Bin Packing Problem with Generalized Time Lags: A Branch-Cut-and-Price Approach

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Bin Packing With Time Lags Problem

Classic Bin Packing Problem

- Set of items to pack into bins.
- Items have positive weight, and bins have capacity.
- Objective: Minimize number of bins used.

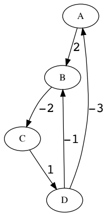
Bin Packing Problem with Time Lags

- Bins are assigned to time periods.
- Number of bins in each period is unbounded
- Pairs of items have precedence constraints with lags.

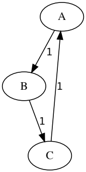
Precedence Constraints

- Precedences are represented by a directed graph $G = (I, A)$.
- Each arc $(i, j) \in A$ has a lag $l_{ij} \in \mathbb{Z}$.
- Bins are assigned to time periods, and items are assigned to the time period of the bin it belongs to.
- Each lag l_{ij} imposes the following constraint: The time period that item j is assigned must be at least l_{ij} time periods after the time period item i is assigned.

The graph is not necessarily acyclic.



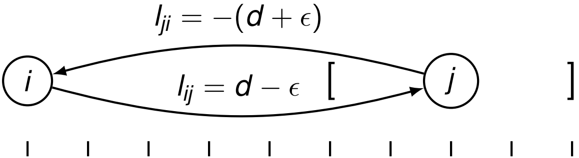
An instance is infeasible if and only if there is a cycle of positive length in the graph.



Motivation

Applications

- Performing a set of periodic tasks using rented capacitated resources



- Flexible periodic vehicle routing (generalisation)

Special cases

- Simple Assembly Line Balancing Problem of type 1 ($l_{ij} = 0$) [Becker and Scholl, 2006]
- Bin Packing with Precedences ($l_{ij} = 1$) [Pereira, 2016]
- Bin Packing with Generalized Precedences ($l_{ij} \geq 0$) [Kramer et al., 2017]

An IP formulation: variables and objective

Notation

- The bin capacity $W \in \mathbb{Z}^+$.
- A weight $w_i \in \mathbb{Z}^+$, $w_i \leq W$, for each $i \in V$.
- $\mathcal{B} = \{1, 2, \ldots, B\}$ the set of potential bins in a period.
- $\mathcal{T} = \{1, 2, \ldots, T\}$ the set of time periods.

Variables

- $x_{ibt} \in \{0, 1\}$ for each $i \in V, j \in \mathcal{B}, t \in \mathcal{T}$. Takes value 1 iff item i is assigned to bin b of time period t.
- $u_{bt} \in \{0, 1\}$ for each $j \in \mathcal{B}, t \in \mathcal{T}$. Takes value 1 iff bin b of time period t is in use.

Objective

$$\min \sum_{b \in \mathcal{B}} \sum_{t \in \mathcal{T}} u_{bt}$$

An IP formulation: constraints

Basic Structure

$$\sum_{b \in \mathcal{B}} \sum_{t \in \mathcal{T}} x_{ibt} = 1 \qquad \forall i \in I,$$

$$x_{ibt}, u_{bt} \in \{0, 1\} \qquad \forall i \in I, b \in \mathcal{B}, t \in \mathcal{T}.$$

Bin use and capacity

$$\sum_{i \in I} w_i x_{ibt} \leq W \, u_{bt} \qquad \forall b \in \mathcal{B}, t \in \mathcal{T}.$$

Precedence Constraints

$$l_{ij} + \sum_{t \in \mathcal{T}} t \cdot \sum_{b \in \mathcal{B}} x_{ibt} \leq \sum_{t \in \mathcal{T}} t \cdot \sum_{b \in \mathcal{B}} x_{jbt} \qquad \forall (i, j) \in A.$$

Symmetry-breaking constraints

$$u_{b-1,t} \geq u_{bt} \qquad \forall t \in \mathcal{T}, \forall b \in \mathcal{B} \setminus \{1\}.$$

Suitable partitions

Suitable partition

Partition $\mathcal{P}$ of I is suitable if graph $G'_{\mathcal{P}} = (I, A \cup A'_{\mathcal{P}})$ has no cycle of positive length, where $A'_{\mathcal{P}}$ contains arcs (i, j) with $l_{ij} = 0$ for all $i, j \in P$, $P \in \mathcal{P}$.

Proposition

Partition $\mathcal{P}$ induces a feasible solution if and only if

- $\mathcal{P}$ contains all items in I
- $\mathcal{P}$ is a suitable partition.
- $\sum_{i \in P} w_i \leq W$ for each $P \in \mathcal{P}$.

Distance

d_{ij} — the total lag of the longest directed path from i to j in G.
If no path between i and j in G, $d_{ij} = -\infty$.

Sufficient condition

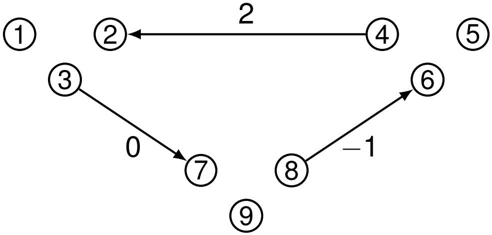
Any partition $\mathcal{P}$ containing set $B \supseteq \{i, j\}$, $d_{ij} > 0$, is non-suitable

Set partitioning formulation

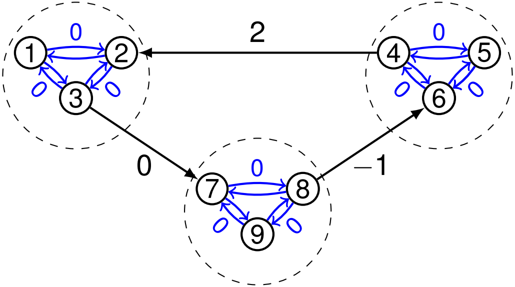
- $\mathcal{B}$ — set of all items set which can be put to the same bin
- Variable λ_B, $B \in \mathcal{B}$, — whether set B is put to the same bin
- $\mathbb{1}_B(i) = 1 \Leftrightarrow i \in B$
- $\mathcal{N} \subset \mathcal{B}$ — set of non-suitable partitions

$$
\begin{aligned}
\min \ & \sum_{B \in \mathcal{B}} \lambda_B \\
\text{s.t.} \ & \sum_{B \in \mathcal{B}} \mathbb{1}_B(i) \lambda_B = 1, && \forall i \in I, \\
& \sum_{B \in \mathcal{P}} \lambda_B \leq |\mathcal{P}| - 1, && \forall \mathcal{P} \in \mathcal{N}, \\
& \lambda_B \in \{0, 1\}, && \forall B \in \mathcal{B}.
\end{aligned}
$$

Characterising non-suitable partitions

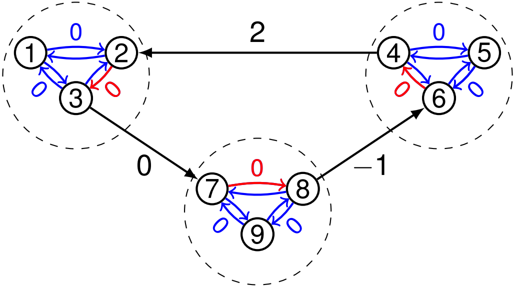


Characterising non-suitable partitions



- Partition $\mathcal{P}$ is non-suitable $\Rightarrow$ there is a cycle of positive length in graph $G'_{\mathcal{P}} = (I, A \cup A'_{\mathcal{P}})$.

Characterising non-suitable partitions



- Partition $\mathcal{P}$ is non-suitable $\Rightarrow$ there is a cycle of positive length in graph $G'_{\mathcal{P}} = (I, A \cup A'_{\mathcal{P}})$.
- Let $C_{\mathcal{P}} \subseteq A \cup A'_{\mathcal{P}}$ be such a cycle, and $F_{\mathcal{P}} = (C_{\mathcal{P}} \setminus A) \subseteq A'_{\mathcal{P}}$ be the set of arcs in the cycle induced by the partition
- Then constraint $\sum_{B \in \mathcal{P}} \lambda_B \leq |\mathcal{P}| - 1$ can be replaced by

$$\sum_{(i,j) \in F_{\mathcal{P}}} \sum_{\substack{B \in \mathcal{B}: \\ \{i,j\} \in B}} \lambda_B \leq |F_{\mathcal{P}}| - 1$$

Pricing problem

- π_i, $i \in I$, — dual values from the set partitioning constraints
- $\mu_{\mathcal{P}}$, $\mathcal{P} \in \bar{\mathcal{N}}$, — dual values from the active “suitability” constraints

Binary knapsack problem with hard and soft conflicts

$$
\begin{aligned}
\max \ & \sum_{i \in I} \pi_i z_i + \sum_{\mathcal{P} \in \bar{\mathcal{N}}} \sum_{(i,j) \in F_{\mathcal{P}}} \mu_{\mathcal{P}} y_{ij} \\
\text{s.t.} \ & \sum_{i \in I} w_i z_i \leq W, \\
& z_i + z_j \leq 1, && \forall i, j \in I, d_{ij} > 0, \\
& z_i + z_j \leq 1 + y_{ij}, && \forall \mathcal{P} \in \bar{\mathcal{N}}, \forall (i,j) \in F_{\mathcal{P}}, \\
& z_i \in \{0, 1\}, && \forall i, j \in I. \\
& y_{ij} \geq 0, && \forall \mathcal{P} \in \bar{\mathcal{N}}, \forall (i,j) \in F_{\mathcal{P}}.
\end{aligned}
$$

Solution is using a MIP solver.

Separation of “non-suitability” constraints

Integer solution $\mathcal{P}$

We search for a positive cycle in $G'_{\mathcal{P}}$ in $O(|I|^2)$ time.

Fractional solution $(\bar{\mathcal{P}}, \bar{\lambda})$

1. We create valued directed graph $\bar{G}'_{\bar{\mathcal{P}}} = (I, A \cup A'_{\bar{\mathcal{P}}})$:

$$v_{ij} = \begin{cases} 1 - \sum_{B \in \bar{\mathcal{P}}: \{i,j\} \in B} \bar{\lambda}_B, & (i,j) \in A'_{\bar{\mathcal{P}}}, \\ 0, & (i,j) \in A. \end{cases}$$

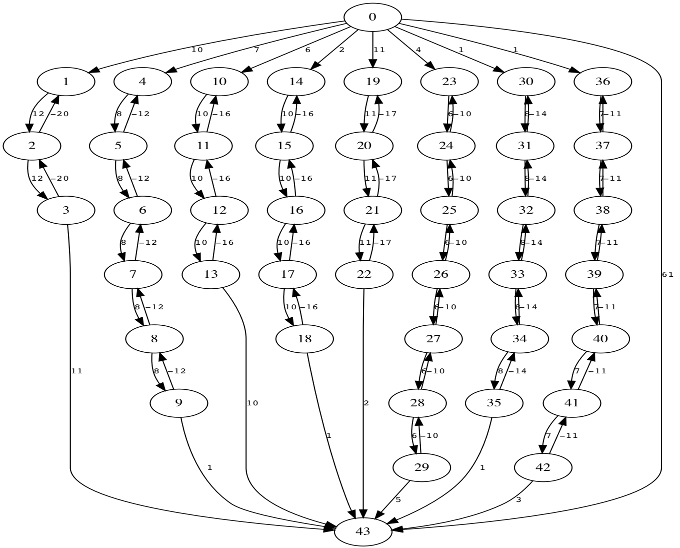
2. We search (by enumeration) in $\bar{G}'_{\bar{\mathcal{P}}}$ for cycles C such that

$$\begin{cases} \sum_{(i,j) \in C} l_{ij} > 0, \\ \sum_{(i,j) \in C} v_{ij} < 1. \end{cases}$$

Other components of the Branch-Cut-and-Price

- Automatic dual price smoothing stabilization [Pessoa et al., 2018]
- Ryan & Foster branching [Ryan and Foster, 1981]
- Multi-phase strong branching [Pecin et al., 2017]
- Strong diving heuristic with Limited Discrepancy Search [Sadykov et al., 2018]
  - 10 dives are performed
  - 10 candidates are evaluated before each fixing
  - Each time a set of items is fixed, we update the hard conflicts

Structure of test instances



Dimension of test instances

1386 instances

- Same flexibility (relative interval for the distance between consecutive tasks)
- Number of time periods $\in \{20, 30, \ldots, 110, 120\}$
- Number of chains $\in \{3, 4, \ldots, 9\}$.
- Average number of items per chain $\in \{5, 6, \ldots, 10\}$.
- Average number of items per bin $\in \{2, 3, 4\}$.
- As a result, number of items $\in [15, 117]$ with $\approx$ normal distribution.

Main experiment results

Solved to optimality within 3 hours

Method	**% Solved**
BCP	69.5%
CPLEX 12.8	46.2%

On the set of instances solved by both methods,
BCP is 9 times faster on average

Other experiment results (1)

Percentage of solved instance by number of chains

# of chains	% BCP	% CPLEX
3	100.0%	93.4%
4	98.0%	75.3%
5	83.8%	58.1%
6	65.2%	35.4%
7	55.1%	26.8%
8	43.4%	17.7%
9	40.9%	16.7%

Other experiment results (1)

Percentage of solved instances by number of periods

# of periods	% BCP	% CPLEX
20	72%	67%
30	75%	63%
40	79%	67%
50	75%	53%
60	70%	48%
70	67%	41%
80	66%	34%
90	61%	33%
100	66%	37%
110	67%	35%
120	66%	31%

Perspectives

Ongoing work

- Support of Chvátal-Gomory rank-1 cuts
- Custom branch-and-bound algorithm for the pricing problem
- Tests on the instances of the special cases of the problem

Research directions

- Limit on the number of bins per period
- Makespan objective
- (Flexible) Periodic Vehicle Routing

References I

Becker, C. and Scholl, A. (2006).
A survey on problems and methods in generalized assembly line balancing.
European Journal of Operational Research, 168(3):694 – 715.

Kramer, R., Dell'Amico, M., and Iori, M. (2017).
A batching-move iterated local search algorithm for the bin packing problem with generalized precedence constraints.
International Journal of Production Research, 55(21):6288–6304.

Pecin, D., Pessoa, A., Poggi, M., and Uchoa, E. (2017).
Improved branch-cut-and-price for capacitated vehicle routing.
Mathematical Programming Computation, 9(1):61–100.

Pereira, J. (2016).
Procedures for the bin packing problem with precedence constraints.
European Journal of Operational Research, 250(3):794 – 806.

References II

Pessoa, A., Sadykov, R., Uchoa, E., and Vanderbeck, F. (2018).
Automation and combination of linear-programming based stabilization techniques in column generation.
INFORMS Journal on Computing, 30(2):339–360.

Ryan, D. M. and Foster, B. A. (1981).
An integer programming approach to scheduling.
In Wren, A., editor, *Computer Scheduling of Public Transport Urban Passenger Vehicle and Crew Scheduling*, pages 269 – 280. North-Holland, Amsterdam.

Sadykov, R., Vanderbeck, F., Pessoa, A., Tahiri, I., and Uchoa, E. (2018).
Primal heuristics for branch-and-price: the assets of diving methods.
INFORMS Journal on Computing, (Forthcoming).