{\sigma}_g = \widetilde{\sigma}'_g \circ \theta$ for all $g \in G$.

(5) We aim to prove that σ is an ergodic Γ-action if and only $\widetilde{\sigma}$ is an ergodic G-action. For this we give another point of view on the induced action. On $G \times X$, we may consider the G-action by left multiplication on the first variable. We can also consider a Γ-action on this space:

$$\gamma \cdot (g, x) := (g\gamma^{-1}, \sigma_\gamma(x)) \text{ for all } \gamma \in \Gamma, g \in G, x \in X.$$

a) Check that this action $\Gamma \curvearrowright G \times X$ is well defined and properly discontinuous (assuming that X is a lcsc space and the initial action σ is continuous). So we may consider the quotient space, denoted by $G \times_\Gamma X$, which is a nice (Hausdorff) lcsc space. Check that the action $G \curvearrowright G \times X$ defined above factors to an action on the quotient $G \curvearrowright G \times_\Gamma X$.

b) Prove that this action $G \curvearrowright G \times_\Gamma X$ is conjugate with the induced action $\widetilde{\sigma}$.

c) Prove that a G-invariant set $A \subset G \times X$ is of the form $G \times Y$ (up to a co-null set), where $Y \subset X$ is any set.

d) Conclude that if σ is ergodic, so is $\widetilde{\sigma}$.

(6) Is it true that σ is mixing if and only if $\widetilde{\sigma}$ is mixing?

Part 2. Induced representations.

(5) Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, prove that the following formula defines a unitary representation $\widetilde{\pi} : G \to \mathcal{U}(\widetilde{\mathcal{H}})$ on the space $\widetilde{\mathcal{H}} := L^2(G/\Gamma, \mathcal{H})$ of $\mathcal{H}$-valued L2-functions on G/Γ (modulo almost sure equality):

$$\widetilde{\pi}(g)(f)(a) := \pi(c(g, g^{-1}a))(f(g^{-1}a)), \quad \text{for all } g \in G, f \in L^2(G/\Gamma, \mathcal{H}), a \in G/\Gamma.$$

Let us specify that $\widetilde{\mathcal{H}}$ is a Hilbert space with respect to the inner product

$$\langle f_1, f_2 \rangle := \int_{G/\Gamma} \langle f_1(a), f_2(a) \rangle_{\mathcal{H}} d\lambda_{G/\Gamma}(a).$$

(6) Prove that $\widetilde{\pi}$ does not depend, up to conjugacy (to be specified), on the choice of the section s.

(7) Prove that the induced representation of the Koopman representation of an action σ identifies with the Koopman representation of its induced action $\widetilde{\sigma}$.