Equivalent systems of kinetic relaxation schemes

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Workshop Schémas numériques de type Boltzmann

Introduction

We consider the scalar conservation law in d dimensions

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = 0, \qquad (\mathcal{E})$$

with $w(\boldsymbol{x}, t) \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{q}(w) \in \mathbb{R}^d$.

- We use a kinetic relaxation scheme $DdQn_v$ which approximates $(\mathcal{E})$ with n_v equations and n_v unknowns. Kinetic models are efficient numerical schemes which use transport at **constant velocities**. However, it can be difficult to analyze them directly.

Introduction

- The solution given by the kinetic model can be approximated by an **equivalent equation** with one unknown w, for example in [Dub08,Gra14].

- In this project, we have proposed an **equivalent system** of n_v variables: w and $n_v - 1$ additional variables.

- We will compare the **subcharacteristic stability condition** given by the analysis of the equivalent equation and the **hyperbolicity condition** given by the equivalent system.

1. Kinetic scheme
   - Kinetic approximation
   - Splitting method
   - Kinetic velocities
   - Flux errors

2. Derivation and comparison of the equivalent equations
   - Computation of the equivalent system and the equivalent equation
   - The $D1Q2$ model
   - The $D2Q3$ model
   - The $D2Q4$ model

Plan

Kinetic approximation

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = 0 \qquad (\mathcal{E})$$

We consider the BGK kinetic model

$$\partial_t f_i + \nabla \cdot (\boldsymbol{\lambda}_i f_i) = \frac{1}{\varepsilon}\left(f_i^{eq} - f_i\right), \qquad \text{for } i = 1, \ldots, n_v, \qquad (\mathcal{K})$$

where

- $\boldsymbol{\lambda}_i$ are the kinetic velocities,
- $\boldsymbol{f} = (f_i)$ is the kinetic unknown,
- $\boldsymbol{f}^{eq} = (f_i^{eq})$ is the equilibrium kinetic vector which satisfies the **consistency relations**

$$w = \sum_{i=1}^{n_v} f_i^{eq} \quad \text{and} \quad \boldsymbol{q}(w) = \sum_{i=1}^{n_v} \boldsymbol{\lambda_i} f_i^{eq}.$$

In the limit $\varepsilon \to 0$, $\sum_{i=1}^{n_v} f_i$ tends to the solution w.

Splitting method

To solve in time the kinetic model

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = \frac{1}{\varepsilon}(f_i^{eq} - f_i), \qquad (\mathcal{K})$$

we apply a splitting method:

- **Transport step** :

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = 0. \qquad (\mathcal{T})$$

We solve exactly these transport equations with the translation

$$f_i^*(\boldsymbol{x}, t + \Delta t) = f_i(\boldsymbol{x} - \Delta t \boldsymbol{\lambda_i}, t).$$

- **Relaxation step** :

$$\partial_t f_i = \frac{1}{\varepsilon}(f_i^{eq} - f_i). \qquad (\mathcal{R}_\omega)$$

We do the relaxation

$$f_i^{n+1} = f_i^* + \omega \left(f_i^{*,eq} - f_i^* \right), \qquad \text{with } \omega \in [1, 2].$$

The kinetic velocities

- In the $D1Q2$ model, we have $n_v = 2$ opposite kinetic velocities :

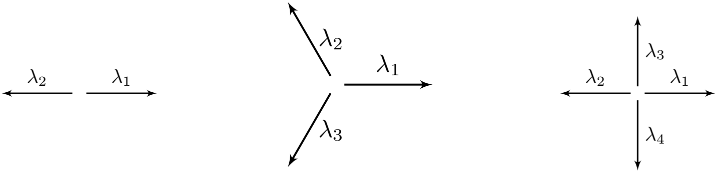
$$\boldsymbol{\lambda}_1 = (\lambda), \quad \boldsymbol{\lambda}_2 = (-\lambda).$$

- In the $D2Q3$ model, we have $n_v = 3$ kinetic velocities :

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} -\frac{\lambda}{2} \\ \frac{\lambda\sqrt{3}}{2} \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} -\frac{\lambda}{2} \\ -\frac{\lambda\sqrt{3}}{2} \end{pmatrix}.$$

- In the $D2Q4$ model, we have $n_v = 4$ velocities along the Cartesian axes :

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} -\lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}, \quad \boldsymbol{\lambda}_4 = \begin{pmatrix} 0 \\ -\lambda \end{pmatrix}.$$



Equilibrium vectors

The **consistency conditions** gives us the system

$$\begin{pmatrix} w \\ q_1(w) \\ q_2(w) \\ z_3^{eq} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & \lambda_{4,1} \\ \lambda_{1,2} & \lambda_{2,2} & \lambda_{3,2} & \lambda_{4,2} \\ m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \end{pmatrix}}_{M} \begin{pmatrix} f_1^{eq} \\ f_2^{eq} \\ f_3^{eq} \\ f_4^{eq} \end{pmatrix}.$$

With the $D2Q4$ model, we are free to choose the third moment and its equilibrium. We choose:

$$m_{i,3} = (\lambda_{i,1})^2 - (\lambda_{i,2})^2 \qquad \text{and} \qquad z_3^{eq} = 0.$$

By inverting the matrix M, we obtain the expression of the equilibrium kinetic functions f_i^{eq}.

Flux errors

We define the **approximated fluxes** as

$$z_k = \sum_{i=1}^{n_v} \lambda_{i,k} f_i, \qquad \text{for } 1 \leqslant k \leqslant d,$$

and the **flux errors** as

$$y_k = z_k - q_k(w), \qquad \text{for } 1 \leqslant k \leqslant d.$$

For the $D2Q4$ model, we add a fourth variable

$$z_3 = \sum_{i=1}^{n_v} (\lambda_{i,1}^2 - \lambda_{i,2}^2) f_i.$$

We will compute the equivalent system in the

$$(w, \boldsymbol{y}) = \Big(w, y_1, y_2, z_3\Big)$$

variables.

Change of variables

The transport step can be rewritten in these variables by

$$\begin{pmatrix} w^*(\boldsymbol{x}, t+\Delta t) \\ y_1^*(\boldsymbol{x}, t+\Delta t) \\ y_2^*(\boldsymbol{x}, t+\Delta t) \\ z_3^*(\boldsymbol{x}, t+\Delta t) \end{pmatrix} = MD(\Delta t)M^{-1} \begin{pmatrix} w \\ y_1 + q_1(w) \\ y_2 + q_2(w) \\ z_3 \end{pmatrix} (\boldsymbol{x}, t) - \begin{pmatrix} 0 \\ q_1(w) \\ q_2(w) \\ 0 \end{pmatrix} (\boldsymbol{x}, t)$$

where D is a diagonal matrix where D_{ii} is a translation operator in the $\boldsymbol{\lambda}_i$ direction. And the relaxation step is

$$\begin{pmatrix} w(\boldsymbol{x}, t+\Delta t) \\ y_1(\boldsymbol{x}, t+\Delta t) \\ y_2(\boldsymbol{x}, t+\Delta t) \\ z_3(\boldsymbol{x}, t+\Delta t) \end{pmatrix} = \begin{pmatrix} w^*(\boldsymbol{x}, t+\Delta t) \\ (1-\omega)y_1^*(\boldsymbol{x}, t+\Delta t) \\ (1-\omega)y_2^*(\boldsymbol{x}, t+\Delta t) \\ (1-\omega)z_3^*(\boldsymbol{x}, t+\Delta t) \end{pmatrix}.$$

Plan

Computation of the equivalent system

We compute the Taylor expansion of

$$\partial_t \begin{pmatrix} w(t) \\ \boldsymbol{y}(t) \end{pmatrix} = \frac{\begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix}(t+\Delta t) - \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix}(t-\Delta t)}{2\Delta t} + O(\Delta t^2).$$

We obtain an **equivalent system on $(w, \boldsymbol{y})$** of the form

$$\partial_t \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} - \frac{\alpha}{\Delta t} \begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix} + \sum_{i=1}^{d} A_i \partial_i \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} + \Delta t \sum_{i,j=1}^{d} B_{ij} \partial_{ij} \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} = O(\Delta t^2).$$

Computation of the equivalent substitution equation

$$\partial_t \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} - \frac{\alpha}{\Delta t} \begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix} + \sum_{i=1}^{d} A_i \partial_i \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} + \Delta t \sum_{i,j=1}^{d} B_{ij} \partial_{ij}^2 \begin{pmatrix} w \\ \boldsymbol{y} \end{pmatrix} = O(\Delta t^2).$$

Now, we assume that $\boldsymbol{y} = O(\Delta t)$, i.e. $\boldsymbol{y} = \Delta t \tilde{\boldsymbol{y}}$. By replacing $\boldsymbol{y}$ in the system, we obtain

$$y_k = \frac{\Delta t}{\alpha} \sum_{i=1}^{d} A_i[k,1] \partial_{x_i} w + O(\Delta t^2).$$

Then, by replacing the y_k in the first equation of the system, we retrieve **the equivalent equation on w** given in [Dub08,Gra14].

$$\partial_t w + \sum_{i=1}^{d} a_i \partial_i w + \Delta t \sum_{i,j=1}^{d} b_{ij} \partial_{ij}^2 w = O(\Delta t^2).$$

Study of the scheme

$(\mathcal{E})$		$DdQn_v$		Equivalent system		Equivalent equation
w	$\rightsquigarrow$	$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$	$\leftrightarrow$	$\begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix}$	$\xrightarrow[\boldsymbol{y}=O(\Delta t)]{}$	w
				$\downarrow$		$\downarrow$
				Hyperbolic stability condition		Diffusive stability condition

Equivalent system for the $D1Q2$ model

For a linear flux $q(w) = cw$, we have the equivalent system

$$\partial_t \begin{pmatrix} w \\ y \end{pmatrix} - \frac{1}{\Delta t} \frac{\omega(2-\omega)(\omega^2 - 2\omega + 2)}{2(\omega-1)^2} \begin{pmatrix} 0 \\ y \end{pmatrix}$$
$$+ \begin{pmatrix} c & \frac{(\omega-2)^2(\omega^2-2\omega+2)}{8(\omega-1)^2} \\ \frac{(\lambda^2-c^2)(\omega-2)^2(\omega^2-2\omega+2)}{8(\omega-1)^2} & -c\frac{\omega^4-4\omega^3+6\omega^2-4\omega+2}{2(\omega-1)^2} \end{pmatrix} \partial_x \begin{pmatrix} w \\ y \end{pmatrix} = O(\Delta t).$$

Hyperbolicity condition

The matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2 - c^2 \end{pmatrix},$$

symmetrize the equivalent system when

$$|c| < \lambda.$$

Therefore, the equivalent system is **hyperbolic** if

$$|c| < \lambda.$$

Equivalent equation of the $D1Q2$ model

By considering $y = O(\Delta t)$, we obtain

$$y = \Delta t \frac{(\lambda^2 - c^2)(\omega - 2)}{4\omega} \partial_x w,$$

which gives us **the equivalent equation on w**

$$\partial_t w + c\partial_x w + \frac{\Delta t}{2} \left(\frac{1}{\omega} - \frac{1}{2} \right) (\lambda^2 - c^2) \partial_{xx} w = O(\Delta t^2).$$

When $\omega \neq 2$, the substitution equation is **stable** under the subcharacteristic condition:

$$|c| < \lambda.$$

Comparative study of the two models

We consider monochromatic exact solutions

$$\begin{pmatrix} w \\ ye^{-\alpha\frac{t}{\Delta t}} \end{pmatrix} = \begin{pmatrix} w_0 \\ y_0 \end{pmatrix} e^{ikx+\gamma t}, \qquad \text{with } k \in \mathbb{N} \text{ and } \gamma \in \mathbb{C}.$$

We obtain the following dispersion relation by injecting this solution

- in the equivalent equation on w

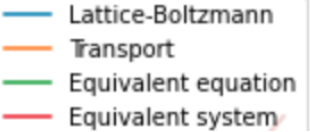
$$(\gamma_{eq} + aik_{eq} - \Delta t bk_{eq}^2)w = 0,$$

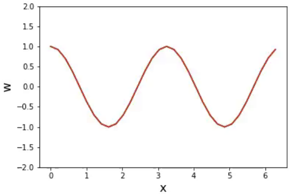
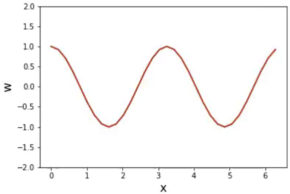
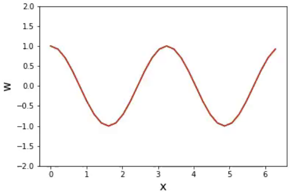
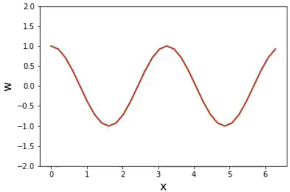
- in the equivalent system on $\left(w, \boldsymbol{y}e^{-\alpha\frac{t}{\Delta t}}\right)$

$$\left(\gamma_{sys} I_2 + Aik_{sys} - \Delta t Bk_{sys}^2\right) \begin{pmatrix} w \\ ye^{-\alpha\frac{t}{\Delta t}} \end{pmatrix} = 0.$$

For the equivalent system, we obtain two γ_{sys}. We choose the one that makes the solution w decrease slowly.

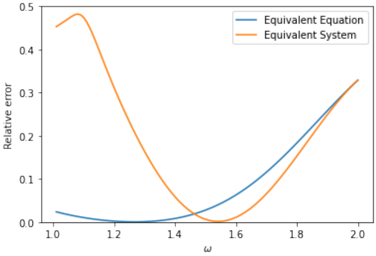
Comparison of the solutions



$\omega = 1.9$	$\omega = 1.7$
	
$\omega = 1.5$	$\omega = 1.3$
	

Comparative study of the two equivalent equations

We computed the relative errors $\dfrac{\sum_{i=0}^{Nx}\sum_{n=0}^{Nt}(w_{LB}^{i,n}-w_{eq}^{i,n})^2}{\sum_{i=0}^{Nx}\sum_{n=0}^{Nt}(w_{LB}^{i,n})^2}$:



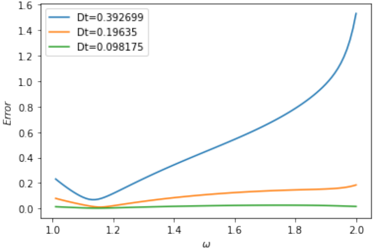
- For little values of ω, the substitution equation appears to be the most accurate.
- For greater values of ω, the equivalent system is more relevant.

Numerical validation of $y = O(\Delta t)$

When we compute the equivalent equation from the system, we suppose that $y = O(\Delta t)$, and it leads to

$$y(t) = \frac{(\lambda^2 - c^2)(\omega - 2)}{4\omega} \Delta t \partial_x w.$$

We can verify if the solution $\begin{pmatrix} w \\ y \end{pmatrix}$ given by the Lattice-Boltzmann method verifies this equation, according to the value of ω. We obtain the following error



Equivalent system when $\omega = 2$

When $\omega = 2$, **the equivalent system of the $D2Q3$ model** is

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} q_1'(w) & 0 & 0 \\ 0 & \frac{\lambda}{2} - q_1'(w) & 0 \\ 0 & -q_2'(w) & -\frac{\lambda}{2} \end{pmatrix}}_{A_1} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} q_2'(w) & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{2} - q_1'(w) \\ 0 & -\frac{\lambda}{2} & -q_2'(w) \end{pmatrix}}_{A_2} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} = O(\Delta t^2).$$

- In green, we retrieve the initial equation $(\mathcal{E})$.

Numerical validation of the equivalent equation

We can compare

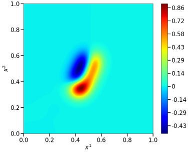
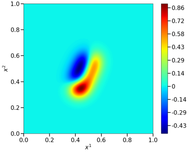
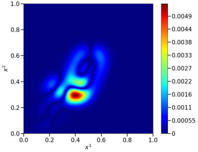
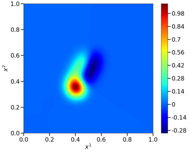
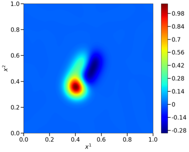
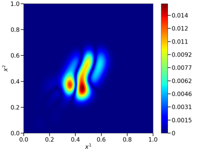
- $\boldsymbol{y}^{vf}$: solution of the equivalent equation with a finite volume method,
- $\boldsymbol{y}^{kin} = \sum_{i=1}^{3} \boldsymbol{\lambda}_i f_i - q(\sum_{i=1}^{3} f_i)$, with $\boldsymbol{f}$ the solution of $(\mathcal{E})$ with the $D2Q3$ model.

We choose $\Omega = [0,1] \times [0,1]$ with a mesh of size 800×800, $\boldsymbol{q}'(w) = (1,1)$, $\lambda = 3$, $T_f = 0.06$ and a Gaussian initialization

$$w(\boldsymbol{x}, 0) = \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{x}_0^w\|^2}{2\sigma^2}\right) \quad \text{and} \quad y_k(\boldsymbol{x}, 0) = \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{x}_0^y\|^2}{2\sigma^2}\right),$$

with $\sigma = 0.05$, $\boldsymbol{x}_0^w = (0.25, 0.25)$ and $\boldsymbol{x}_0^y = (0.5, 0.5)$.

Validation of the equivalent equation

	$\boldsymbol{y}^{kin}$	$\boldsymbol{y}^{vf}$	$\|\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\|$
y_1			
y_2			

$$\|\boldsymbol{y}_1^{kin} - \boldsymbol{y}_1^{vf}\| = 5.64567 \times 10^{-4} \quad \text{and} \quad \|\boldsymbol{y}_2^{kin} - \boldsymbol{y}_2^{vf}\| = 1.95625 \times 10^{-3}$$

- The equivalent equation is a good approximation of the scheme, and therefore it gives useful information in its behavior.

Equivalent system of the $D2Q3$ model

Now, let us consider any relaxation parameter $\omega \in [1, 2]$ and a linear flux $\boldsymbol{q}(w) = \begin{pmatrix} aw \\ bw \end{pmatrix}$. We have

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} - \frac{1}{\Delta t} \frac{\omega(\omega-2)(\omega^2-2\omega+2)}{4(\omega-1)^2} \begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}$$

$$+ \begin{pmatrix} a & -2\gamma_1 & 0 \\ \gamma_1(2a+\lambda)(a-\lambda) & \gamma_2(-a+\frac{\lambda}{2}) & 0 \\ \gamma_1 b(2a+\lambda) & -\gamma_2 b & -\gamma_2 \frac{\lambda}{2} \end{pmatrix} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix}$$

$$+ \begin{pmatrix} b & 0 & -2\gamma_1 \\ \gamma_1 b(2a+\lambda) & 0 & -\gamma_2(a+\frac{\lambda}{2}) \\ \gamma_1(a\lambda+2b^2-\lambda^2) & -\gamma_2\frac{\lambda}{2} & -\gamma_2 b \end{pmatrix} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} = O(\Delta t),$$

with $\gamma_1 = -\frac{(\omega^2-2\omega+2)(\omega-2)^2}{16(\omega-1)^2}$ and $\gamma_2 = \frac{\omega^4-4\omega^3+6\omega^2-4\omega+2}{2(\omega-1)^2}$.

Hyperbolicity condition

We consider a system of the form

$$\partial_t v + A_1 \partial_1 v + A_2 \partial_2 v = 0.$$

- This system is **hyperbolic** if for all unit vector $\boldsymbol{n} = (n_1, n_2)$, the matrix $n_1 A_1 + n_2 A_2$ is diagonalizable in $\mathbb{R}$.

- This system is **symmetrizable** if it exists a symmetric positive definite matrix P such as for all unit vector $\boldsymbol{n} = (n_1, n_2)$, the matrix $P(n_1 A_1 + n_2 A_2)$ is symmetric, or, more simply, such as PA_1 and PA_2 are symmetric.

A symmetrizable system is hyperbolic.

Hyperbolicity condition of the $D2Q3$ model

The matrix

$$P = \begin{pmatrix} \frac{1}{2}\lambda(a^2-2a\lambda-3b^2+\lambda^2)(2a+\lambda) & 0 & 0 \\ 0 & -(a\lambda+2b^2-\lambda^2) & b(2a+\lambda) \\ 0 & b(2a+\lambda) & -(a-\lambda)(2a+\lambda) \end{pmatrix},$$

verifies that PA_1 and PA_2 are symmetric.

Therefore, the equivalent system is **hyperbolic** if

$$\lambda^2 - a^2 - b^2 - \sqrt{(a^2+b^2)^2 + \lambda(-2a^3+6ab^2) + \lambda^2(a^2+b^2)} > 0.$$

Equivalent equation on w

If we assume that $\boldsymbol{y} = O(\Delta t)$, we obtain

$$y_1 = \frac{\Delta t}{2}\left(\frac{1}{\omega} - \frac{1}{2}\right)(2a+\lambda)\left((a-\lambda)\,\partial_1 w + b\partial_2 w\right) + O(\Delta t^2),$$

and

$$y_2 = \frac{\Delta t}{2}\left(\frac{1}{\omega} - \frac{1}{2}\right)\left((2ba+b\lambda)\,\partial_1 w + \left(\lambda a + 2b^2 - \lambda^2\right)\partial_2 w\right) + O(\Delta t^2).$$

By reinjecting these expressions of the y_i in the equivalent system, we retrieve the **equivalent equation on w**

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = \frac{\Delta t}{2}\left(\frac{1}{\omega} - \frac{1}{2}\right)\nabla \cdot (\mathcal{D}_3 \nabla w) + O(\Delta t^2),$$

with the diffusion matrix

$$\mathcal{D}_3 = \begin{pmatrix} \frac{\lambda}{2}(\lambda + a) - a^2 & -\frac{\lambda}{2}b - ab \\ -\frac{\lambda}{2}b - ab & \frac{\lambda}{2}(\lambda - a) - b^2 \end{pmatrix}.$$

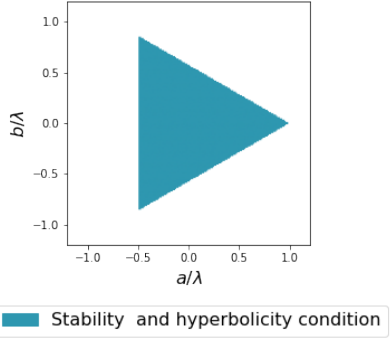
Subcharacteristic stability condition

The model is stable if the diffusion matrix is positive.

When $\omega \neq 2$, the equivalent equation is **stable** if

$$\lambda^2 - a^2 - b^2 - \sqrt{(a^2+b^2)^2 + \lambda(-2a^3 + 6ab^2) + \lambda^2(a^2+b^2)} > 0.$$

- We retrieve exactly the hyperbolicity condition.



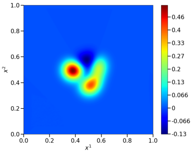
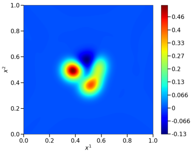
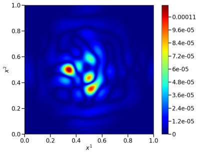
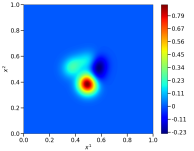
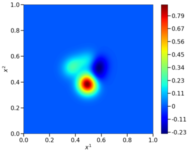
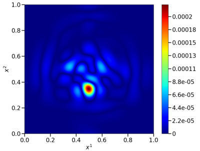
Equivalent system of the $D2Q4$ model when $\omega = 2$

When $\omega = 2$, **the equivalent equation of the $D2Q4$ model** is

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} + \underbrace{\begin{pmatrix} q_1'(w) & 0 & 0 & 0 \\ 0 & -q_1'(w) & 0 & \frac{1}{2} \\ 0 & -q_2'(w) & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \end{pmatrix}}_{A_1} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} q_2'(w) & 0 & 0 & 0 \\ 0 & 0 & -q_1'(w) & 0 \\ 0 & 0 & -q_2'(w) & -\frac{1}{2} \\ 0 & 0 & -\lambda^2 & 0 \end{pmatrix}}_{A_2} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} = O(\Delta t^2).$$

Numerical validation of the equivalent equation

	$\boldsymbol{y}^{kin}$	$\boldsymbol{y}^{vf}$	$\lVert\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\rVert$
y_1			
y_2			

$$\lVert\boldsymbol{y}_1^{kin} - \boldsymbol{y}_1^{vf}\rVert = 1.21999 \times 10^{-5} \quad \text{and} \quad \lVert\boldsymbol{y}_2^{kin} - \boldsymbol{y}_2^{vf}\rVert = 1.57384 \times 10^{-5}$$

- The equivalent equation is a good approximation of the scheme, and therefore it gives useful information in its behavior.

Equivalent system for the $D2Q4$ system

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} - \frac{1}{\Delta t} \frac{\omega(\omega-2)(\omega^2-2\omega+2)}{4(\omega-1)^2} \begin{pmatrix} 0 \\ y_1 \\ y_2 \\ z_3 \end{pmatrix}$$

$$+ \begin{pmatrix} a & 2\gamma_1 & 0 & 0 \\ \gamma_1(\lambda^2-2a) & -a\gamma_2 & 0 & \frac{\gamma_2}{2} \\ -2ab\gamma_1 & -b\gamma_2 & 0 & 0 \\ 2\lambda^2 a\gamma_1 & \lambda^2\gamma_2 & 0 & 0 \end{pmatrix} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix}$$

$$+ \begin{pmatrix} b & 0 & 2\gamma_1 & 0 \\ -2ab\gamma_1 & 0 & -a\gamma_2 & 0 \\ \gamma_1(\lambda^2-2b^2) & 0 & -b\gamma_2 & -\frac{\gamma_2}{2} \\ -2\lambda^2 b\gamma_1 & 0 & -\lambda^2\gamma_2 & 0 \end{pmatrix} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} = O(\Delta t)$$

with $\gamma_1 = \frac{(\omega-2)^2(\omega^2-2\omega+2)}{16(\omega-1)^2}$ and $\gamma_2 = \frac{\omega^4-4\omega^3+6\omega^2-4\omega+2}{2(\omega-1)^2}$.

Hyperbolicity condition for the $D2Q4$ model

The matrix

$$P = \begin{pmatrix} \lambda^2(4a^2-\lambda^2)(4b^2-\lambda^2) & 0 & 0 & 0 \\ 0 & -2\lambda^2(4b^2-\lambda^2) & 0 & 2a(4b^2-\lambda^2) \\ 0 & 0 & -2\lambda^2(4a^2-\lambda^2) & -2b(4a^2-\lambda^2) \\ 0 & 2a(4b^2-\lambda^2) & -2b(4a^2-\lambda^2) & -2a^2-2b^2+\lambda^2 \end{pmatrix},$$

verifies that PA_1 and PA_2 are symmetric.

The equivalent system is **hyperbolic** if

$$\max(|a|, |b|) < \frac{\lambda}{2}.$$

Equivalent equation on w

If we assume that the $\boldsymbol{y} = O(\Delta t)$, then we obtain an expression of $\boldsymbol{y}$ that we can reinject in the equivalent system.

Then, we retrieve the **equivalent equation on** w

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = \frac{\Delta t}{2} \left(\frac{1}{\omega} - \frac{1}{2} \right) \nabla \cdot (\mathcal{D}_4 \nabla w) + O(\Delta t^2),$$

with the diffusion matrix

$$\mathcal{D}_4 = \begin{pmatrix} \frac{\lambda^2}{2} - a^2 & -ab \\ -ab & \frac{\lambda^2}{2} - b^2 \end{pmatrix}.$$

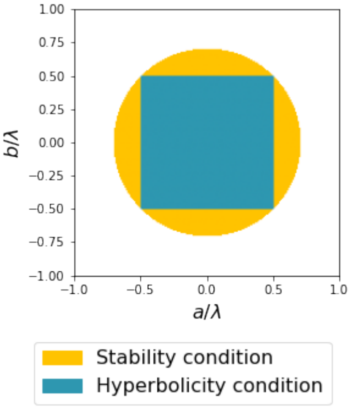
Subscharacteristic stability condition

When $\omega \neq 2$, the **subcharacteristic diffusive stability condition** is

$$a^2 + b^2 \leqslant \frac{\lambda^2}{2}.$$

- This condition is less restrictive than the hyperbolicity condition.

What happens when the diffusive stability condition is satisfied but not the hyperbolicity condition ?

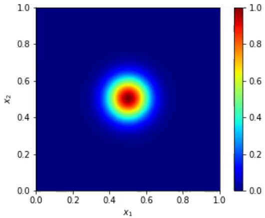
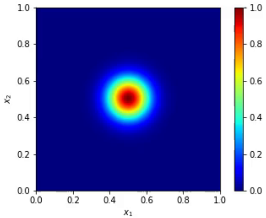


$\omega = 2$

We choose the velocity $\begin{cases} a = 1, \\ b = 0. \end{cases}$

Diffusive stability condition: $\lambda > \sqrt{2(a^2+b^2)} = \sqrt{2}$

Hyperbolic stability condition: $\lambda > 2\max(|a|,|b|) = 2$

$\lambda = 1.6$	$\lambda = 2.2$
	

$\omega = 1.6$

We choose the velocity $\begin{cases} a = 1, \\ b = 0. \end{cases}$

Stability condition:	$\lambda > \sqrt{2(a^2 + b^2)} = \sqrt{2}$
Hyperbolicity condition:	$\lambda > 2\max(\lvert a\rvert, \lvert b\rvert) = 2$

<table>
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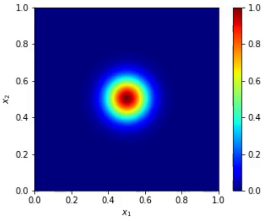
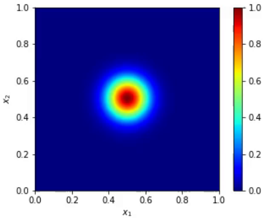
<tr><td></td><td></td></tr>
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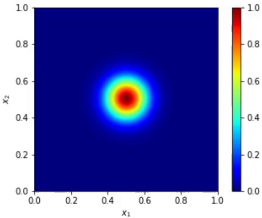
$\omega = 1.2$

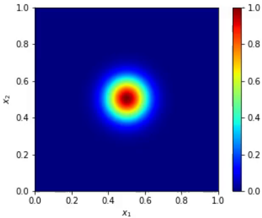
We choose the velocity $\begin{cases} a = 1, \\ b = 0. \end{cases}$

Stability condition: $\lambda > \sqrt{2(a^2 + b^2)} = \sqrt{2}$
Hyperbolicity condition: $\lambda > 2\max(|a|, |b|) = 2$

$\lambda = 1.6$



$\lambda = 2.2$



Conclusion and perspectives

- We have shown that the classical stability condition does not give a stable solution in some case.

- We have computed an equivalent system on n_v variables that gives us a more restrictive condition to choose λ. This condition gives us stable solution.

- We have compared the domains of validity of the equivalent equation and the equivalent system, and we would like now to find a quantitative criterion which gives us the best choice according to ω.

Thank you for your attention !

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