Modelization and numerical scheme for a bitemperature 2D transverse plasma

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Introduction

- Plasmas : state of matter where cohabitates neutral particles and ions,
- Plasma modelization $\rightarrow$ great opportunity for energetic purposes (ex: nuclear fusion, etc..),
- Equations derivation and numerical modelization :
  - 1D, E, B, Suliciu relaxation (Brull, Dubroca, Lhébrard),
  - 2D, E, isentropic (Estibals, Guillard, Sangam),
  - 2D, E, Aregba-Natalini (Arebga, Brull, Prigent)

1. Derivation of the fluid system
   - Physical framework and kinetic modelization
   - Derivation for the bi-fluid system
   - Dimensionless equations

2. Kinetic scheme

Plan

Physical framework

- We consider a plasma as a bi-fluid system $\mathcal{P} := \{e^-, A^{Z+}\}$, with respective charges $q_e := -e$ (electron), $q_i := -Ze$ (ion) ans masses m_e, m_i,
- Z is the ionization rate, assumed constant,
- The mesoscopic charge is $\mathcal{Q} \simeq 0$ (quasi-neutrality assumption),
- A^{Z+} may be whether monoatomic or polyatomic,
- No chemical reaction assumed, only mechanical collisions,
- Each fluid is submitted to a Lorentz force due to the electromagnetic field (E, B):

$$\mathcal{F}_\alpha(v) = \frac{q_\alpha}{m_\alpha}\Big(E + v \times B\Big),$$

Maxwell's equations

The electromagnetic field (E, B) fulfills the so-called Maxwell equations:

- Maxwell-Gauss:

$$\varepsilon_0 \nabla \cdot E = \mathcal{Q},$$

- Maxwell-Flux:

$$\nabla \cdot B = 0,$$

- Maxwell-Faraday:

$$\nabla \times E = -\partial_t B,$$

- Maxwell-Ampère:

$$\nabla \times B = \mu_0 J + \frac{1}{c^2} \partial_t E,$$

with μ_0 the vacuum permeability, ε_0 the vacuum permittivity, c the speed of light in vacuum, and J the current density.

$$\mu_0 \varepsilon_0 c^2 = 1.$$

Kinetic modelization

- Lorentz forces $\Rightarrow$ kinetic operator of Vlasov type: for $\alpha \in \{i, e\}$,

$$\boxed{(\partial_t + v \cdot \nabla_X + \mathcal{F}_\alpha \cdot \nabla_v) f_\alpha = \Big(\frac{\partial f_\alpha}{\partial t}\Big)_{coll},}$$

with $f_\alpha(x, v, t)$ the density of the species α at $(x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \infty)$.

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- Collision operator of BGK type (for numerical purposes) with the form :

$$\Big(\frac{\partial f_\alpha}{\partial t}\Big)_{coll} = \underbrace{\frac{1}{\tau_\alpha}\Big(\mathcal{M}(f_\alpha) - f_\alpha\Big)}_{\alpha \leftrightarrow \alpha} + \underbrace{\frac{1}{\tau_{\alpha\beta}}\Big(\tilde{\mathcal{M}}(f_\alpha, f_\beta) - f_\alpha\Big)}_{\alpha \leftrightarrow \beta},$$

  with $\{\alpha, \beta\} = \{i, e\}$.

Kinetic modelization

- Monoatomic conservation laws :

$$\int dv \begin{pmatrix} 1 \\ m_e v^2 \\ \frac{1}{2} m_e v^2 \end{pmatrix} (M_e(f_e) - f_e) = 0,$$

- Polyatomic conservation laws :

$$\iint dv dI \begin{pmatrix} 1 \\ m_e v^2 \\ \frac{1}{2} m_e v^2 + I_i^{\frac{2}{\delta_i}} \end{pmatrix} (M_i(f_i) - f_i) = 0,$$

$$M_e(f_e) = \frac{n_e}{\left(\frac{2\pi k T_e}{m_e}\right)^{\frac{3}{2}}} \exp\Big(- m_e \frac{(v-u)^2}{2kT_e} \Big)$$

$$M_i(f_i) = \frac{\rho}{\left(\frac{2\pi k T_i}{m_i}\right)^{\frac{3}{2}}} \frac{1}{\Lambda_i} \exp\Big(- \frac{m_i (v-u)^2 + I_i^{\frac{2}{\delta_i}}}{2kT_i} \Big)$$

Kinetic modelization

- Plasma conservation laws :

$$\iint \begin{pmatrix} m_e & 0 \\ 0 & m_i \\ m_e v & m_i v \\ \frac{1}{2} m_e v^2 & \frac{1}{2} m_i v^2 + I_i^{\frac{2}{\delta_i}} . \end{pmatrix} \cdot \begin{pmatrix} (\widetilde{\mathcal{M}}_{ei}(f_e, f_i) - f_e) dv d\delta_0 \\ (\widetilde{\mathcal{M}}_{ie}(f_e, f_i) - f_i) dv dI_i \end{pmatrix} = 0$$

$$\widetilde{\mathcal{M}}_{\alpha\beta} = \mathcal{M}[\rho_\alpha, u_\#, T_\#, m_\alpha, \mathfrak{m}_\alpha], \{\alpha, \beta\} = \{i, e\},$$

with $u_\#, T_\#$ well-chosen fictitious quantities.

Masses and momenta equations

- Masses equations :

$$\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha u_\alpha) = 0.$$

By summation $\sum_\alpha$:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0.$$

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- Momentum equation for α :

$$\partial_t(\rho_\alpha u_\alpha) + \nabla \cdot (\rho_\alpha u_\alpha \otimes u_\alpha + p_\alpha) - n_\alpha q_\alpha (E + u_\alpha \times B) = \frac{A_\alpha A_\beta}{A_\alpha + A_\beta}(u_\beta - u_\alpha).$$

Total momentum equation

The current density J is defined by :

$$J := \sum_{\alpha} q_\alpha \iint v f_\alpha dv dI_\alpha = n_i e(u_i - u_e).$$

$$(u, J) \mapsto (u_e, u_i) : \begin{cases} u_e = u - \frac{c_i}{n_i e} J, \\ u_i = u + \frac{c_e}{n_i e} J. \end{cases}$$

Hence the total momentum equation :

$$\partial_t(\rho u) + \nabla \cdot \left(\rho u \otimes u + \frac{c_e c_i}{n_i^2 e^2} j \otimes j + p\mathbb{I} \right) + B \times J = 0.$$

Ohm's Law

Operating $\iint \sum_\alpha \frac{q_\alpha}{m_\alpha}(\mathcal{K}_\alpha) dv dI_\alpha$ provides Ohm's Law:

$$\partial_t J + \nabla \cdot \Big(n_i e u \otimes u + (J \otimes u)^s + (c_e - c_i) \frac{J \otimes J}{n_i^2 e^2} + e \big(\frac{p_e}{m_e} - \frac{p_i}{m_i} \big) \mathbb{I} \Big)$$
$$- n_i e^2 \big(\frac{1}{m_e} + \frac{1}{m_i} \big) E - n_i e^2 \Big[\big(\frac{1}{m_e} + \frac{1}{m_i} \big) u + \frac{1}{n_i e} \big(\frac{c_e}{m_i} - \frac{c_i}{m_e} J \big) \Big] \times B$$
$$= \frac{A_e A_i}{A_e + A_i} \frac{\big(\frac{1}{m_i} - \frac{1}{m_e} \big)}{n_i e} J.$$

The energies equations

Partial mechanical energy equation for α with $\iint \frac{1}{2} m_\alpha v^2 (\mathcal{K}_\alpha) dv dI_\alpha$:

$$\partial_t \mathcal{E}_\alpha + \nabla \cdot \Big((\mathcal{E}_\alpha + p_\alpha) u_\alpha \Big) - n_\alpha q_\alpha u_\alpha \cdot E = S_{\alpha\beta},$$

with :

$$S_{\alpha\beta} := \nu_{T,\alpha\beta}(T_\beta - T_\alpha) + \nu_{J\cdot u,\alpha\beta} J \cdot u + \frac{1}{2} \nu_{J^2,\alpha\beta} J^2 = -S_{\beta\alpha},$$

Total mechanical energy equation :

$$\partial_t \mathcal{E} + \sum_\alpha \nabla \cdot \Big((E_\alpha + p_\alpha) u_\alpha \Big) - J \cdot E = 0$$

Introduction

In order to estimate and compare the different parameters, we need dimensionless equations and reference quantities.

Notation

If X is a quantity, $\mathcal{D}(X)$ will be its dimension, $[X]$ its reference counterpart (with the same dimension) and $\{X\} := \frac{X}{[X]}$ the dimensionless quantity derived from X.
If (E) is an equation, we will denote $\mathcal{D}\,|(E)$ its dimensionless counterpart.

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If (E) is an equation, we will denote $\mathcal{D}\,|(E)$ its dimensionless counterpart.

$\rightarrow$ Ex : $\mathcal{D}(u) = L.T^{-1}$. The two reference quantities we will use shall be :

- The reference length for the plasma L_0,
- The reference magnetic amplitude B_0.

First reference quantities

- We will chose $m_0 := [m] := m_i$, and denote $\boxed{\mu := \{m_e\} = \frac{m_e}{m_i}}$ the mass ratio.
- The reference

$$u_0 := \{u\} := \frac{B_0}{\sqrt{m_0 n_0 \mu_0}},$$

  will be the Alfvén velocity.
- $\omega_0 := u_0 L_0^{-1}$,
- The Debye length is the following quantity:

$$\lambda_D := \frac{\varepsilon_0 k_B [T_e]}{[n] e^2},$$

  with k_B the Boltzmann constant.

$\rightarrow \lambda_D << L_0 \Rightarrow$ the plasma can be assumed quasi-neutral.

Dimensionless Maxwell equations

The dimensionless Maxwell equations are :

- Maxwell-Gauss:
$$\mathcal{D}| \; \delta_i^* r^2 \nabla \cdot E = \sigma_0 n,$$
- Maxwell-Flux:
$$\mathcal{D}| \; \nabla \cdot B = 0,$$
- Maxwell-Faraday:
$$\mathcal{D}| \; \partial_t B = -\nabla \times E,$$
- Maxwell-Ampère:
$$\mathcal{D}| \; \nabla \times B = J + r^2 \partial_t E,$$

with $\delta_i^* := \left(\sqrt{\frac{m_0}{[n]\mu_0 e^2}}\right)/L_0$ the plasma skin depth and $r := \frac{u_0}{c}$.

$$\rightsquigarrow \lambda_D/L_0 = r^2 \delta_i^* << 1.$$

Dimensionless Ohm's Law

⇝ Reminder:

$$\partial_t J + \nabla \cdot \Big(n_i e u \otimes u + (J \otimes u)^s + (c_e - c_i) \frac{J \otimes J}{n_i^2 e^2} + e(\frac{p_e}{m_e} - \frac{p_i}{m_i}) \mathbb{I} \Big)$$
$$- n_i e^2 (\frac{1}{m_e} + \frac{1}{m_i}) E - n_i e^2 \Big[(\frac{1}{m_e} + \frac{1}{m_i}) u + \frac{1}{n_i e} (\frac{c_e}{m_i} - \frac{c_i}{m_e} J) \Big] \times B$$
$$= \frac{A_e A_i}{A_e + A_i} \frac{(\frac{1}{m_i} - \frac{1}{m_e})}{n_i e} J.$$

- $\tau_{\alpha\beta}^{-1} \propto \frac{e^4 [n]}{(\varepsilon_0 m_\alpha)^{\frac{1}{2}} T_\alpha^{\frac{3}{2}}}$, so :
- $R_T := \frac{\tau_{ei}}{\tau_{ie}} \propto \sqrt{\mu}$,
- $R_T^0 := \frac{R_T}{\sqrt{\mu}} \propto 1$,
- $c_e = \frac{\mu}{1+\mu}$, $c_i = \frac{1}{1+\mu}$.

Dimensionless Ohm's Law

- Dimensionless equation for J:

$$\mu\partial_t J + \mu\nabla\cdot\Big((J\otimes u)^s\Big) + (\mu-1)\mu\nabla\cdot\Big(\frac{J\otimes J}{n}\Big) + \frac{1}{\delta_i^*}\nabla(\mu p_i - p_e)$$
$$-\frac{1}{(\delta_i^*)^2}(\mu+1)n(E+u\times B) - \frac{1}{\delta_i^*(1+\mu)}(1-\mu)J\times B$$
$$= \frac{1}{\delta_i^*\{\tau_{ie}\}}\frac{R_T^0\sqrt{\mu}}{1+R_T^0\sqrt{\mu}}(-\mu+1)J$$

- Massless approximation:

$$\boxed{\mu \simeq 0.}$$

- Massless Ohm's Law:

$$\rho E = \rho B\times u - \delta_i^*\Big(B\times J + \nabla p_e\Big)$$

Dimensionless Ohm's Law

- Dimensionless equation for J:

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- Massless approximation:

$$\boxed{\mu \simeq 0.}$$

- Massless Ohm's Law:

$$\rho E = \rho B\times u - \delta_i^*\Big(B\times J + \nabla p_e\Big) - \sqrt{\mu}R_T^0\delta_i^*\frac{J}{\{\tau_{ie}\}}.$$

Mechanical energies equations

- We have the source terms estimates in μ :

$$\nu := \nu_{T,\alpha\beta} = \Theta(1),\ \nu_{J\cdot u,\alpha\beta} = O(\mu),\ \nu_{J^2,\alpha\beta} = O(\mu).$$

Mechanical energies equations

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$$\nu := \nu_{T,\alpha\beta} = \Theta(1),\ \nu_{J\cdot u,\alpha\beta} = O(\mu),\ \nu_{J^2,\alpha\beta} = O(\mu).$$

- Quasi-neutral approximation:

$$\boxed{\delta_i^* \simeq 0}$$

- Partial dimensionless mechanical energy equations:

$$\mathcal{D}|\ \partial_t \mathcal{E}_e + \nabla \cdot \Big((\mathcal{E}_e + p_e) u \Big) - u \cdot \nabla p_e = \nu \rho (T_i - T_e),$$

$$\mathcal{D}|\ \partial_t \mathcal{E}_i + \nabla \cdot \Big((\mathcal{E}_i + p_i) u \Big) + u \cdot \nabla p_e + \frac{J}{\rho} \cdot (B \times u) = \nu \rho (T_e - T_i).$$

Electromagnetic energy equations

The dimensionless | electrostatic energy is $\mathcal{E}_E = r^2 \frac{E^2}{2}$. magnetic energy is $\mathcal{E}_m := \frac{B^2}{2}$.

- Equation for $\mathcal{E}_E$:

$$\mathcal{D}| \; \partial_t \mathcal{E}_E - (\nabla \times B) \cdot E + J \cdot E = 0,$$

- Equation for $\mathcal{E}_m$:

$$\mathcal{D}| \; \partial_t \mathcal{E}_m + (\nabla \times E) \cdot B = 0,$$

- Total energy equation :

$$\partial_t(\mathcal{E}_e + \mathcal{E}_i + \mathcal{E}_E + \mathcal{E}_m) + \nabla \cdot \Big((\mathcal{E}_e + p_e + \mathcal{E}_i + p_i)u + E \times B \Big) = 0,$$

- Alternative cationic energy equation:

$$\partial_t(\mathcal{E}_i + \mathcal{E}_E + \mathcal{E}_m) + \nabla \cdot \Big((\mathcal{E}_i + p_i)u + E \times B \Big) + u \cdot \nabla p_e = \nu \rho (T_e - T_i).$$

Asymptotic bi-fluid system

Transverse (E, B) assumptions:

- $E = (E_1, E_2, 0)$, $B = (0, 0, B_3 =: B)$,
- $\partial_3 \equiv 0$.

Dimensionless fluid system for the plasma:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$

$$\partial_t(\rho u) + \nabla \cdot \Big(\rho u \otimes u + p_e + p_i + \mathcal{E}_m\Big) = 0,$$

$$\partial_t \mathcal{E}_e + \nabla \cdot \Big((\mathcal{E}_e + p_e)u\Big) - u \cdot \nabla p_e = \nu\rho(T_i - T_e),$$

$$\partial_t(\mathcal{E}_i + \mathcal{E}_E + \mathcal{E}_m) + \nabla \cdot \Big((\mathcal{E}_i + p_i)u + E^{\perp}B\Big) + u \cdot \nabla p_e = \nu\rho(T_e - T_i),$$

$$\partial_t B + \nabla \cdot (Bu) = 0,$$

with the closure adiabatic relations :

$$p_e = \rho T_e = \frac{2}{3}\mathcal{E}_e,\ p_i = \rho T_i = (\gamma_i - 1)(\mathcal{E}_i - \frac{1}{2}\rho u^2).$$

Plan

The Aregba Natalini procedure

- We consider a conservative hyperbolic system, for the unknown $u : \mathbb{R}^D \times [0, \infty) \to \mathbb{R}^K$:

$$\partial_t u + \sum_d \partial_d F_d(u) = 0,$$

- Aim : construct a kinetic scheme for this system, i.e a kinetic approximation such that the (ε, n) diagramm commutes.

The Aregba Natalini procedure

- Aregba Natalini schemes : class of systematic kinetic splitting schemes procedure by discretization of a kinetic approximation:

$$\partial_t f + \sum_d \Lambda_d \partial_d f = \frac{1}{\varepsilon}\Big(M[Pf := U] - f\Big)$$

- Only requires the choice of :
  - Projection P such that: $Pf = U$,
  - Maxwellian functions M_d such that :

$$\begin{cases} PM(U) = U \\ P\Lambda_d M_d(U) = F_d(U). \end{cases}$$

Modifications of the procedure

Here, two differences :

- We are considering two discretization for each system :

$$\begin{cases} \partial_t \rho^\alpha + \nabla \cdot (\rho^\alpha u^\alpha) = 0, \\ \partial_t (\rho^\alpha u^\alpha) + \nabla \cdot (\rho^\alpha u^\alpha \otimes u^\alpha + p_\alpha) - \frac{n_\alpha q_\alpha}{m_\alpha \delta_i^*}(E - u^{\alpha,\perp} B) = 0, \\ \partial_t \mathcal{E}^\alpha + \nabla \cdot ((\mathcal{E}^\alpha + p^\alpha) u^\alpha) - \frac{n_\alpha q_\alpha}{\delta_i^*} u_\alpha \cdot E = 0, \end{cases}$$

- We need to take into account the nonconservative and source terms, via a splitting for the kinetic discretization :

$$f_j^{\alpha,n+1} = \underbrace{f_j^{\alpha,n} + \sum_d \frac{\Delta t}{\Delta x_d} \Phi_j^n(f^{\alpha,n})}_{\text{Step 1}} + \underbrace{N(E^{n+1}, B^{n+1}) f^{\alpha,n+1} - S(f^{\alpha,n+1})}_{\text{Step 2}}.$$

Choice of the parameters

$$\Lambda_{dl} = \begin{cases} \lambda_d^- \text{ if } l = d \\ \lambda_d^+ \text{ if } l = d+3 \\ 0 \text{ else} \end{cases},$$

$$M^\alpha(U) := \begin{bmatrix} \frac{1}{\lambda_1^+ - \lambda_1^-}\left(\frac{\lambda_1^+}{3}U - F_1(U)\right) \\ \frac{1}{\lambda_2^+ - \lambda_2^-}\left(\frac{\lambda_2^+}{3}U - F_2(U)\right) \\ \frac{\lambda_3^+}{3(\lambda_3^+ - \lambda_3^-)}U \\ \frac{1}{\lambda_1^+ - \lambda_1^-}\left(-\frac{\lambda_1^-}{3}U + F_1(U)\right) \\ \frac{1}{\lambda_2^+ - \lambda_2^-}\left(-\frac{\lambda_2^-}{3}U + F_2(U)\right) \\ -\frac{\lambda_3^-}{3(\lambda_3^+ - \lambda_3^-)}U \end{bmatrix}, P =^T \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 1 : Computing the conservative part

We start from the maxwellian projection :

$$f^{\alpha,n} := M^\alpha(U^{\alpha,n}).$$

We operate a FDS:

$$f_j^{\alpha,n+\frac{1}{2}} = f_j^{\alpha,n} + \sum_{d=1}^{2} \frac{\Delta t}{\Delta x_d}\Big(h^{\alpha,n}_{j-\frac{e_d}{2}} - h^{\alpha,n}_{j+\frac{e_d}{2}}\Big),$$

with $h^{\alpha,n}_{j+\frac{e_d}{2}} := \Lambda_d^{\alpha,+} f_j^{\alpha,n} - \Lambda_d^{\alpha,-} f^{\alpha,n}_{j+\frac{e_d}{2}}$. Applying the projection :

$$U_j^{\alpha,n+\frac{1}{2}} = U_j^{\alpha,n} + \sum_{d} \frac{\Delta t}{\Delta x_d}\Big(Ph^{\alpha,n}_{j-\frac{e_d}{2}} - Ph^{\alpha,n}_{j+\frac{e_d}{2}}\Big).$$

Step 2 : Adding nonconservative and source terms

$$(\rho, B)^{n+1} = (\rho, B)^{n+\frac{1}{2}},$$

$$u^{e,n+1} = u^{e,n+\frac{1}{2}} + \frac{\Delta t}{\delta_i^* \mu}\Big(E^{n+1} - u^{e,n+1,\perp} B^{n+1}\Big),$$

$$u^{i,n+1} = u^{i,n+\frac{1}{2}} - \frac{\Delta t}{\delta_i^*}\Big(E^{n+1} - u^{i,n+1,\perp} B^{n+1}\Big),$$

$$E^{n+1} = E^{n+\frac{1}{2}} + \frac{\rho^{n+1}\Delta t}{\delta_i^*}\Big(u^{e,n+1} - u^{i,n+1}\Big).$$

$\rightsquigarrow$ The three last equations form a closed system at time $n+1$ for $\mathcal{V}^{n+1} :=^T (u^{e,n+1}, u^{i,n+1}, E^{n+1})$.

Adding nonconservative and source terms

$$\begin{aligned}\mathcal{E}^{e,n+1} &= \mathcal{E}^{e,n+\frac{1}{2}} + \frac{\Delta t}{\delta_i^*}\rho^{n+\frac{1}{2}} u^{e,n+\frac{A(e)+1}{2}} \cdot E^{n+1} \\ &+ \frac{2}{3}\nu\Delta t\Big(\mathcal{E}^{e,n+1} - \frac{1}{2}\rho^{n+\frac{1}{2}}\mu u^{e,n+1} \cdot u^{e,n+\frac{1+B(e)}{2}}\Big) \\ &+ (\gamma_i - 1)\nu\Delta t\Big(- \mathcal{E}^{i,n+1} + \frac{1}{2}\rho^{n+\frac{1}{2}} u^{i,n+1} \cdot u^{i,n+\frac{1+B(i)}{2}}\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{E}^{i,n+1} &= \mathcal{E}^{i,n+\frac{1}{2}} - \frac{\Delta t}{\delta_i^*}\rho^{n+\frac{1}{2}} u^{i,n+\frac{A(i)+1}{2}} \cdot E^{n+1} \\ &+ \frac{2}{3}\nu\Delta t\Big(- \mathcal{E}^{e,n+1} + \frac{1}{2}\rho^{n+\frac{1}{2}}\mu u^{e,n+1} \cdot u^{e,n+\frac{1+B(e)}{2}}\Big) \\ &+ (\gamma_i - 1)\nu\Delta t\Big(\mathcal{E}^{i,n+1} - \frac{1}{2}\rho^{n+\frac{1}{2}} u^{i,n+1} \cdot u^{i,n+\frac{1+B(i)}{2}}\Big),\end{aligned}$$

⇒ Necessarily, $A(\alpha), B(\alpha) = 1$.

⇒ The implicit step is well-defined for $\begin{cases} 0 < \Delta t < \frac{1}{-1+\nu(\gamma_i - \frac{1}{3})} \\ 0 < \mu < \frac{1}{2} \end{cases}$.

Letting $\mu \to 0$

- $u^{e,n+\frac{1}{2}} = \Theta(\frac{1}{\mu})$,
- Every other quantity at time $n+\frac{1}{2}, n+1$ is $O(1)$.

We obtain the Ohm Law for $\mu = 0$:

$$0 = \underbrace{\lim_{\mu\to 0}(\mu u^{e,n+\frac{1}{2}})}_{=:l} + \frac{\Delta t}{\delta_i^*}(E^{n+1} - u^{e,n+1,\perp} B^{n+1}).$$

$$\mathcal{E}^{e,n+1} = \mathcal{E}^{e,n+\frac{1}{2}} + \frac{\Delta t}{\delta_i^*}\rho^{n+\frac{1}{2}} u^{e,n+\frac{1}{2}} \cdot E^{n+1} + \frac{2}{3}\nu\Delta t\mathcal{E}^{e,n+1}$$
$$+ (\gamma_i - 1)\nu\Delta t\Big(- \mathcal{E}^{i,n+1} + \frac{1}{2}\rho^{n+\frac{1}{2}}(u^{i,n+1})^2\Big),$$

$$\mathcal{E}^{i,n+1} = \mathcal{E}^{i,n+\frac{1}{2}} - \frac{\Delta t}{\delta_i^*}\rho^{n+\frac{1}{2}} u^{i,n+\frac{1}{2}} \cdot E^{n+1} - \frac{2}{3}\nu\Delta t\mathcal{E}^{e,n+1}$$
$$+ (\gamma_i - 1)\nu\Delta t\Big(\mathcal{E}^{i,n+1} - \frac{1}{2}\rho^{n+\frac{1}{2}}(u^{i,n+1})^2\Big),$$

Letting $\delta_i^* \to 0$

- $u^{e,n+1} - u^{i,n+1} = \Theta(\delta_i^*)$,
- Every quantity at time $n + \frac{1}{2}, n+1$ is $O(1)$.

Expressions for the a priori δ_i^*-stiff terms :

$$0 = \rho^{n+\frac{1}{2}} l \cdot u^{e,n+1} + \frac{\Delta t}{\delta_i^*} \rho u^{e,n+1} \cdot E^{n+1},$$

$$-\frac{\Delta t}{\delta_i^*} \rho^{n+\frac{1}{2}} E^{n+1} \cdot u^{i,n+1} = -\frac{\Delta t}{\delta_i^*} \rho E^{n+1} \cdot u^{e,n+1} + E^{n+1} \cdot (E^{n+\frac{1}{2}} - E^{n+1}).$$

Letting $\delta_i^* \to 0$

Passing to the limit, we obtain the sought discretization :

$$\rho^{n+1} = \rho^{n+\frac{1}{2}},$$

$$u^{n+1} = l + u^{i,n+\frac{1}{2}} (= u^{n+\frac{1}{2}}),$$

$$E^{n+1} = -u^{n+1,\perp} B^{n+1},$$

$$\mathcal{E}^{e,n+1} = \mathcal{E}^{e,n+\frac{1}{2}} - \rho l \cdot u^{e,n+1} + \nu\rho(T^{e,n+1} - T^{i,n+1}),$$

$$\mathcal{E}^{i,n+1} = \mathcal{E}^{i,n+\frac{1}{2}} + \rho l \cdot u^{e,n+1} + E^{n+1} \cdot (E^{n+\frac{1}{2}} - E^{n+1}) + \nu\rho(T^{i,n+1} - T^{e,n+1}).$$