HOMEWORK: HOMOGENEOUS SPACES

Orbits and homogeneous spaces. Consider a lcsc group G and a continuous group action $G \curvearrowright X$ on a l.c.s.c. space X. Fix a point $x \in X$ and denote by $H < G$ the stabilizer of x inside G: $H = \{g \in G \mid gx = x\}$. Note that H is a closed subgroup of G. Assume that the orbit Gx, endowed with the induced topology is again locally compact.

Show that the map $g \in G \mapsto gx \in X$ induces a homeomorphism from G/H onto the the orbit Gx of x in X. Here G/H is endowed with the quotient topology.
Hint. Use Baire theorem.

Application: prove that $\mathrm{SO}(n, \mathbb{R})$ is a connected group for all $n \geq 1$.
Hint. Use the fact that $\mathrm{SO}(n, \mathbb{R})$ acts on the $n-1$-sphere $\mathbb{S}^{n-1}$ by homeomorphisms.

Measures on homogeneous spaces: existence. Let G be a lcsc group and $H < G$ be a closed subgroup.

(1) Prove that there exists on G a Borel probability measure $\mu \in \mathrm{Prob}(G)$ which is equivalent to the Haar measure. Prove that the push forward of μ via the map $G \mapsto G/H$ is a G-quasi-invariant probability measure on G/H.
(2) Let $G = \mathrm{SL}_2(\mathbb{R})$ and let $H < G$ be the subgroup consisting of upper triangular matrices in G. We aim to prove that there is no G-invariant probability measure on G/H.
  a. Check that G/H identifies with the projective line $\mathbb{P}^1$, in such a way that the translation action $G \curvearrowright G/H$ coincides with the natural action $G \curvearrowright \mathbb{P}^1$ by homographies.
  b. Let $\mu \in \mathrm{Prob}(\mathbb{P}^1)$ be a G-invariant probability measure. Prove that μ has no atom, i.e. that $\mu(\{x\}) = 0$, for all $x \in \mathbb{P}^1$.
  c. Using the diagonal matrices $\mathrm{diag}(n, 1/n)$, $n \geq 1$, prove that any neighborhood $A \subset \mathbb{P}^1$ of $\mathbb{R}e_1$ has measure 1. Using the regularity of μ, prove that this implies that $\mu(\{\Re_1\}) = 1$, contradicting the previous question.

  With the same argument, one can show that more generally, $\mathbb{P}^1$ does not carry any G-invariant regular σ-finite measure.

So even though there always exists a quasi-invariant Radon measure on G/P, this is not the case of invariant measures.

Measures on homogeneous spaces: uniqueness. Let G be a lcsc group and $H < G$ be a closed subgroup. Denote by μ_G the Haar measure on G.

(1) Let $f : G/H \to \mathbb{R}_+$ be a Borel function. Prove that if there exists $x \in G/H$ such that $\int_G f(gx) d\mu_G(g) = 0$, then this is the case for every $x \in G/H$.
(2) Consider a non-zero Radon measure ν on G/H which is quasi-invariant under the natural action $G \curvearrowright G/H$. Let $f : G/H \to \mathbb{R}_+$ be a Borel function. Prove that $\nu(f) = 0$ if and only if the condition of the previous question holds true. Conclude that any other non-zero Radon measure ν' on G/H which is quasi-invariant is equivalent to ν.
(3) Prove that there exists at most one G-invariant Radon measure on G/H, up to possible scaling.

Applying this exercise when H is the trivial subgroup seems to give a shorter proof of the uniqueness of the Haar measure. But this uniqueness is implicitly used to solve the exercise. Do you agree?