Factorisation of linear differential operators in positive characteristic

–PhD Defense–

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Linear differential operator algebra

Definition

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Exemple :

$(2x^6 + 2)\partial^6 + (x^6 + 2)\partial^3 + 2x^6 + 2x^3 + 2 \in \mathbb{F}_3(x)\langle\partial\rangle$

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Objective: How to factor L as a product of irreducible linear differential operators?

Contribution 1

Theorem (P., 2021)

It is possible to compute, for a given linear differential equation with coefficients in $\mathbb{Z}[x]$, the characteristic polynomials of its p-curvatures for all primes $p < N$ using $\tilde{O}(N)$ binary operations.

Contribution 1

Theorem (P., 2021)

It is possible to compute, for a given linear differential equation with coefficients in $\mathbb{Z}[x]$ of order m, with polynomial coefficients of degree at most d and integer coefficients of bit size at most n, all the characteristic polynomials of its p-curvatures for all primes $p < N$ using

$$\tilde{O}(Nd((n+d)(m+d)^{\omega} + (m+d)^{\Omega}))$$

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- $\omega < 2,373$ is an exponent of matrix multiplication in any ring.
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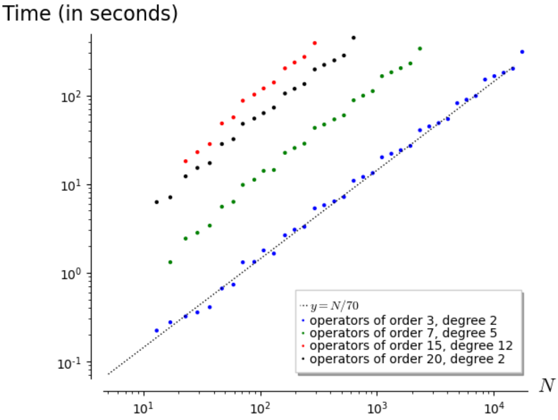
- Azumaya algebra structure
- Fast factorial computation techniques (Harvey 2014)

Implementation

Implementation of the algorithm in SageMath



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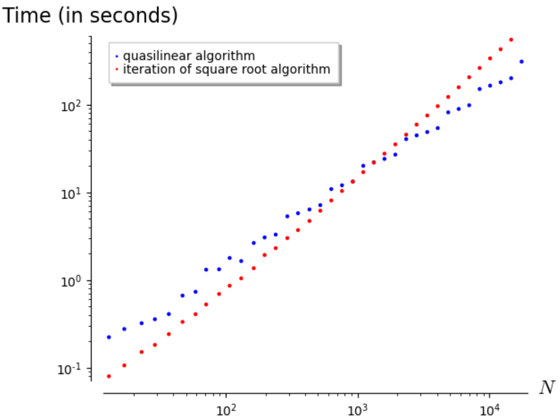

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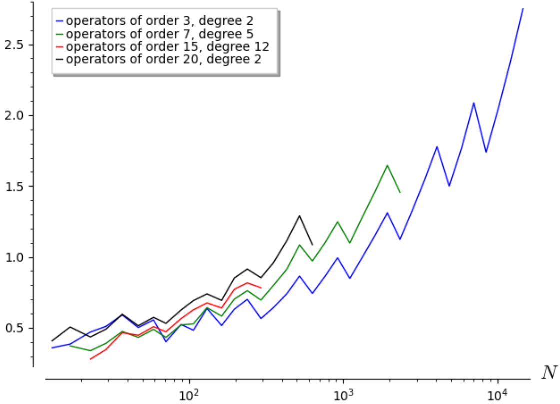

State of the art on factorisation

In characteristic 0:

- D. Yu. Grigoriev, Complexity of factoring and calculating the GCD of linear ordinary differential operators, JSC. 10 (1990).
- M. van Hoeij, Factorization of differential operators with rational functions coefficients, JSC. 24 (1997).
- M. van Hoeij. Rational solutions of the mixed differential equation and its application to factorization of differential operators. ISSAC 1996.
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- F. Chyzak, A. Goyer, and M. Mezzarobba, Symbolic-numeric factorization of differential operators. ISSAC 2022.

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In characteristic p:

- M. van der Put, Differential equations in characteristic p. 1995.
- M. van der Put. Modular methods for factoring differential operators. Unpublished manuscript, 1997.
- M. Giesbrecht, Y. Zhang, Factoring and decomposing Ore polynomials over $\mathbb{F}_p(t)$, ISSAC 2003.
- T. Cluzeau, Factorisation of differential systems in characteristic p, ISSAC 2003.
- X. Caruso, J. Le Borgne. A new faster algorithm for factoring skew polynomials over finite fields. JSC 2017.
- J. Gomez-Torrecillas, F. J. Lobillo, G. Navarro, Computing the bound of an Ore polynomial. Applications to factorisation, JSC 2019.

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$C := \mathbb{F}_p(x^p)$ is the field of constants of $\mathbb{F}_p(x)$.
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- Is $N(\partial^p)$ irreducible?

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Let $N \in C[Y]$ be an irreducible polynomial. **Two questions:**

- Is $N(\partial^p)$ irreducible?
- If it isn't, how to determine an irreducible divisor of it?

Contribution 2

Theorem (P. 2023)

Let $N_* \in \mathbb{F}_p[X, Y]$ be an irreducible polynomial of bidegree (d_x, d_y). There exists an algorithm testing the irreducibility of $N_*(x^p, \partial^p)$ in polynomial time in d_x, d_y and $\log(p)$.

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Remark

The algorithm works for operators with coefficients in algebraic function fields.

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We seek to factor $N(\partial^p)$, with $N(Y)$ irreducible separable over C.

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Theorem (Artin-Wedderburn)

Any central simple C_N-algebra is isomorphic to a matrix ring over a division algebra.

$\mathcal{D}_{N(\partial^p)}$ is either a division algebra or it is isomorphic to $M_p(C_N)$.

Structural results

If $\mathcal{D}_{N(\partial^p)}$ is a division algebra then $N(\partial^p)$ has no nontrivial divisor.

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y_N is the image of Y in $C_N = {}^{C[Y]}/_{N(Y)}$

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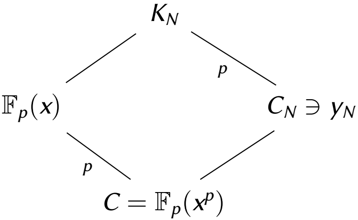

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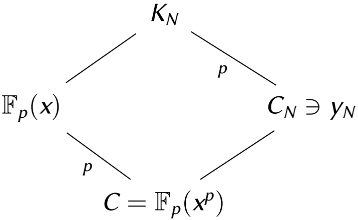
$$\varphi_N : {}^{\mathbb{F}_p(x)\langle\partial\rangle}/_{N(\partial^p)} \xrightarrow{\sim} {}^{K_N\langle\partial\rangle}/_{\partial^p - y_N}$$



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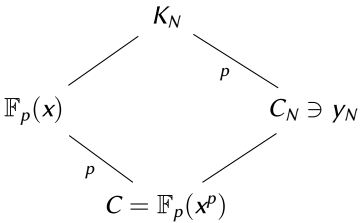

Lemma (Jacobson, van der Put)

If $\mathcal{D}_{N(\partial^p)} \simeq M_p(C_N)$ then $L \in K_N\langle\partial\rangle$ is an irreducible divisor of $\partial^p - y_N$ iff $L = \partial - f$ with $f^{(p-1)} + f^p = y_N$.

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Theorem (P. 2023)

$N(\partial^p)$ is reducible iff the equation $f^{(p-1)} + f^p = y_N$ has a local solution in every place of K_N.

$$\left(g\frac{\mathrm{d}}{\mathrm{d}t}\right)^{p-1}(f) + f^p = y_N \qquad \text{in } \mathbb{F}_q((t)),$$

where g is the derivative of a prime element of the place considered.

Hensel lemma

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Theorem (P. 2023)

If $f \in \mathbb{F}_q((t))$ verifies $\left(g\frac{\mathrm{d}}{\mathrm{d}t}\right)^{p-1}(f) + f^p = y_N + O(t^{pn})$ then there exists $f_* \in \mathbb{F}_q((t))$ such that

$$\left(g\frac{\mathrm{d}}{\mathrm{d}t}\right)^{p-1}(f_*) + f_*^p = y_N + O(t^{p(pn+(p-1)(1-\nu(g)))})$$

Irreducibility test

Corollary

- $\left(g\frac{\mathrm{d}}{\mathrm{d}t}\right)^{p-1}(f) + f^p = y_N$ has a solution in $\mathbb{F}_q((t))$ iff there exists $f_* \in \mathbb{F}_q((t))$ such that $\left(g\frac{\mathrm{d}}{\mathrm{d}t}\right)^{p-1}(f) + f^p = y_N + O(t^{p\nu(g)})$.

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Theorem (P. 2023)

Let $N_* \in \mathbb{F}_p[X, Y]$ be an irreducible polynomial of bidegree (d_x, d_y). We can test the irreducibility of $N_*(x^p, \partial^p)$ in $\mathbb{F}_p(x)\langle\partial\rangle$ at the cost of:

- *a factorisation of x and a root of N_* in K_N,*
- *$O(d_x + d_y)$ evals of functions in K_N of size $O(d_y \times (d_x^2 d_y + d_x d_y^2))$,*
- *$O_\varepsilon((d_x + \deg a \log(p)^2 + (d_x^3 d_y^2 + d_x^2 d_y^3)\log(p))$ bit operations.*

Factoring $N(\partial^p)$: when $K_N = \mathbb{F}_p(x)$

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Remove poles by adding multiple of $\frac{h'}{h}$.

p-Riccati equation in the general case

Same ideas:

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Problem: By adding $\frac{h'}{h}$ we may add more poles.

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How close is the divisor of poles of f from being a $\frac{h'}{h}$?

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Definition

- $D \sim D'$ iff $D - D'$ is principal i.e. equal to $(h) := \sum_{\mathfrak{P} \in \mathbb{P}_{K_N}} \nu_{\mathfrak{P}}(h) \cdot \mathfrak{P}$ for some $h \in K_N$.
- The group of equivalence classes of divisors is the divisor class group of $\mathrm{Cl}(K_N)$

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$\mathrm{Cl}(K_N)$ is a finitely generated commutative group.

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Theorem (P. 2023)

Let Σ_N be the set of solutions of the p-Riccati equation. If S is a generating family of ${}^{\mathrm{Cl}(K_N)}/_{p\mathrm{Cl}(K_N)}$ then

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where

$$A(S) = \max(\mathrm{Diff}(K_N) - 2(x)_\infty + \sum_{\mathfrak{P} \in S} \mathfrak{P}, (a)_\infty).$$

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- ~~Compute S a generating family of ${}^{\mathrm{Cl}(K_N)}/_{p\mathrm{Cl}(K_N)}$.~~
  Pick sufficiently many random places.
- Compute a $\mathbb{F}_p$-basis of $\mathcal{L}(A(S))$.
- Solve a linear system.

Perspectives

- lclm decomposition. Writing L as the lclm of $L_1, \ldots, L_n$ with $\sum_{i=1}^{n} \operatorname{ord}(L_i) = \operatorname{ord}(L)$.

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- Loewy decomposition: $L = L_1 \ldots L_n$ with L_n being the lclm of all the irreducible right factors of L. Give a lclm decomposition of each L_i.
- Study the fine-grained algorithmic aspects of the algorithm (OM-factorization, Riemann-Roch spaces)

Thank you for your attention

p-Riccati equation in the general case

Denote $K_N = \mathbb{F}_p(x)[a]$ with $a^p = y_N$.
Let $f \in K_N$ verify $f^{(p-1)} + f^p = a^p$. Let $\mathfrak{P}$ be a pole of f, $\nu_{\mathfrak{P}}$ the associated valuation and $e(\mathfrak{P})$ its ramification index.

Lemma

$$\nu_{\mathfrak{P}}(f) \geqslant \min(-e(\mathfrak{P}), \nu_{\mathfrak{P}}(a))$$

Lemma

If $\nu_{\mathfrak{P}}(a) > -e(\mathfrak{P})$, then there exists a unique $k \in \mathbb{F}_p$ such that for all $g \in K_N$ verifying $\nu_{\mathfrak{P}}(g) \equiv k \mod p$,

$$\nu_{\mathfrak{P}}(f - \frac{g'}{g}) \geqslant 1 - e(\mathfrak{P}).$$

$$\mathfrak{Re}_{\mathfrak{P}}(f) := k$$

p-Riccati equation in the general case

Theorem

$$\mathfrak{Re}(f) = \sum_{\mathfrak{P}} \mathfrak{Re}_{\mathfrak{P}}(f) \cdot \mathfrak{P}$$

If $\mathfrak{Re}(f) \sim D'$ *then there exists another solution* f_* *verifying*

$$\nu_{\mathfrak{P}}(f_*) \geqslant \min(\nu_{\mathfrak{P}}(a), 1 - e(\mathfrak{P}))$$

for all $\mathfrak{P} \notin \mathrm{Supp}(D')$.

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Corollary

If S is a family of places of K_N generating $\mathrm{Cl}(K_N)$ then there exists another solution f_* of the p-Riccati equation verifying $\nu_{\mathfrak{P}}(f_*) \geqslant \min(\nu_{\mathfrak{P}}(a), 1 - e(\mathfrak{P}))$ for all $\mathfrak{P} \notin S$.

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If $\mathfrak{Re}(f) \sim D' + pD_p$ then there exists another solution f_* verifying

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