Constraint Programming

Lecture 2. Local consistency.

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Lignes directrices

Binary constraints and CSPs

Local consistency : an overview

Arc-consistency

Other local consistencies

Definitions

- A constraint is binary if it contains at most two variables.
- A CSP is binary if all its constraints are binary.

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- A CSP is binary if all its constraints are binary.

Network of binary constraints

A constraint network is a set of constraints on variables with discrete and finite domain.

A binary constraint network can be represented by a special graph :

- Vertices represent variables.
- Edges represent constraints.

If two vertices are adjacent, there are constraints containing the corresponding variables.

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Making any CSP binary

For each non-binary CSP, there exists an equivalent binary CSP.

Example

$\Rightarrow$

$D_x = D_y = D_z = \{1, 2, 3\}$

$D_w = \{abc\}_{a \neq b \neq c}$

$C_x = \{(a, abc)\}_{a \in D_x, abc \in D_w}$

$C_y = \{(b, abc)\}_{b \in D_y, abc \in D_w}$

$C_z = \{(c, abc)\}_{c \in D_z, abc \in D_w}$

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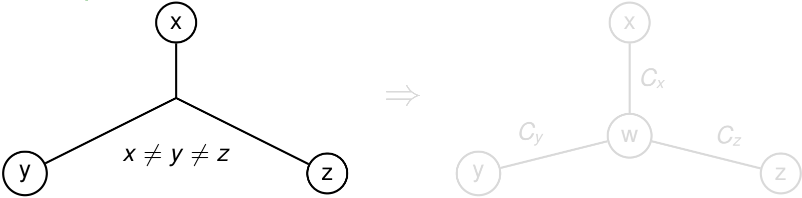

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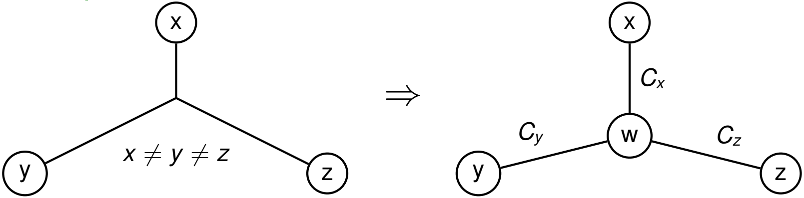

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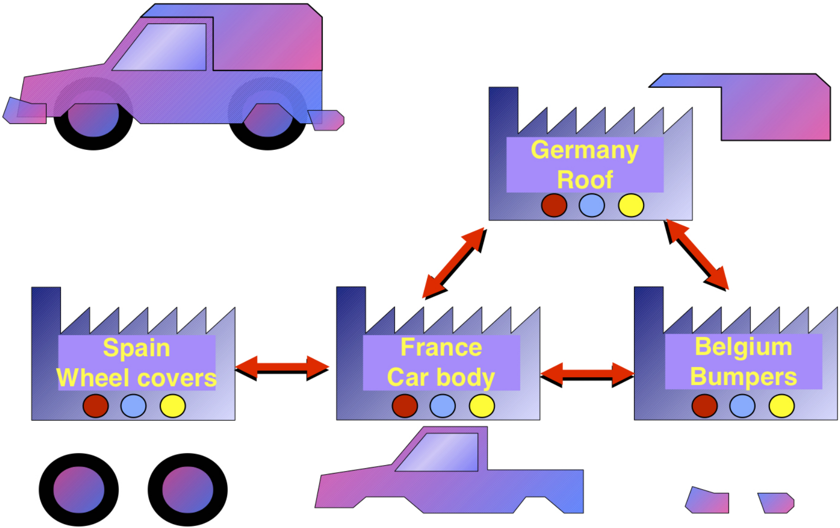
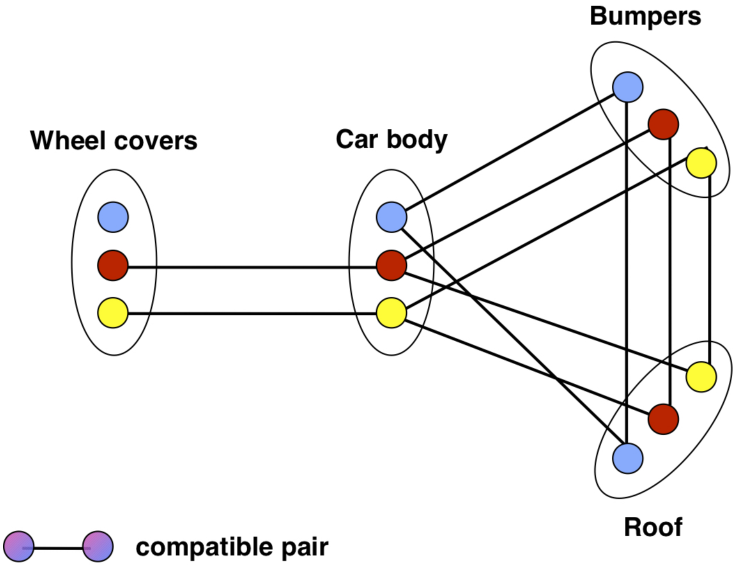
A trivial example

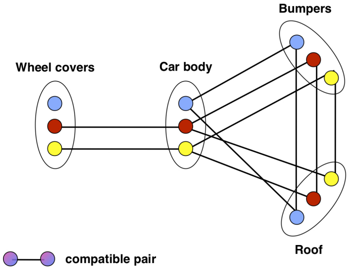


Source : Philippe Baptiste

A trivial example : the extended constraint network



Trivial solution algorithm for our example



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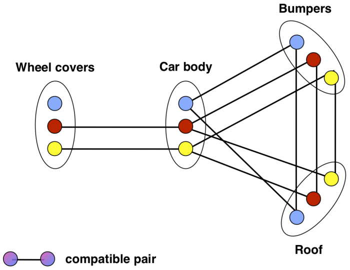

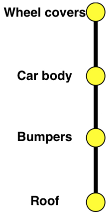

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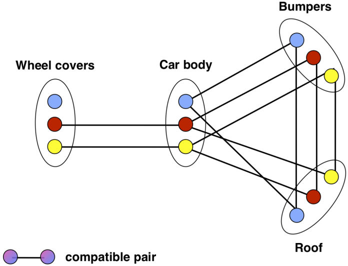

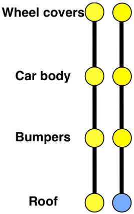

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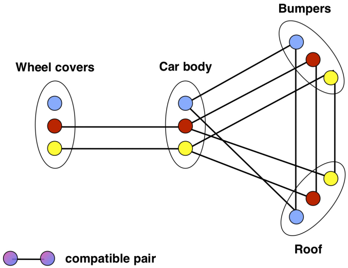

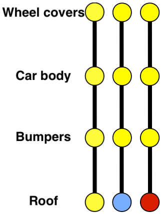

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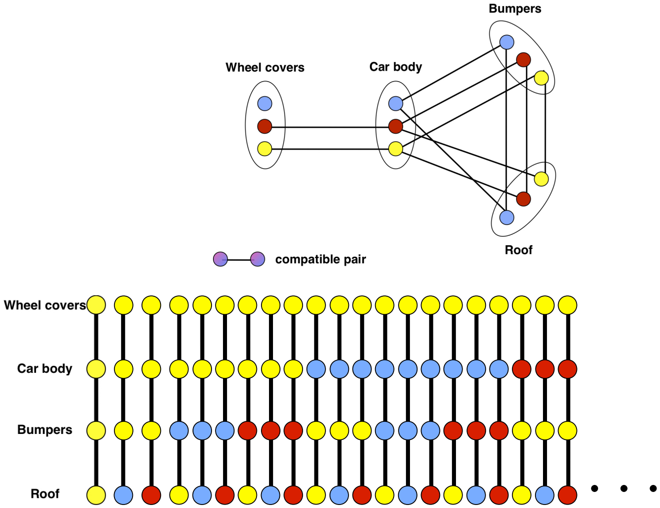

Trivial solution of discrete CSPs

- Suppose that all domains are finite
- Then the number $|A|$ of different assignments of values in variable domains is finite too :

$$|A| = |D_{x_1}| \times \cdots \times |D_{x_n}|.$$

- Then, we can consider these assignments one by one and verify whether at least one of them satisfies the constraints.
- Computational complexity (number of operations to do) is at least $|A| \times |C|$.
- It is too large !
  - 8 queens : $8^8 \times 96 \approx 10^{10}$
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Inconsistency

To solve faster, we try to remove values (from domains of variables) which do not lead to any solution.

Example

- $D_x = \{1, 2, 3\}$, $D_y = \{2, 3, 4\}$, $x \geq y$.
- If $x = 1$, there are no values in D_y which satisfy the constraint.
- Therefore, we can delete 1 from D_x.

We say that these values are « inconsistent » with one or several constraints.

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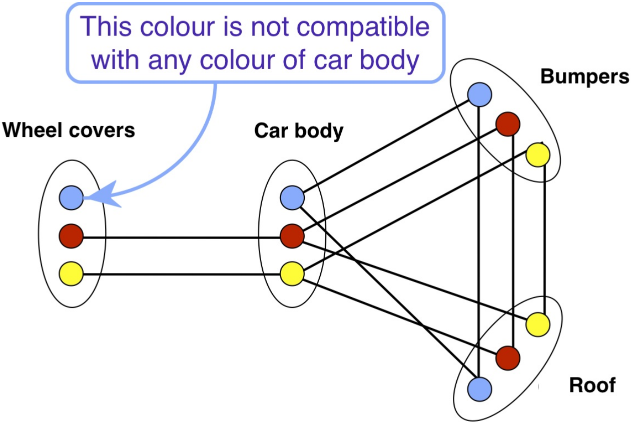
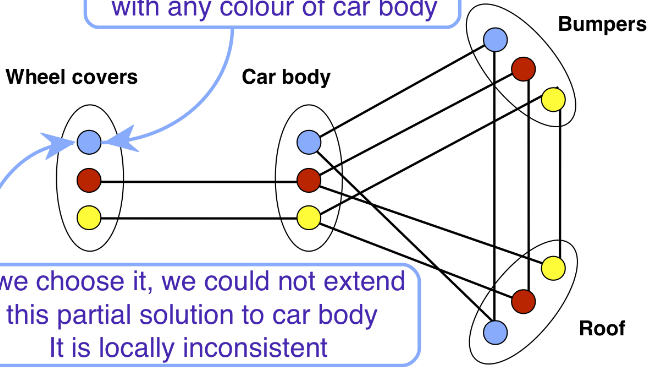
Inconsistency : illustration I



Inconsistency : illustration I

This colour is not compatible with any colour of car body



If we choose it, we could not extend this partial solution to car body
It is locally inconsistent

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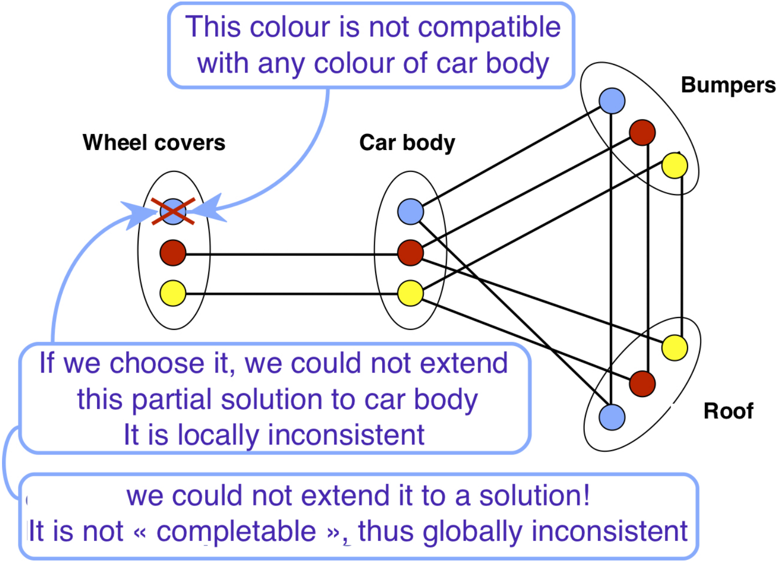
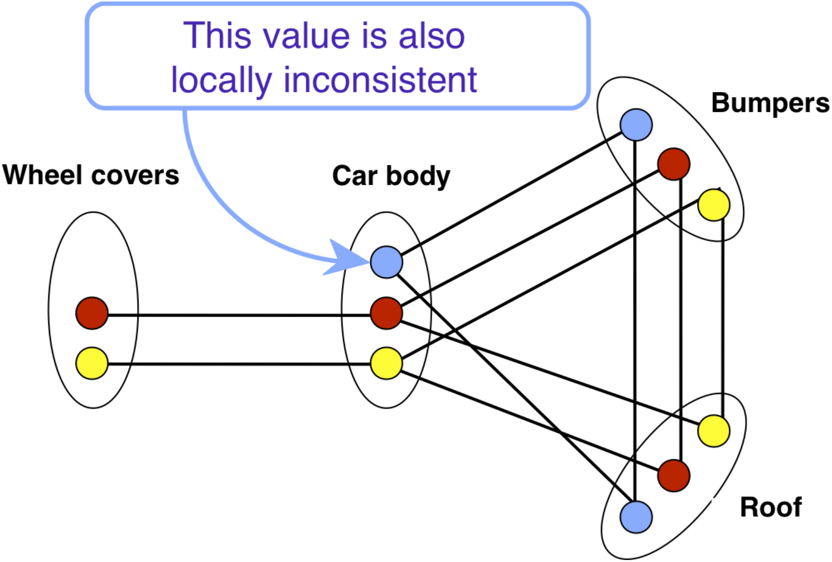


Inconsistency : illustration II



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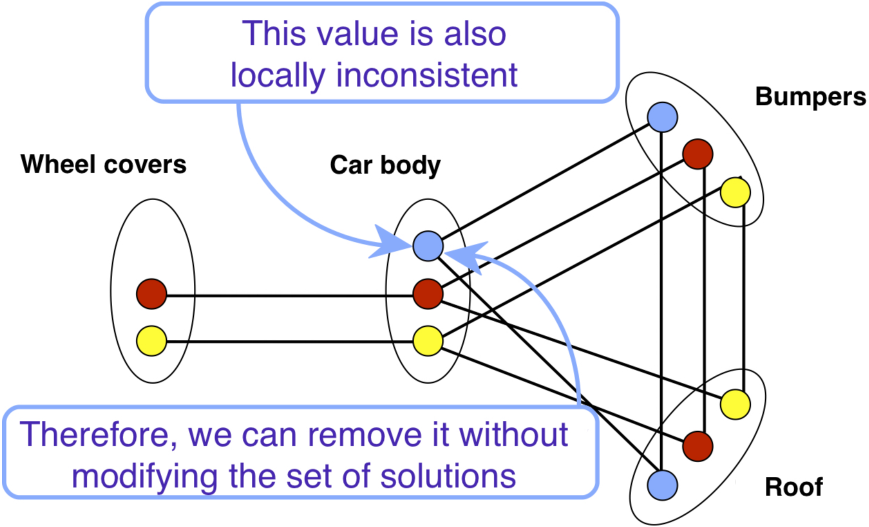
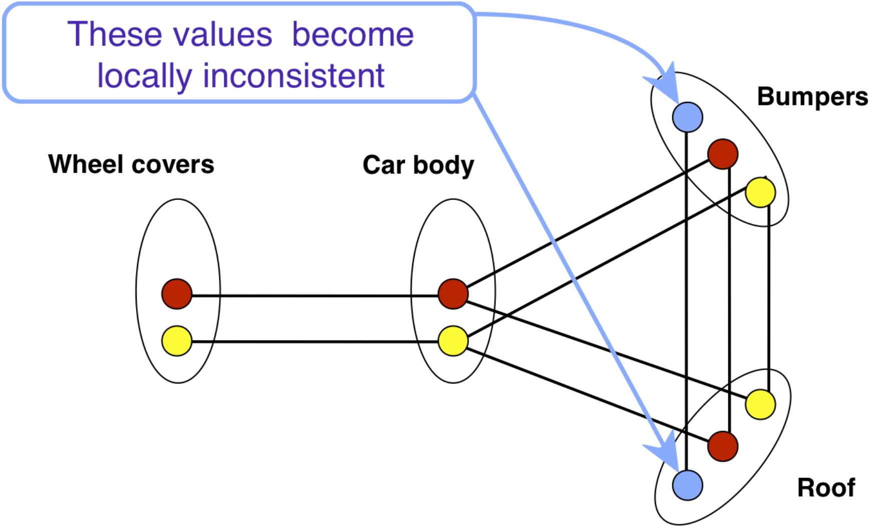
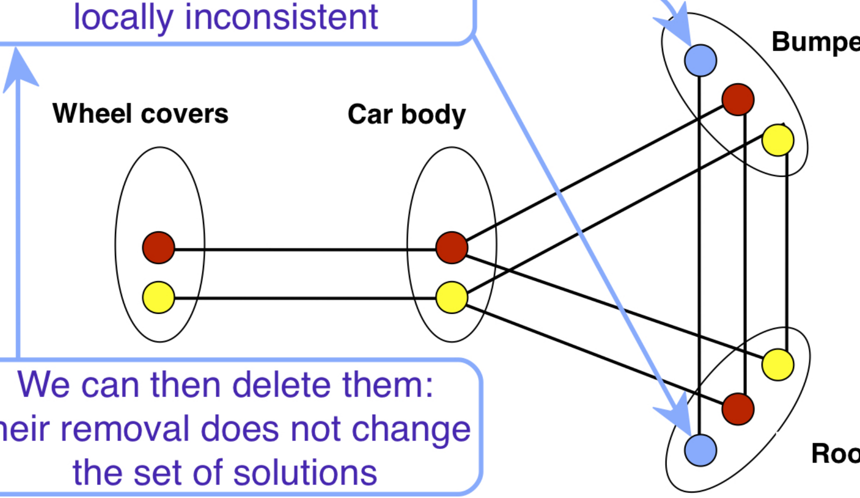


Inconsistency : illustration III



Inconsistency : illustration III

These values become locally inconsistent



We can then delete them: their removal does not change the set of solutions

Local consistency

- We do not check consistency of values with all constraints at the same time (this would mean solving the whole CSP)
- We check consistency with a subset of constraints (usually, one), therefore the consistance is local.
- A CSP is locally consistent if all values in all variable domains are locally consistent
- When a CSP achieves local consistency, the set of solutions does not change. Therefore, the CSP remains equivalent, but simpler (or smaller).

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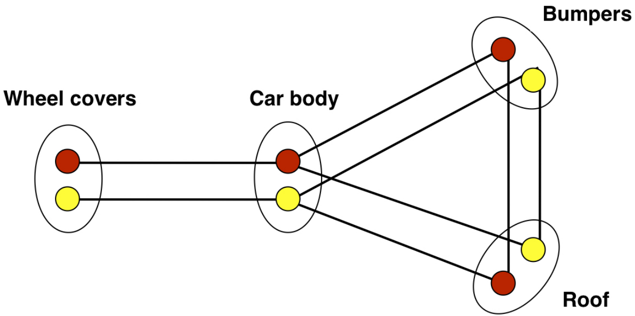

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- We reduced the search space !

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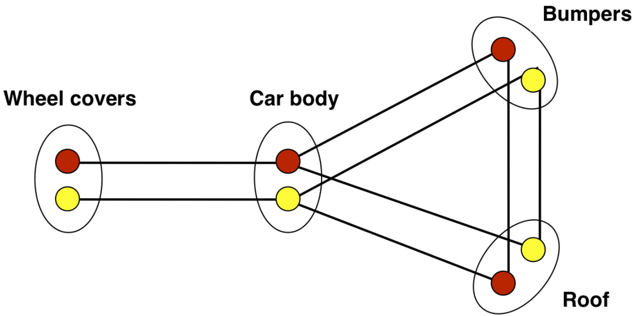

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Lignes directrices

Définitions I

Here we concentrate on the binary case.

- There exist different levels of local consistency
- If we check consistency of a value with only binary constraint at a time, it is the arc-consistency.
- Value a of variable x est arc-consistent if and only if it has at least one compatible value (support) in each neighbour domain.
- **Formally :**

$$\langle x, a\rangle \text{ is arc-consistent } \Leftrightarrow \forall C(x,y)\ \exists b \in D_y :\ C(a,b).$$

$$\langle x, a\rangle \text{ is not arc-consistent } \Leftrightarrow \exists C(x,y) :\ \forall b \in D_y\ \neg C(a,b).$$

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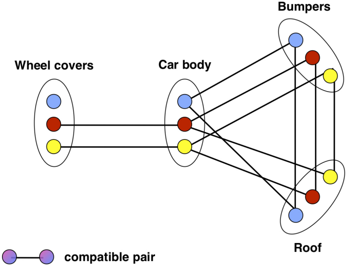
Arc-consistency : illustration



(Wheel covers, 🔴) is arc-consistent as there exists a compatible value for Car body.
(Car body, 🔴) is arc-consistent as

- there exists a compatible value (🔴) for wheel covers,
- there exists a compatible value (🔴) for bumpers,
- there exists a compatible value (🟡) for the roof

(Car body, 🔵) is not arc-consistent as it does not have a compatible value for wheel covers.
Removal of (Car body, 🔵) makes (Roof, 🔵) and (Bumpers, 🔵) arc-inconsistent

Definitions II

- A contraint is arc-consistent if and only if all values in domains of its variables are arc-consistent.
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Why arc-consistency ?

There exist polynomial (and efficient !) algorithms to achieve the arc-consistency for a binary CSP.

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Algorithm AC-1

repeat
  finished $\leftarrow$ TRUE;
  foreach *contraint* $C(x, y)$ **do**
    if *there exist values in* D_x *which do not have a support in* D_y **then**
      remove them;
      finished $\leftarrow$ FALSE;
until *finished* $=$ *TRUE*;

Computational complexity in the worst case :

$$O(nd \times ed^2) = O(ned^3),$$

where n — number of variables, e — number of constraints, d — size of the largest domain.

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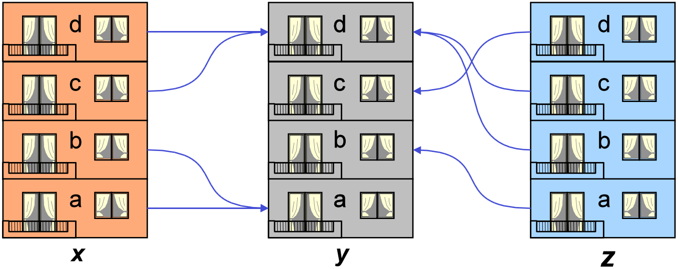
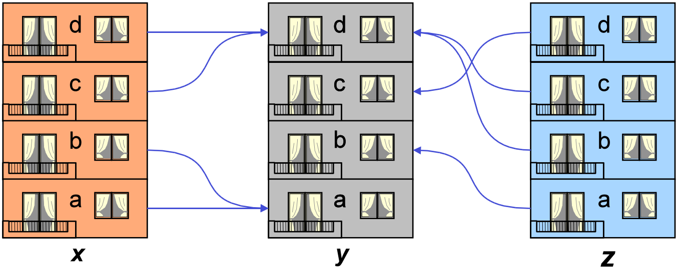
Friends analogy



Town↔CSP
Building↔Variable
Neighbour buildings↔Variables « connected » by a constraint
Resident↔Value
Friends↔Pair of values satisfying the constraint

A resident is arc-consistent while it has at least on friend in each neighbour building.

Friends analogy



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Friends analogy — AC-1

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AC-1 :

1. Each arc-inconsistent resident leaves the town and sends a letter « I leave the town - Anonymous » to every resident of the town.
2. When a resident receives a letter, he verifies whether he is still arc-consistent.

Algorithm AC-3

toTest $\leftarrow \{C(x,y)\}_{C(x,y)\in C}$;
foreach $C(x,y) \in$ *toTest* **do**
    toTest $\leftarrow$ *toTest* $\setminus\{C(x,y)\}$;
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Spacial complexity in the worst case :

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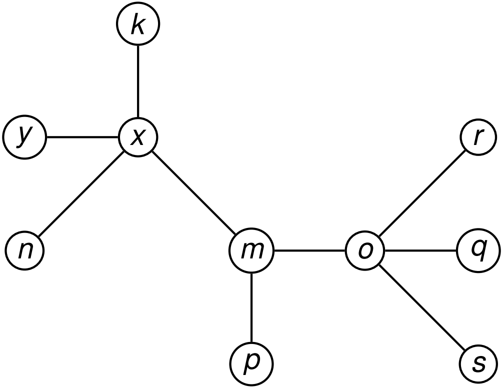
Algorithm AC-3 : illustration

If a value is removed, we verify arc-consistency only for values of neighbour variables.



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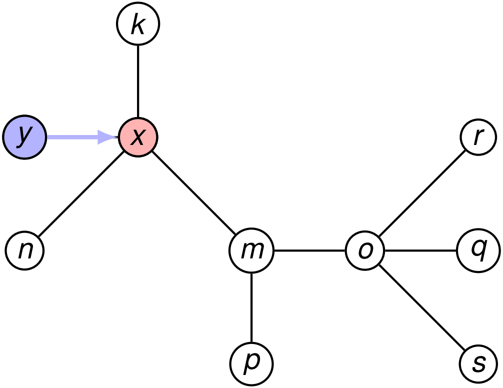

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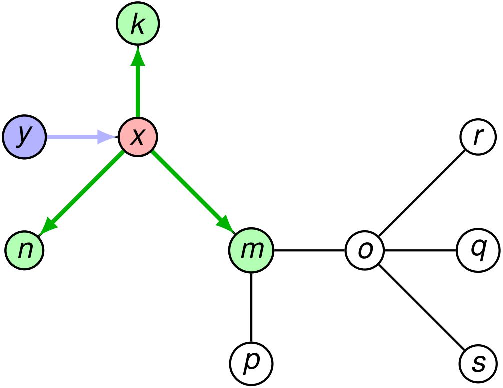

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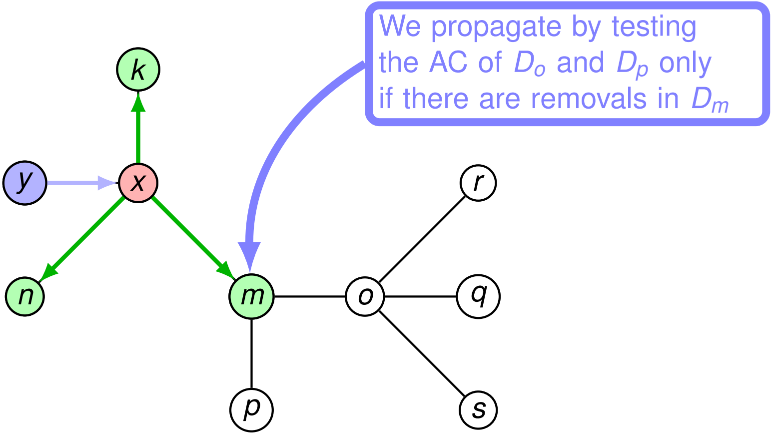

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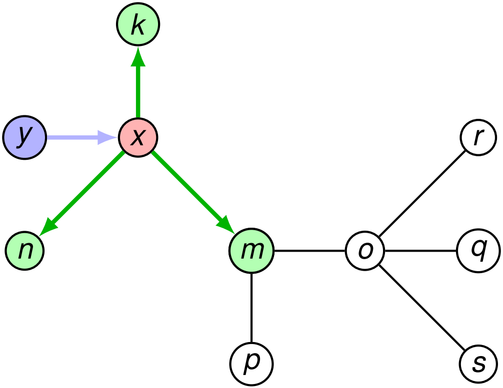
AC-3 :

1. Each arc-inconsistent resident leaves the town and sends a letter « I leave building x - Anonymous » to **residents of neighbour buildings**.
2. When a resident receives a letter, he verifies whether he still has friends **in building** x.

Algorithm AC-4 : illustration

If a value is removed, we verify arc-consistency only for values of neighbour variables which are compatible with the removed value.



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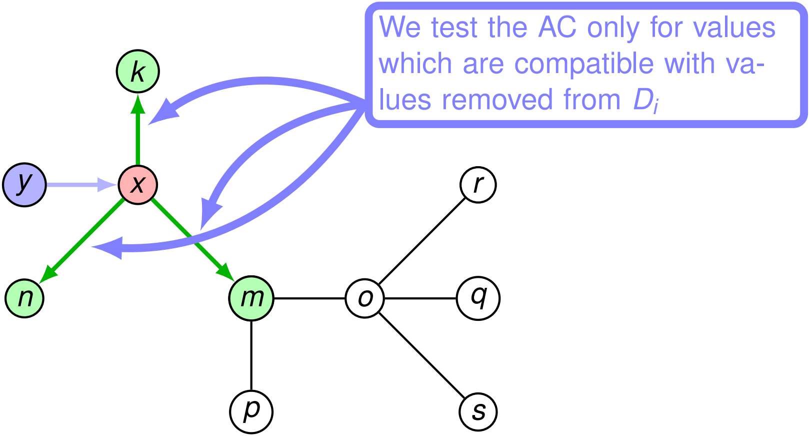

Friends analogy — AC-4

AC-3 :

1. Each arc-inconsistent resident leaves the town and sends a letter « I leave building x - Anonymous » **to all residents** of neighbour buildings.
2. When a resident receives a letter, he verifies whether he still has friends **in building** x.

AC-4 :

1. Each arc-inconsistent resident leaves the town and sends a letter « I leave building x - Signed $\langle x, a \rangle$ » **to its friends** in neighbour buildings.
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Algorithm AC-4

Overview

- We use structure $\mathcal{S}$: $\mathcal{S}_{i,a}$ is the set of pairs (variable, value) which support (i, a)
- We count the number of supports $\texttt{counter}[(i,j),a]$
- As soon as one $\texttt{counter}[(i,j),a]$ becomes zero, value a is removed from D_i and the pair (i, a) is added to list Q of revisions to do

Complexity analysis

- Spatial complexity is $O(ed^2)$ (memory needed to keep structure $\mathcal{S}$)
- Initialization of structure $\mathcal{S}$ takes time $O(ed^2)$
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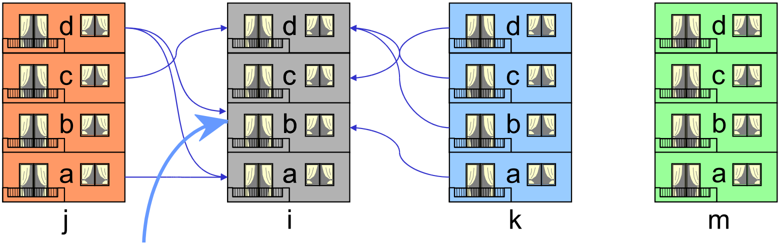
AC-6 :

1. Each arc-inconsistent resident leaves the town and sends a letter « I leave building x - Signed $\langle x, a\rangle$ » **to its friends for which he is the friend living in the smallest apartement number in** c
2. When a resident receives a letter, he **searches another friend in** x **who lives in an apartment with larger number than** $\langle x, a\rangle$

Algorithm AC-6 : illustration



- If resident $\langle i, b \rangle$ leaves, only $\langle k, a \rangle$ will be notified.
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- We will search for another friend of $\langle k, a \rangle$ living on a higher floor than $\langle i, b \rangle$.

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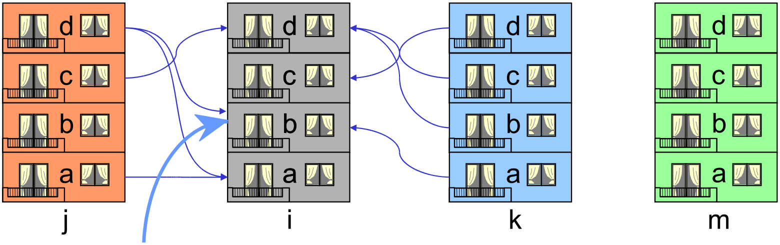

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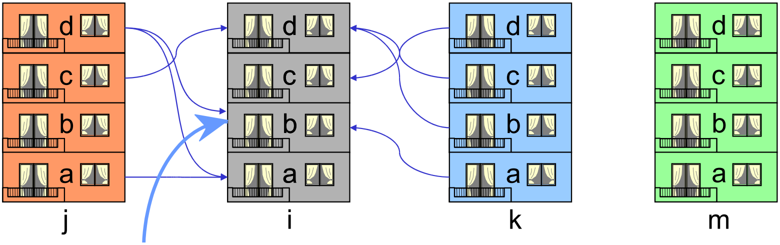

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Arc-consistency algorithms in practice

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Time complexity	$O(ed^3)$	$O(ed^2)$	$O(ed^2)$
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Implementation difficulty	easy	medium	hard

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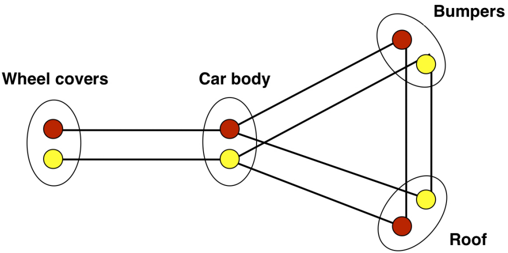

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Lignes directrices

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A pair of variables are path consistent with the third variable if each arc-consistent pair of values can be extended to another variable in such a way that all binary constraints are satisfied.

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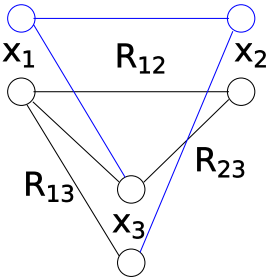

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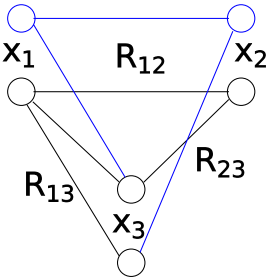

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There exist « stronger » local consistencies for finite CSPs..

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And if the domains are not discrete? Intervals?

Arc-B-consistency is the arc-consistency limited to bounds of intervals.

- Weaker that the arc-consistency.
- Easy to implement, therefore broadly used.

Definition

A constraint $C(x_1, \ldots, x_k)$ is arc-B-consistent if and only if

$$\forall x_i \quad \forall a_i \in \{\min(D_{x_i}), \max(D_{x_i})\}$$

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Arc-B-consistency : example

Variables : x, y, z. Domains are intervals.
Constraint : $z = x + y$

$\min(D_z)$ and $\max(D_z)$ are arc-consistent if

$$\begin{cases} \min(D_z) \geq \min(D_x) + \min(D_y) \\ \max(D_z) \leq \max(D_x) + \max(D_y) \end{cases}$$

otherwise

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