The Kato square root problem on locally uniform domains

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Evolution Equations: Applied and Abstract Perspectives

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- $a_{ij}, b_i, c_j, d : O \to \mathbb{C}^{m\times m}$ bounded and measurable
- define sesquilinear form

$$a(u,v) = \int_O \sum_{i,j=1}^{d} a_{ij}\partial_j u \cdot \overline{\partial_i v} + \sum_{i=1}^{d} b_i u \cdot \overline{\partial_i v} + \sum_{j=1}^{d} c_j \partial_j u \cdot \overline{v} + du \cdot \overline{v}\, dx$$

- form a coercive in Gårding's sense

$$Re\, a(u,u) \geq \lambda(\|u\|^2_{\mathrm{L}^2(O)} + \|\nabla u\|^2_{\mathrm{L}^2(O)})$$

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Problem

For which spaces $\mathcal{V}$ do we have $\mathrm{D}(L^{\frac{1}{2}}) = \mathcal{V}$ with equivalent norms?

What is known for mixed boundary conditions

Theorem (AKM '06, EHT '14/16)

Suppose:

- *O bounded domain*
- *O is d-regular*
- *∂O is $(d-1)$-regular*
- *$D \subseteq \partial O$ is $(d-1)$-regular*
- *O is bi-Lipschitz near $\partial O \setminus D$*

Then the Kato property holds for $\mathcal{V} = \mathrm{H}^{1,2}_D(O)$.

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(The first two conditions are grouped as "domain"; the last three as boundary.)

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Aim: only demand for boundary regularity!

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Suppose:

- *O ~~bounded domain~~* } "domain"
- *O is d-regular*
- *∂O is ~~$(d-1)$-regular~~ porous* } boundary
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"domain" (first two conditions); boundary (last three conditions)

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- better extension and approximation theory: no charts

~~What~~ This is known for mixed boundary conditions~~?~~ !

Theorem (AKM '06, EHT '14/16, B.–Egert–Haller-Dintelmann '19)

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(first two items: ~~"domain"~~; last three items: boundary)

Then the Kato property holds for $\mathcal{V} = \mathrm{H}^{1,2}_D(O)$.

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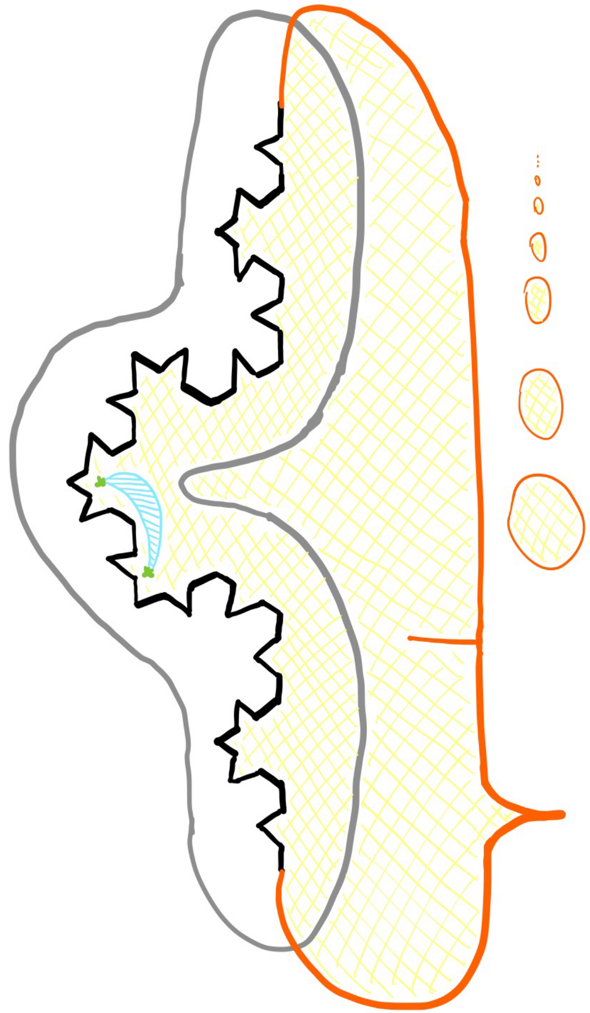
- thickening of O and "localization": no d-regularity

A wild geometry that is admissible

- Neumann boundary part *N* (black) is fractal



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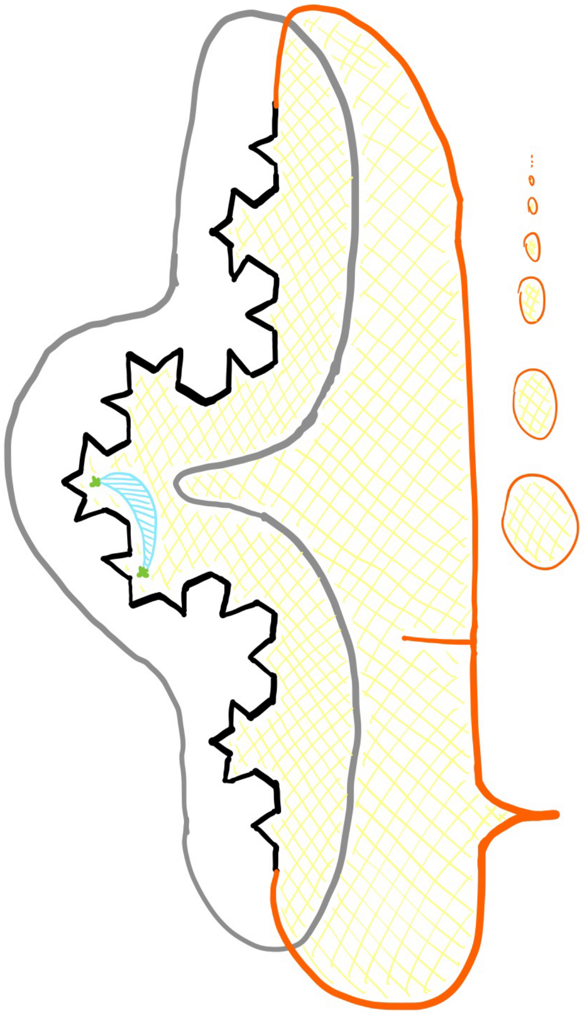

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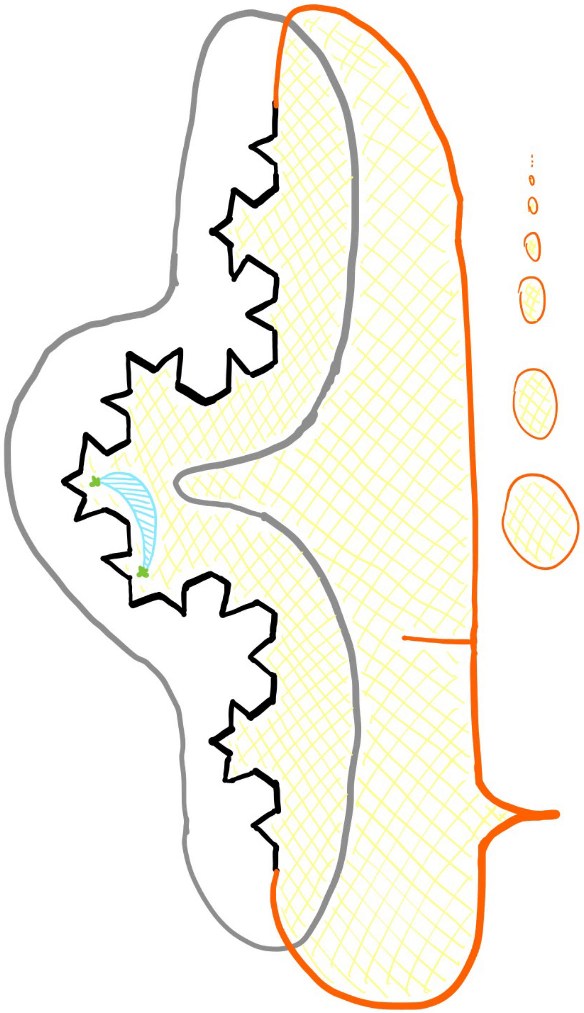

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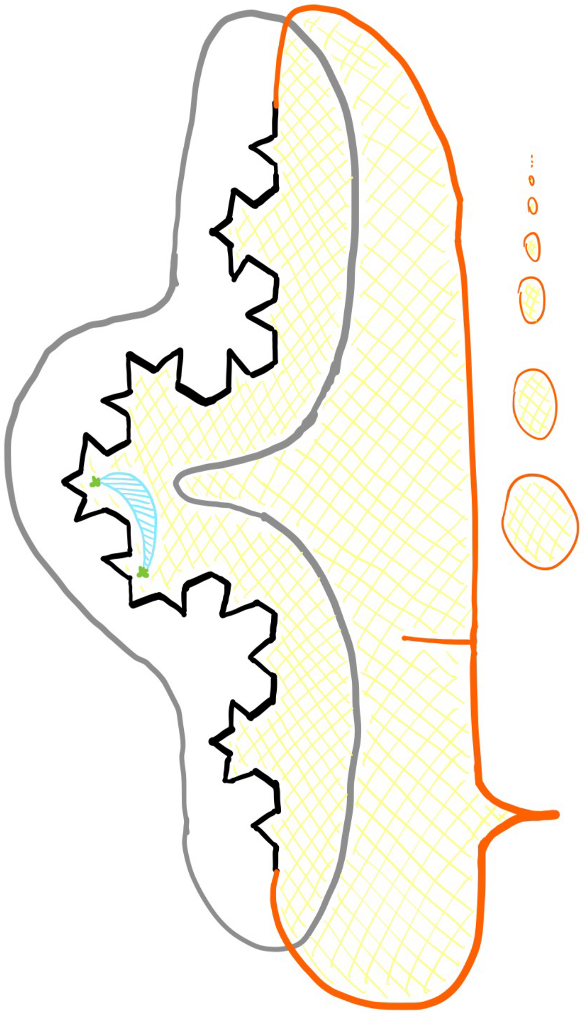

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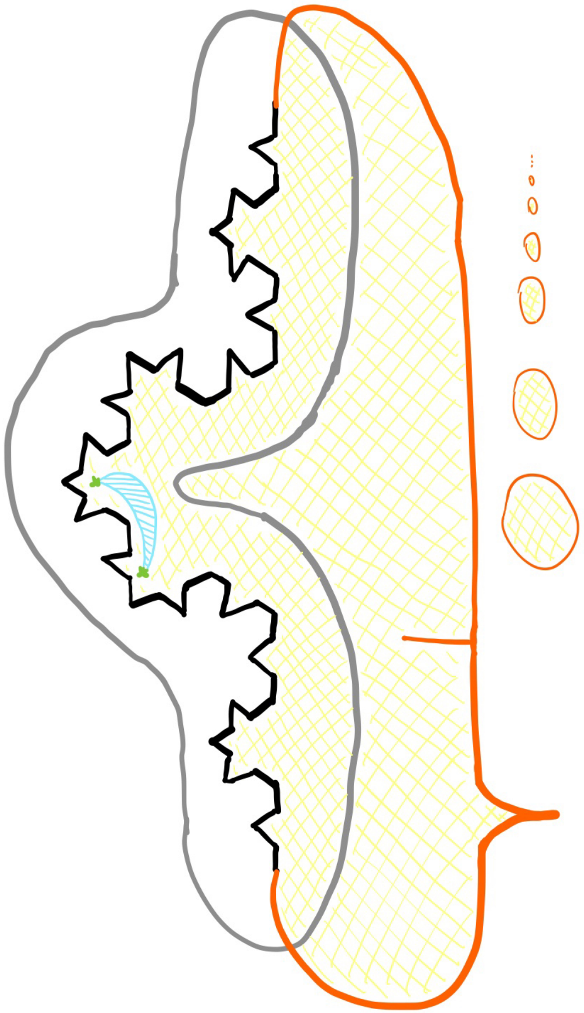
- diameter of connected components away from N degenerates



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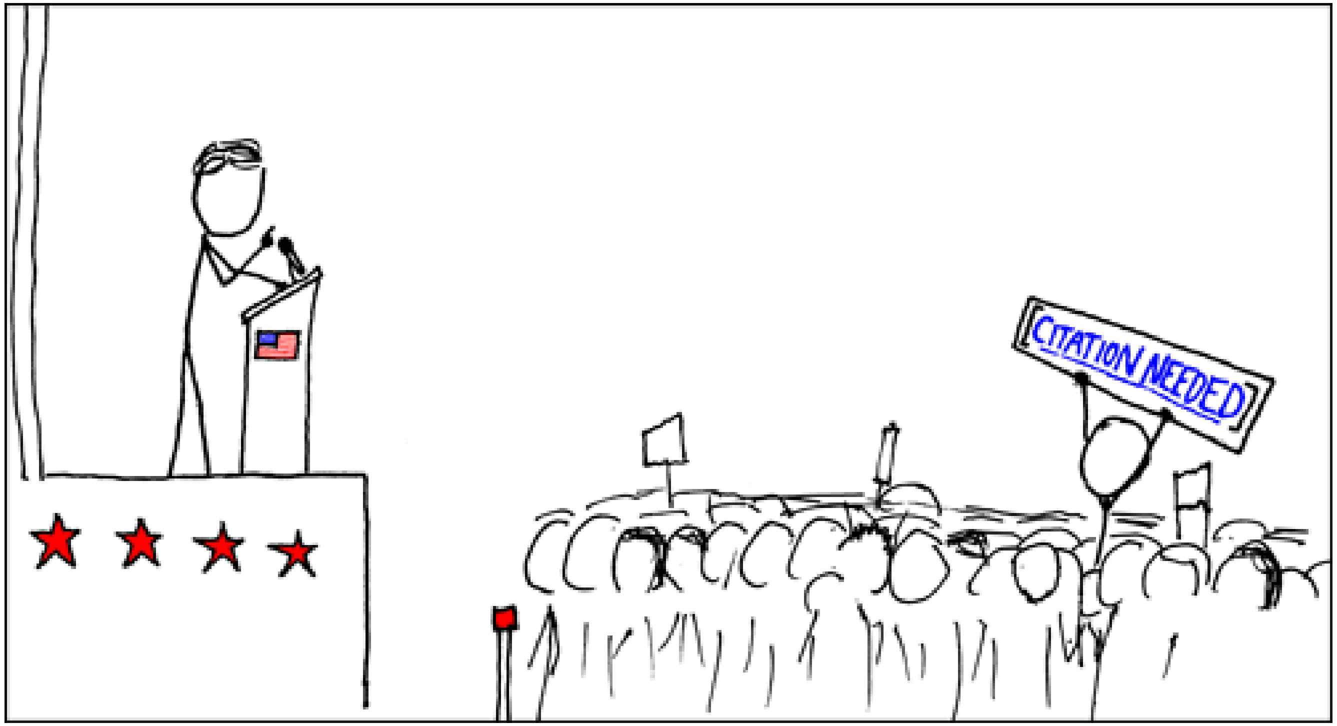
- Decomposition/Localization of functional calculi

Thank you for your attention!



S. Bechtel, M. Egert and R. Haller-Dintelmann.
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Available on arXiv.