Computing Characteristic Polynomials of p-Curvatures in Average Polynomial Time

Functional Equation in Limoges 2022

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Differential equations in characteristic p

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Idea: Such an equation has an algebraic basis of solutions iff the ”p-curvature” of this equation is zero.

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Theorem (Cartier)

For any such linear differential equation we have an equality between

- *the dimension of the space of solutions that are algebraic over* $\mathbb{F}_p(z)$
- *the dimension of the kernel of the p-curvature of this differential equation.*

p-curvature in characteristic 0

Conjecture (Grothendieck-Katz)

A linear differential equation in characteristic 0 admits an algebraic basis of solutions over $\mathbb{Q}(z)$ iff its reduction modulo p has an algebraic basis of solutions over $\mathbb{F}_p(z)$ for all primes p except a finite number.

p-curvature in characteristic 0

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Theorem (Chudnovsky^2)

If $f \in \mathbb{Z}[[z]]$ (with non zero convergence radius) is a solution of a linear differential equation, then the minimal differential equation for f only has nilpotent p-curvatures, except for a finite number of primes..

Useful for guessing procedures.

Other applications

- Algorithms for factoring differential operators using p-curvatures [Cluzeau, ISSAC 2003]

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- Algorithms for computing the Lie algebra of differential operators [Barkatou, Cluzeau, Di Vizio, Weil, ISSAC 2016]

Algebra of Differential operators

Definition

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$$\partial f = f\partial + f'$$

The p-curvature

Idea : The p-curvature of an operator is ∂^p modulo this operator

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The p-curvature of an operator $L \in \mathbb{F}_p(z)\langle\partial\rangle$ is the $\mathbb{F}_p(z)$-linear endomorphism of $\mathbb{F}_p(z)\langle\partial\rangle / \mathbb{F}_p(z)\langle\partial\rangle L$ induced by the left multiplication by ∂^p.

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$$A = \begin{pmatrix} 0 & 0 & -\frac{z^3+3}{(z+1)^2} \\ 1 & 0 & \frac{z}{(z+1)^2} \\ 0 & 1 & 0 \end{pmatrix}$$

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Size: A_p is of bit size $\tilde{O}(p)$.
Cost: $\tilde{O}(p^2)$ binary operations.

p-curvature of an operator

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Fact: $\chi(A_p(L)) \in \mathbb{F}_p(z^p)[x]$.

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Fact: $\chi(A_p(L)) \in \mathbb{F}_p(z^p)[x]$.
Size: $\chi(A_p(L))$ is of bit size $O(\log(p))$.

Previous works around the computation of p-curvatures

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My Contribution: Theoretical part

Theorem (P., 2021)

It is possible to compute, for a given linear differential equation with coefficients in $\mathbb{Z}[z]$, the characteristic polynomials of its p-curvatures for all primes $p < N$ in $\tilde{O}(N)$ binary operations.

My Contribution: Theoretical part

Theorem (P., 2021)

It is possible to compute, for a given linear differential equation with coefficients in $\mathbb{Z}[z]$ of order r, with polynomial coefficients of degree at most d and integer coefficients of bit size at most B, all the characteristic polynomials of its p-curvatures for all primes $p < N$ in

$$\tilde{O}(Nd((B+d)(r+d)^{\omega} + (r+d)^{\Omega}))$$

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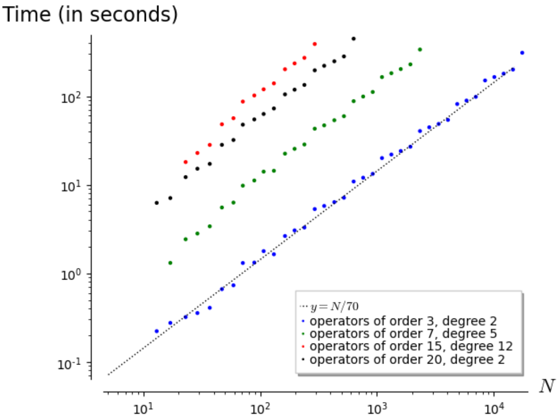
$\omega < 2,373$ is an exponent of matrix multiplication in any ring.
$\Omega < 2,698$ is an exponent for the computation of the characteristic polynomial in any ring.

My Contribution: Practical part

Implementation of the algorithm in Sagemath



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Step 2: Use the factorial computation method of [Costa, Gerbicz, Harvey, Math. Comp. 2014]

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Another p-curvature

Definition

Let $L_\theta \in \mathbb{F}_p(\theta)\langle\partial\rangle$. Its p-curvature $B_p(L_\theta)(\theta)$ is the $\mathbb{F}_p(\theta)$-linear endomorphism of ${}^{\mathbb{F}_p(\theta)\langle\partial\rangle}/_{\mathbb{F}_p(\theta)\langle\partial\rangle L_\theta}$ induced by the left multiplication by ∂^p.

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If $B = B(L_\theta)(\theta)$ is its companion matrix then:

$$B_p(L_\theta) = B(\theta)B(\theta+1)\cdots B(\theta+p-1)$$

Two crucial maps: $\Xi_{z,\partial}$ and $\Xi_{\theta,\partial}$

Let $L_z \in \mathbb{F}_p(z)\langle\partial\rangle$ (resp. $L_\theta \in \mathbb{F}_p(\theta)\langle\partial\rangle$) with leading coefficient l_z (resp. l_θ).

$$\Xi_{z,\partial}(L_z) := l_z(z)^p \chi(A_p(L_z))(\partial^p)$$

$$\Xi_{\theta,\partial}(L_\theta) := \left(\prod_{i=0}^{p-1} l_\theta(\theta + i)\right) \chi(B_p(L_\theta))(\partial^p)$$

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- Multiplicativity: $\Xi_{\cdot,\partial} : \mathbb{F}_p(\cdot)\langle\partial^{\pm 1}\rangle \to \mathbb{F}_p(\cdot)\langle\partial^{\pm p}\rangle$.

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Theorem (Bostan, Caruso, Schost, ISSAC 2014)

The applications $\Xi_{\cdot,\partial}$ commute with the isomorphism Φ_p:

$$\begin{array}{ccc} k[x]\langle\partial^{\pm 1}\rangle & \xrightarrow[\sim]{\Phi_p} & k[\theta]\langle\partial^{\pm 1}\rangle \\ \downarrow{\scriptstyle \Xi_{x,\partial}} & & \downarrow{\scriptstyle \Xi_{\theta,\partial}} \\ k[x^p][\partial^{\pm p}] & \xrightarrow[\sim]{\Phi_p} & k[\theta^p - \theta][\partial^{\pm p}] \end{array}$$

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$$(z+1)^2\partial^3 - z\partial + z^3 + 3 \quad \mapsto \quad \partial^3 + 2\theta\partial^2 + (\theta^2-\theta)\partial - (\theta+3) + (\theta^3 - 3\theta^2 + 2\theta)\partial^{-3}$$

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$$(z^{10}+2z^5+1)\partial^{15}+ (4z^5+2)\partial^5+z^{15}+2z^5$$

$$\partial^{15}+2(\theta^5-\theta)\partial^{10} +((\theta^5-\theta)^2+2)\partial^5+4(\theta^5-\theta) +2(\theta^5-\theta)\partial^{-5}+(\theta^5-\theta)^3\partial^{-15}$$

Extension to integral coefficients

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$B(\theta)$ is the companion matrix of $\Phi(L)$.

$$B(\theta)B(\theta+1)\cdots B(\theta+p-1) \quad \text{mod } p \text{ for all } p < N.$$

A first simplification

If $L \in \mathbb{F}_p[z]\langle\partial\rangle$ has coefficients of degree at most d, then $\Xi_{\theta,\partial}(\Phi_p(L))$ has coefficients of degree at most d in $\theta^p - \theta$.

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Lemma

It is possible to determine entirely $P \in \mathbb{F}_p[\theta^p - \theta]$ of degree dp in θ from its first d coefficients.

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Conclusion: All computations can be done modulo θ^{d+1}.

Computation of $(p-1)! \mod p^s$ [Costa, Gerbicz, Harvey, 2014]

$N = 7$.

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$$((3-1)! \mod 3^s5^s7^s) \times (3 \times 4)$$

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Computation of $B(\theta) \cdots B(\theta + p - 1) \mod p$

$N = 7$.

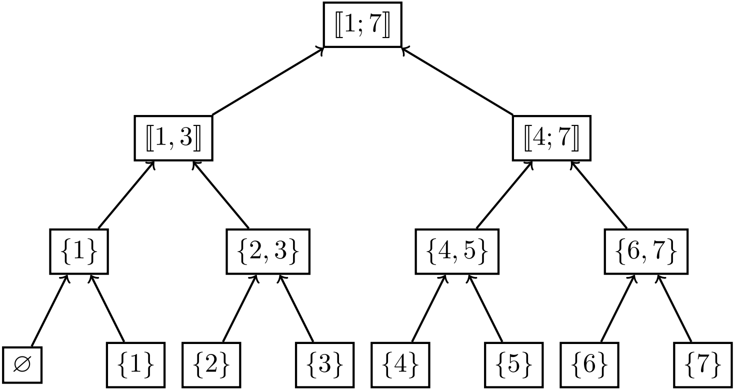
$$B(\theta)B(\theta+1)B(\theta+2) \quad B(\theta)\cdots B(\theta+4) \quad B(\theta)\cdots B(\theta+6)$$
$$\mod 3\times 5\times 7 \qquad\qquad \mod 5\times 7 \qquad\qquad \mod 7$$

$$(B(\theta)B(\theta+1)B(\theta+2) \mod 3\times 5\times 7) \times B(\theta+3)B(\theta+4) \mod 5\times 7$$

$$(B(\theta)\cdots B(\theta+4) \mod 5\times 7) \times B(\theta+5)B(\theta+6) \mod 7$$

Remainder tree.

Computation of $(p-1)! \mod p^s$ [Costa, Gerbicz, Harvey, 2014]



Future works

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Thank you for your attention