Factoring linear differential operators in positive characteristic

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Algebra of differential operators

Object of study

Differential operators in $\mathbb{F}_p(x)\langle\partial\rangle = \{a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x)\}$.

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Running example:

$L = (2x^6 + 2)\partial^6 + (x^6 + 2)\partial^3 + 2x^6 + 2x^3 + 2 \in \mathbb{F}_3(x)\langle\partial\rangle$

State of the art

- M. van der Put. Modular methods for factoring differential operators. Unpublished manuscript, 1997.
- M. Giesbrecht, Y. Zhang, Factoring and decomposing Ore polynomials over $\mathbb{F}_p(t)$, ISSAC 2003.
- T. Cluzeau, factorisation of differential systems in characteristic p, ISSAC 2003.
- X. Caruso, J. Le Borgne. A new faster algorithm for factoring skew polynomials over finite fields. JSC 2017.
- J. Gomez-Torrecillas, F. J. Lobillo, G. Navarro, Computing the bound of an Ore polynomial. Applications to factorisation, JSC 2019.

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Notation

$C := \mathbb{F}_p(x^p)$ is the field of constants of $\mathbb{F}_p(x)$.
$C[\partial^p]$ is the center of $\mathbb{F}_p(x)\langle\partial\rangle$.

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Tool 1: p-curvature and first reduction (van der Put, Cluzeau)

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Contribution

A full factorisation algorithm that extends to operators with coefficients in finite separable extensions of $\mathbb{F}_p(x)$.

A guideline: studying the submodules of $\mathbb{F}_p(x)\langle\partial\rangle / \mathbb{F}_p(x)\langle\partial\rangle L$

Notation

- $\mathcal{D}_L := \mathbb{F}_p(x)\langle\partial\rangle / \mathbb{F}_p(x)\langle\partial\rangle L$.
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Direct consequence of $\mathbb{F}_p(x)\langle\partial\rangle$ being a left principal ideal domain.

p-curvature

Definition

The p-curvature of L is

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Proposition

Its characteristic polynomial $\chi(\psi_p^L)$ belongs to $\boldsymbol{C}[\boldsymbol{Y}]$.

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Operator version of classical classification results of differential modules in positive characteristic (van der Put).

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- $\mathcal{D}_{N(\partial^p)}$ is a C_N-algebra ($Y \mapsto \partial^p$)
- $\mathcal{D}_L$ is a left $\mathcal{D}_{N(\partial^p)}$-module

Structure results

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Corollary

$\mathcal{D}_{N(\partial^p)}$ is either a division algebra or isomorphic to $M_p(C_N)$.

Morita's equivalence

If $\mathcal{D}_{N(\partial^p)}$ is a division algebra then $N(\partial^p)$ has no nontrivial divisor and $L = N(\partial^p)$ is irreducible.

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- If L' is a divisor of $N(\partial^p)$ then $\dim \mathcal{M}\mathrm{or}_N(\mathcal{D}_{L'}) = \mathrm{ord}(L')/\deg(N)$.

Running example

$$L = N(\partial^p) = (2x^6 + 2)\partial^6 + (x^6 + 2)\partial^3 + 2x^6 + 2x^3 + 2$$

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Theorem (P., 2022)

Let $(H_i)_{1 \leqslant i \leqslant p}$ be irreducible divisors of $N(\partial^p)$ such that $N(\partial^p) = \mathrm{lclm}_{i=1}^p H_i$.

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$$\mathrm{lclm}(H_i, R) \cdot R^{-1}$$

is an irreducible divisor of L.

Algorithm

- Compute $\chi(\psi_p^L)$ and factor it $\chi(\psi_p^L) = \prod_{i=1}^{n} N_i^{\nu_i}(Y)$.
- Compute $L = L'_1 \cdots L'_m$ such that each L'_j divides a $N_i(\partial^p)$.

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- Compute $\mathrm{lclm}(R'_j, H_{i,k}) \cdot (R'_j)^{-1}$

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Recall: $\mathcal{D}_{N(\partial^p)} = {}^{\mathbb{F}_p(x)\langle\partial\rangle}/_{N(\partial^p)}$

$$\begin{array}{ccc} \mathbb{F}_p(x)\langle\partial\rangle & \hookrightarrow & K_N\langle\partial\rangle \\ \downarrow & & \downarrow\!\!\downarrow \\ {}^{\mathbb{F}_p(x)\langle\partial\rangle}/_{N(\partial^p)} & \overset{\varphi_N}{\dashrightarrow} & {}^{K_N\langle\partial\rangle}/_{\partial^p - y_N} \end{array}$$

"p-Riccati" equation

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Lemma (P., 2022)

Let $f \in K_N$ be such that $f^{(p-1)} + f^p = y_N$. Then

$$\partial^p - y_N = \mathrm{lclm}_{i=1}^{p}\left(\partial - f - \frac{i}{x}\right)$$

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- For each $N_i(\partial^p)$ compute $(H_{i,k})_{k \in [\![1;p]\!]}$ such that $\mathrm{lclm}_{k=1}^{p} H_{i,k} = N_i(\partial^p)$.

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- For each $N_i(\partial^p)$ compute $(H_{i,k})_{k\in[\![1;p]\!]}$ such that $\mathrm{lclm}_{k=1}^p H_{i,k} = N_i(\partial^p)$.
  - Compute $f_i \in K_N$ such that $f_i^{(p-1)} + f_i^p = y_{N_i}$.
  - Set $H_{i,k} = \mathrm{gcrd}(\varphi_{N_i}^{-1}(\partial - f_i - \frac{k}{x}), N_i(\partial^p))$.
- Compute $\mathrm{lclm}(R'_j, H_{i,k}) \cdot (R'_j)^{-1}$

The case when $L \neq N(\partial^p)$ (after van der Put)

$$L_N := \mathrm{gcrd}(\varphi_N(L), \partial^p - y_N)$$

is a non trivial divisor of $\partial^p - y_N$.

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The space of solutions of p-Riccati is an affine space

Algorithm

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Factoring $N(\partial^p)$: when $K_N = \mathbb{F}_p(x)$

Suppose that $K_N = \mathbb{F}_p(x)$, $y_N = g^p \in \mathbb{F}_p(x^p)$
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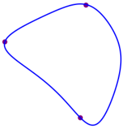

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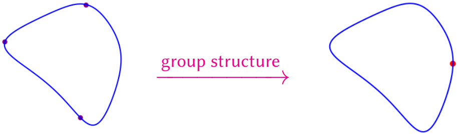

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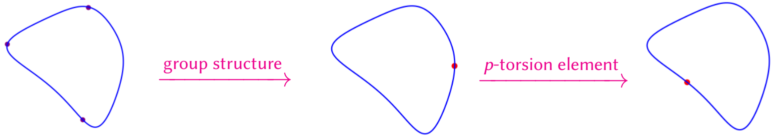

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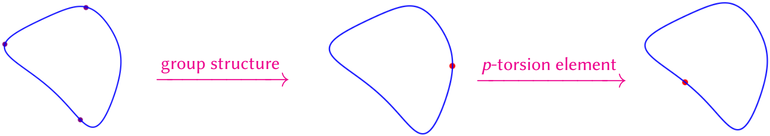

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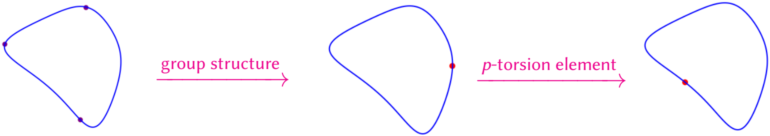

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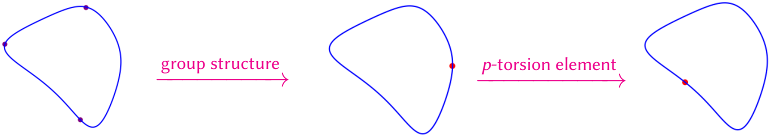

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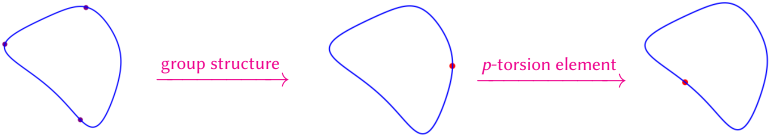

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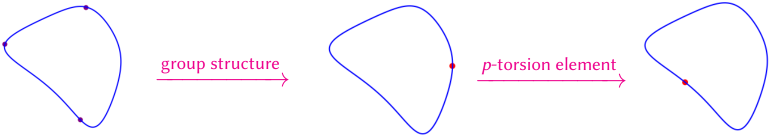

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Thank you for your attention