HOMEWORK: LOCALLY COMPACT GROUPS AND LATTICES

1. Locally compact groups

The first exercise is not the easiest, but it gives many reassuring properties about lcsc groups.

Topological properties of lcsc groups. Let G be an lcsc group.

(1) Prove that G is para-compact, i.e. it is the union of countably many compact subsets. (*Hint.* First prove that G admits a dense countable set $(g_n)_{n\geq 1}$. Then prove that for any compact neighborhood $K \subset G$ of the identity, $G = \bigcup_n g_n K$.) Does the converse hold true?
(2) Prove that there exists an increasing sequence of compact symmetric subsets $(E_n)_{n\in\mathbb{Z}}$ of G such that $G = \bigcup_n E_n$, E_n forms a basis of neighborhoods of the identity in G and $E_n E_n E_n \subset E_{n+1}$ for all $n \in \mathbb{Z}$.
(3) Prove that the function $d : G \times G \to \mathbb{R}_+$ defined as follows is a distance on G:

$$d(x, y) = \inf\{t \in \mathbb{R}_+ \mid \exists n_1, \dots, n_k \in \mathbb{Z}, \exists g_1 \in E_{n_1}, \dots, g_k \in E_{n_k} \text{ such that } t = 2^{n_1} + \dots + 2^{n_k} \text{ and } y^{-1}x = g_1 \cdots g_k\}.$$

Check that d is left invariant, in the sense that $d(gx, gy) = d(x, y)$ for all $g, x, y \in G$.
(4) Prove that any ball $B(g, r) := \{h \in G \mid d(g, h) < r\}$ is a neighborhood of $g \in G$.
(5) Conversely, prove that for any neighborhood U of $g \in G$, there exists $r > 0$ small enough so that $B(g, r) \subset U$. (*Hint.* First reduce to the case where $g = e$ and $U = E_n$ for some n. Then prove that $B(e, 2^n) \subset E_n$. This last part is the most difficult). Conclude that the topology of G coincides with that defined by d, and that the closed balls $\overline{B}(e, R)$ are compact for all radius $R > 0$ (we say that d is proper).

In conclusion, any lcsc group G is metrizable, with a proper, G-invariant distance. Check that G is complete for this distance.

Open subgroups. Prove that an open subgroup in a lcsc group is also closed.

Connected component of the identity. Let G be an lcsc group and denote by G^0 the connected component of the identity element in G. Prove that G^0 is a closed normal subgroup of G. Is it always open in G? And if G is a Lie group?

Prove that if G is a connected group then it has no proper closed subgroup of finite index (By definition a subgroup $H < G$ is said to have *finite index* if G can be covered with finitely many translates of H).

Haar measure. Check that the formulae for the Haar measures on P and on $\mathrm{SL}_2(\mathbb{R})$ given in the notes are correct.

Modular function. Check that if λ is a left Haar measure then $A \mapsto \lambda(A^{-1})$ is a right Haar measure. Moreover, check the formula

$$\int_G f(x^{-1}) d\lambda(x) = \int_G \Delta(x) f(x)\, \mathrm{d}\lambda(x), \text{ for all } f \in C_c(G).$$

2. Lattices

Groups acting on trees. Let $\mathcal{T}$ be a tree, i.e. a graph with no cycle. $V(\mathcal{T})$ will denote its set of vertices. We denote by G the group of all automorphisms of the tree. Consider the family of sets $U_{g,F} \subset G$ indexed by elements $g \in G$ and finite subsets $F \subset V(\mathcal{T})$ and given by

$$U_{g,F} = \{h \in G \mid g(x) = h(x), \text{ for all } x \in F\}.$$

(1) Prove that the family of sets $U_{g,F}$ forms an open basis for a topology on G. This topology coincides with the restriction to G of the product topology on $V(\mathcal{T})^{V(\mathcal{T})}$ (where $V(\mathcal{T})$ is viewed as a discrete set).
(2) Assume that every vertex of $\mathcal{T}$ has the same degree $d < \infty$. Prove that the stabilizer G_v of any vertex $v \in V(\mathcal{T})$ is compact and open in G.
(3) Let Γ be a subgroup of G that acts transitively on $V(\mathcal{T})$ and such that $\Gamma \cap G_v$ is finite for some (and hence any) $v \in V(\mathcal{T})$. Prove that Γ is a co-compact lattice in G.

$\mathrm{SL}_d(\mathbb{Z})$ **is a lattice.** This exercise aims to provide a proof that $\Gamma := \mathrm{SL}_d(\mathbb{Z})$ is a lattice in $G := \mathrm{SL}_d(\mathbb{R})$. Consider the following subgroups of G:

- $K := \mathrm{SO}(d)$;
- $A < G$ the subgroup of diagonal matrices with positive diagonal entries (and determinant 1);
- $N < G$ the subgroup of upper triangular matrices, with 1's on the diagonal.

We denote by λ_K, λ_A and λ_N Haar measures on K, A and N, respectively.

Part I. Computation of the Haar measure on $\mathrm{SL}_d(\mathbb{R})$.

(1) Prove the *Iwasawa decomposition* $G = KAN$: every element $g \in G$ can be uniquely expressed as the product $g = kan$ of elements $k \in K$, $a \in A$ and $n \in N$.

(2) Note that N is parametrized by $d(d-1)/2$ real numbers. Namely, an element n is described by its entries $n_{1,2}, n_{1,3}, \ldots, n_{1,d}, \ldots, n_{d-1,d}$. Prove that λ_N is given (up to a possible rescaling) by the formula

$$\int_N f(n) d\lambda_N(n) = \int_{\mathbb{R}^{d(d-1)/2}} f(n) dn_{1,2} dn_{1,3} \ldots dn_{d-1,d}, \text{ for all } f \in C_c(N).$$

Here $dn_{i,j}$ denotes the Lebesgue measure with respect to the real variable $n_{i,j}$.
(3) Denote by $\rho : A \to \mathbb{R}_+^*$ the homomorphism defined by the formula $\rho(a) = \prod_{i<j}(a_i^{-1}a_j)$, for all $a = \mathrm{diag}(a_1, \ldots, a_d)$. Check that

$$\int_N f(a^{-1}na) d\lambda_N(n) = \rho(a) \int_N f(n) d\lambda_N(n), \text{ for all } f \in C_c(N).$$

(4) Prove that a Haar measure on G is given by the formula

$$\int_G f(g) d\lambda_G(g) = \int_K \int_A \int_N f(kan)\rho(a) d\lambda_K(k) d\lambda_A(a) d\lambda_N(n), \text{ for all } f \in C_c(G).$$

Hint. As an intermediate step, one may check that the push forward measure μ of a Haar measure λ_G under the quotient map $G \to K\backslash G$ is a right Haar measure on the group AN, after identifying $K\backslash G \simeq AN$.

Part II. Finding a nice domain $\mathcal{D} \subset G$ such that $G = \mathcal{D}\Gamma$.

(5) Consider a lattice $\Delta \subset \mathbb{R}^d$ admitting a basis $v_1, \ldots, v_d$. Denote by $w_1, \ldots, w_d$ the orthogonal family deduced from $v_1, \ldots, v_d$ by the Gramm-Schmidt orthogonalization process, that is, $w_1 = v_1$ and $w_{i+1} := P_{V_i^\perp}(v_{i+1})$, where $V_i = \mathrm{span}(v_1, \ldots, v_i)$ and $P_{V_i^\perp}$ is the orthogonal projection on the orthogonal space $V_i^\perp$. Check that the set $\mathcal{F} := \{\sum_{i=1}^d \lambda_i w_i \mid |\lambda_i| \leq 1/2, \text{ for all } 1 \leq i \leq d\}$ satisfies $\Delta + \mathcal{F} = \mathbb{R}^d$.

(6) Prove that Δ admits a basis $v_1, \dots, v_d$ with the following properties. Setting $w_1, \dots, w_d$ the orthogonal family deduced from $v_1, \dots, v_d$ by the Gramm-Schmidt orthogonalization process, then for all $i < d$,
$$\|w_{i+1}\|/\|w_i\| \geq \sqrt{3}/2, \qquad v_{i+1} = w_{i+1} + \sum_{j \leq i} \lambda_j w_j, \text{ with each } |\lambda_j| \leq 1/2,$$
$$\text{and } \|w_1\| \dots \|w_d\| = d_\Delta$$

(7) Deduce that every element of G can be written as a product $g = d\gamma$ for some element $\gamma \in \Gamma$ and $d \in \mathcal{D}$, where $\mathcal{D}$ is described in terms of the Iwasawa decomposition $\mathcal{D} = KA_{2/\sqrt{3}}N_{1/2}$, with
$$A_{2/\sqrt{3}} := \{\mathrm{diag}(a_1, \dots, a_d) \in A \mid a_i/a_{i+1} \leq 2/\sqrt{3}\},$$
$$N_{1/2} := \{n \in N \mid |n_{i,j}| \leq 1/2 \text{ for all } i < j\}.$$

Part III. Conclusion

(8) Prove that Γ is a lattice in G.