A Branch-Cut-and-Price Algorithm for the Location-Routing Problem

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ROADEF 2019  
Le Havre, France, February 21
Location-Routing Problem (LRP)

Data

- \( I \) — set of potential depots with opening costs \( f_i \) and capacities \( w_i \), \( i \in I \)
- \( J \) — set of customers with demands \( d_j \), \( j \in J \)
- Sets of edges: \( E = J \times J \), \( F = I \times J \)
- \( c_e \) — transportation cost of edge \( e \in E \cup F \)
- An unlimited set of vehicles with capacity \( Q \).

The problem

- Decide which depots to open
- Assign every client to an open depot subject to depot capacity
- For every depot, divide assigned clients into routes subject to vehicle capacity
- Minimize the total depot opening and transportation cost
LRP: an illustration

Figure: LRP instance: $G = (I \cup J, E \cup F)$
LRP: a solution

Figure: Location of depots must be jointly decided with vehicle routing.
Literature on LRP

- A combination of two central OR problems
- ≈3000 papers in Google Scholar with both “location” and “routing” in the title

Important recent works

- [Belenguer et al., 2011] — important valid inequalities & Branch-and-Cut;
- [Baldacci et al., 2011b] — exact “enumeration” & column generation approach
- [Contardo et al., 2014] — state-of-the-art exact algorithm
- [Schneider and Löffler, 2019] — state-of-the-art heuristic
- [Schneider and Drexl, 2017] — the latest survey on LRP
Our study

- Recently, large improvement in exact solution of the Capacitated VRP
- Testing these techniques for the LRP
- Testing new families of inequalities specific to the LRP
Formulation

- \( \lambda^i_r, i \in I, r \in R_i, \) equals 1 iff route \( r \) is used for depot \( i \)
- \( a^r_e, e \in E \cup F, r \in \bigcup_{i \in I} R_i, \) equals 1 iff edge \( e \) is used by \( r \)
- \( y_i, i \in I, \) equals 1 iff route depot \( i \) is open
- \( z_{ij}, i \in I, j \in J, \) equals 1 iff client \( i \) is assigned to depot \( i \)

\[
\begin{align*}
\min \ & \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{r \in R_i} \sum_{e \in E \cup F} c_e a^r_e \lambda^i_r \\
\sum_{i \in I} z_{ij} &= 1, \quad \forall \ j \in J, \\
\sum_{r \in R_i} \sum_{e \in \delta(j)} a^r_e \lambda^i_r &= 2z_{ij}, \quad \forall \ i \in I, \ j \in J \\
\sum_{j \in J} d_j z_{ij} &\leq w_i y_i, \quad \forall \ i \in I, \\
z_{ij} &\leq y_i, \quad \forall \ i \in I, \ j \in J, \\
(z, y, \lambda) &\in \{0, 1\}^K
\end{align*}
\]
Given a subset of clients $C \subseteq J$,

$$\sum_{i \in I} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_{er}^i \lambda_r^i \geq 2 \cdot \left\lceil \frac{\sum_{i \in C} d_i}{Q} \right\rceil.$$  

**Separation**

CVRPSEP library [Lysgaard et al., 2004]
Chvátal-Gomory Rank-1 Cuts [Jepsen et al., 2008]

Each cut is obtained by a Chvátal-Gomory rounding of a set $C \subseteq J$ of set packing constraints using a vector of multipliers $\rho$ ($0 < \rho_j < 1, j \in C$):

$$
\sum_{i \in I} \sum_{r \in R_i} \left[ \sum_{j \in C} \rho_j \sum_{e \in \delta(j)} \frac{1}{2} a_{r e}^i \right] \chi_r^i \leq \sum_{j \in C} \rho_j
$$

All dominant vectors of multipliers for $|C| \leq 5$ are given in [Pecin et al., 2017b].

Separation

A local search for each dominant vector of multipliers.
Depot Capacity Cuts [Belenguer et al., 2011]

Given a subset of clients $C \subset J$ which cannot be served by a subset of depots $S \subset I$, $\sum_{j \in C} d_j > \sum_{i \in S} w_i$,

$$\sum_{i \in I \setminus S} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_{r e}^i \lambda_r^i \geq 2.$$

Separation
A heuristic algorithm: combination of GRASP and local search.
Covering inequalities for depot capacities

Let \( W = \sum_{i \in I} w_i \) and \( D = \sum_{j \in J} d_j \)

\[
\sum_{i \in I} w_i y_i \geq d(J)
\]

\[
\Rightarrow \sum_{i \in I} w_i (1 - y_i) \leq W - D
\]

We can generate any valid inequality for this knapsack. For example covering inequalities: given a subset of depots \( S \subset I \), \( \sum_{i \in S} w_i > W - D \),

\[
\sum_{i \in S} (1 - y_i) \leq |S| - 1
\]

Separation

We optimize an LP which looks for an inequality which is satisfied by all integer solutions of the knapsack.
Route Load Knapsack Cuts (RLKC)

\( x_q^i \) — number of routes with load of exactly \( q \leq Q \) units leaving depot \( i \in I \). Then:

\[
\sum_{q=1}^{Q} qx_q^i \leq w_i.
\] (1)

Any valid inequality for (1) is valid for LRP.

**Separation**

Chvátal-Gomory rounding of (1).
1/k-facets of the master knapsack problem

Theorem ([Aráoz, 1974])

The coefficient vectors $\xi$ of the knapsack facets $\xi x \leq 1$ of $\sum_{q=1}^{n} qx_q = n$ with $\xi_1 = 0$, $\xi_Q = 1$ are the extreme points of the following system of linear constraints

\[ \begin{align*}
    \xi_1 &= 0, \quad \xi_Q = 1, \\
    \xi_q + \xi_{Q-q} &= 1 \quad \forall 1 \leq i \leq n/2, \\
    \xi_q + \xi_t &\leq \xi_{q+t} \quad \text{whenever } q + t < n.
\end{align*} \]

Definition

A knapsack facet $\xi x \leq 1$ is called a 1/k-facet if $k$ is the smallest possible integer such that

\[ \xi_q \in \{0/k, 1/k, 2/k, \ldots, k/k\} \cup \{1/2\}. \]

Separation

1/6- and 1/8-facets can be efficiently separated using the algorithm by [Chopra et al., 2019]
Taking into account of RLKCs in the pricing

- Pricing problem is the **Resource Constrained Shortest Path**
- It is solved by a labelling algorithm, each label $L$ is $(\bar{c}^L, j^L, q^L)$
- Dominance relation:
  \[
  L \succ L' \quad \text{if} \quad \bar{c}^L \leq \bar{c}^{L'}, \quad j^L = j^{L'}, \quad q^L \leq q^{L'}
  \] (2)
- Let $\bar{\mu}(q)$ be the sum of dual functions corresponding to active RLKCs
- Label $L$ is now $(\bar{c}^L + \bar{\mu}(q^L), j^L, q^L)$
- Dominance (2) is still valid, as $\bar{\mu}(q)$ is non-decreasing
- Completion bounds can still be efficiently used as $\bar{\mu}(q)$ is super-additive
Other components of the Branch-Cut-and-Price

- Bucket graph-based labelling algorithm for the RCSP pricing [Righini and Salani, 2006] [Sadykov et al., 2017]
- Partially elementary path (ng-path) relaxation [Baldacci et al., 2011a]
- Automatic dual price smoothing stabilization [Wentges, 1997] [Pessoa et al., 2018]
- Reduced cost fixing of (bucket) arcs in the pricing problem [Ibaraki and Nakamura, 1994] [Irnich et al., 2010] [Sadykov et al., 2017]
- Enumeration of elementary routes [Baldacci et al., 2008]
- Multi-phase strong branching [Pecin et al., 2017a]
  - On depot openings (largest priority)
  - On number of vehicles for each depot
  - On number of clients per depot
  - On assignment of clients to depots
  - On edges of the graph
**Computational results (without DCCs)**

Open instances solved to optimality. Could not be solved by the state-of-the-art [Contardo et al., 2014] in 5-97 hours.

<table>
<thead>
<tr>
<th>Set</th>
<th>Instance</th>
<th>Optimum</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Prins et al., 2006]</td>
<td>100x5-1b</td>
<td>213568</td>
<td>10m05s</td>
</tr>
<tr>
<td></td>
<td>100x10-1a</td>
<td>287661</td>
<td>1h32m</td>
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<tr>
<td></td>
<td>100x10-1b</td>
<td>230989</td>
<td>1h38m</td>
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<tr>
<td></td>
<td>100x10-3a</td>
<td>250882</td>
<td>1h17m</td>
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<tr>
<td></td>
<td>100x10-3b</td>
<td>203114</td>
<td>11h01m</td>
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<tr>
<td></td>
<td>200x10-1a</td>
<td>474702</td>
<td>20m42s</td>
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<tr>
<td></td>
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<td>375177</td>
<td>1h55m</td>
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<td>448005</td>
<td>4h45m</td>
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<tr>
<td></td>
<td>200x10-2b</td>
<td>373696</td>
<td>5h53m</td>
</tr>
<tr>
<td>[Tuzun and Burke, 1999]</td>
<td>P113112</td>
<td>1238.24</td>
<td>2h29m</td>
</tr>
<tr>
<td></td>
<td>P131112</td>
<td>1892.17</td>
<td>36m52s</td>
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<tr>
<td></td>
<td>P131212</td>
<td>1960.02</td>
<td>34m59s</td>
</tr>
</tbody>
</table>

**Underlined**: improved solutions over [Schneider and Löffler, 2019]
Sensitivity analysis of cuts specific to LRP

26 instances by [Prins et al., 2006] with 5-10 depots and 50-200 clients. Time limit 3 hours

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Solved</th>
<th>Root gap</th>
<th>Nodes</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>All but DCCs</td>
<td>22/26</td>
<td>0.87%</td>
<td>27.4</td>
<td>611</td>
</tr>
<tr>
<td>All but RLKCs</td>
<td>22/26</td>
<td>0.51%</td>
<td>10.5</td>
<td>480</td>
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<tr>
<td>All but $y$-knapsack</td>
<td>21/26</td>
<td>0.69%</td>
<td>12.3</td>
<td>578</td>
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<tr>
<td>All cuts</td>
<td>22/26</td>
<td>0.47%</td>
<td>9.9</td>
<td>521</td>
</tr>
</tbody>
</table>
Perspectives

Ongoing work

- Improving separation of Route Load Knapsack Cuts
- Parameterisation of separation of Depot Capacity Cuts
- Column generation-based heuristic for fair comparison with [Contardo et al., 2014]
- Testing on larger instances by [Schneider and Löffler, 2019]

Perspectives

- Other families of cuts for the LRP
- Using Route Load Knapsack Cuts for other problems (Multi Capacitated Depot Vehicle Routing?)
References I


References III


References IV


Symmetry helps: Bounded bi-directional dynamic programming for the elementary shortest path problem with resource constraints.

A bucket graph based labeling algorithm with application to vehicle routing.
*Cadernos do LOGIS 7*, Universidade Federal Fluminense.

A survey of the standard location-routing problem.

Large composite neighborhoods for the capacitated location-routing problem.