# Asymptotic-Preserving scheme for the $M_1$-Maxwell system in the quasi-neutral regime.

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**Abstract.** This work deals with the numerical resolution of the $M_1$-Maxwell system in the quasi-neutral regime. In this regime the stiffness of the stability constraints of classical schemes causes huge calculation times. That is why we introduce a new stable numerical scheme consistent with the transitional and limit models. Such schemes are called Asymptotic-Preserving (AP) schemes in literature. This new scheme is able to handle the quasi-neutrality limit regime without any restrictions on time and space steps. This approach can be easily applied to angular moment models by using a moments extraction. Finally, two physically relevant numerical test cases are presented for the Asymptotic-Preserving scheme in different regimes. The first one corresponds to a regime where electromagnetic effects are predominant. The second one on the contrary shows the efficiency of the Asymptotic-Preserving scheme in the quasi-neutral regime. In the latter case the illustrative simulations are compared with kinetic and hydrodynamic numerical results.

**Key words**: Asymptotic-Preserving scheme, Fokker-Planck-Landau equation, Maxwell equations, quasi-neutral limit, angular $M_1$ moments model.

## 1 Introduction

This work deals with non-homogeneous collisional plasmas described by a kinetic model. The plasma is considered as a mixture of electrons and ions. Each species is characterised by its distribution function which corresponds to particles density in the phase space. In this work the $M_1$ angular moments model describes electron transport and considers binary collisions between particles whereas Maxwell equations are used to describe the evolution of electromagnetic fields. For the sake of simplicity, we assume that the plasma consists of electrons and one ion species considered as immobile. This approximation is

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relevant due to the important mass of ions compared to the electrons mass. This means our model is valid on time scales during which the ions motion can be neglected.

For the study of collisional processes, the two important physical scales are the mean free path and the electron-ion collision frequency. The mean free path represents the average distance travelled by an electron between two collisions with an ion. The electron-ion collision frequency represents the number of electron-ion collision per unit of time. When the electronic plasma period is very small compared to the electron-ion collisional period and the Debye length is very small compared to the mean free path, the plasma is designated as quasi-neutral and the Maxwell-Gauss (also called Maxwell-Poisson) and Maxwell-Ampere equations degenerate into algebric equations on collisional time scales.

Therefore to handle this type of situation a new class of methods, called Asymptotic-Preserving (AP) methods has been developed. These methods have been introduced firstly by Shi Jin ( [25]) in the context of diffusive limits for kinetic equations. Consider a system $(S_\alpha)$ depending on a parameter $\alpha$ and $(S_0)$ the corresponding limit system when $\alpha$ tends to zero. In our case $\alpha$ is the ratio between the Debye length and the mean free path. A numerical scheme with time step $\Delta t$ and space step $\Delta x$ is called Asymptotic-Preserving in the limit $\alpha$ tends to zero for the system $(S_\alpha)$ if the scheme is stable independently of the values taken by $\alpha$ and if the limit scheme obtained for $\alpha=0$ is consistent with the limit problem $(S_0)$. In this work the system $(S_\alpha)$ corresponds to the Fokker-Planck-Landau-Maxwell system and $(S_0)$ corresponds to the Fokker-Planck-Landau-Maxwell system in the quasi-neutral limit. This regime has been already studied in the context of fluid models ( [10, 12, 17]). For example in ( [12]), the authors considered a two fluid isentropic Euler system coupled with the Poisson equation. It is shown that the Maxwell-Poisson equation can be reformulated into an elliptic equation which does not degenerate at the quasi-neutral limit. In ( [11]), this approach is generalised to the Euler-Maxwell model with a strong magnetic field. A kinetic model consisting in a two fluid Vlasov-Poisson system has also been investigated in ( [14]). In ( [16]), an Asymptotic-Preserving scheme is proposed for the Euler-Maxwell system in the quasi-neutral regime. The Maxwell equations are reformulated to enable the computation of the electrostatic field even in the limit regime. The development followed to express the electric field is well known in physics ( [9], [3]).

The present paper deals with the construction of an Asymptotic-Preserving scheme or the $M_1$-Maxwell system in the quasi-neutral limit. The strategy adopted is similar to the one in ( [16]), nevertheless to our knowledge, it is the first time that such schemes are considered for kinetic models with true collision operators. This fact is very important to deal with collisional plasma because the collision frequency $\nu$ must follow the Coulombian interaction law $(\nu \approx 1/|v|^3)$. To perform realistic simulations in plasma physics, Coulombian interactions must be used. Therefore, relaxation operators are not relevant from a physical point of view. Moreover up to now, Asymptotic-Preserving schemes for the quasi-neutral limit have been developed either for fluid description of plasma or for collisionless plasmas. Asymptotic-Preserving schemes have been recently used for numerous applications in the context of strong magnetic fields ( [5–7, 15]) for the gyro-fluid

limit as well as in fluid mechanics for the hydrodynamic limit ( [25–27]). Other applications can be found for example in [13,18,23].

Kinetic descriptions are accurate but can be too numerically expensive to be used for many real physical applications. An alternative way could be to consider a fluid description based on average quantities. Nevertheless, macroscopic descriptions are often not accurate enough. For example, in the context of inertial confinement fusion, the distribution functions considered can be far from equilibrium and in this case the fluid description is not adapted. Moreover kinetic effects like non local transport ( [4,31]) or the development of some instabilities ( [19]) can be important on long collisional time scales and are not captured by fluid simulations. At the same time, kinetic codes are usually limited to short time scales and cannot reach time scales studied by fluid simulations. It is therefore an important challenge to describe kinetic effects using reduced kinetic codes on fluid time scales. Angular moments models represent intermediate models between the kinetic and fluid levels. They are less numerically expensive than kinetic models and more accurate than fluid models. They are constructed by using an angular moments extraction ( [28,32]) from the kinetic equations. But, there exists several moment models whose differences come from the choice of the closure. For example, the very popular $P_N$ closure ( [24]) does not ensure the positivity of the distribution function. Hence we consider in this paper a $M_1$ moments model ( [20,21,29]) based on an entropy minimisation principle. A $M_1$-Asymptotic-Preserving scheme is therefore derived following the same method as in the kinetic case.

The paper is organised as follows. Section 2 introduces the Fokker-Planck-Landau-Maxwell system and its quasi-neutral limit. A reformulation of the Fokker-Planck-Landau-Maxwell system is presented in the case of one dimension in space and one dimension in velocity. The model is considered with electric fields and collision operators. Then, the method is generalised for full multi-dimensions problems with electromagnetic fields and collision operators. Section 3 introduces in detail the numerical construction of an Asymptotic-Preserving scheme for the reformulated system of section 2. Section 4 deals with the construction of an Asymptotic-Preserving scheme for the $M_1$ moments model from the kinetic one. Finally, section 5 presents two physically relevant numerical test cases for the $M_1$-Asymptotic-Preserving scheme for different regimes. The first one corresponds to a regime where electromagnetic effects are predominant whereas the second one on the contrary shows the efficiency of the Asymptotic-Preserving scheme in the collisional quasi-neutral regime. The numerical results are compared with kinetic and hydrodynamic numerical results.

# 2 The Fokker-Planck-Landau-Maxwell system and its quasi-neutral limit

We firstly present the kinetic model then we introduce the quasi-neutral limit. It is important to notice, we work with true Coulombian collision operators.

## 2.1 The Fokker-Planck-Landau-Maxwell system.

Consider a kinetic model for a plasma constituted of electrons and one ion species considered fixed. Therefore the description is performed with a non-negative distribution function for electrons $f_e(\mathbf{x},\mathbf{v},\mathbf{t})$, $\mathbf{x}\in\mathbf{R^n}$ represents the space variable, $\mathbf{v}\in\mathbf{R^n}$ is the velocity variable, $n=1,2$ or 3 and $t$ is the time. The mass and the charge of the electron are respectively written $m_e$ and $q_e$. The coefficient $c$ represents the speed of light in vacuum. The Fokker-Planck-Landau-Maxwell system writes

$$\frac{\partial f_e}{\partial t}+\mathbf{v}.\nabla_{\mathbf{x}}f_e+\frac{q}{m_e}(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}}f_e=C_{ee}(f_e,f_e)+C_{ei}(f_e), \tag{2.1}$$

$$\frac{\partial \mathbf{E}}{\partial t}-c^2\nabla_{\mathbf{x}}\times\mathbf{B}=-\frac{\mathbf{j}}{\varepsilon_0}, \tag{2.2}$$

$$\nabla_{\mathbf{x}}.\mathbf{E}=\frac{q_e}{\varepsilon_0}(n_e-n_i), \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla_{\mathbf{x}}\times\mathbf{E}=0, \tag{2.4}$$

$$\nabla_{\mathbf{x}}.\mathbf{B}=0, \tag{2.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ represent respectively electric and magnetic fields. $C_{ee}$ and $C_{ei}$ are the electron-electron and electron-ion collision operators. The expression of $C_{ee}$ and $C_{ei}$ are given by

$$C_{ee}(f_e,f_e)=\nu_{ee}div_{\mathbf{v}}\Big(\int_{\mathbf{v}'\in\mathbb{R}^n}S(\mathbf{v}-\mathbf{v}')[\nabla_{\mathbf{v}}f_e(\mathbf{v})f_e(\mathbf{v}')-f_e(\mathbf{v})\nabla_{\mathbf{v}}f_e(\mathbf{v}')]d\mathbf{v}'\Big), \tag{2.6}$$

$$C_{ei}(f_e)=\nu_{ei}div_{\mathbf{v}}\Big[S(\mathbf{v})\nabla_{\mathbf{v}}f_e(\mathbf{v})\Big], \tag{2.7}$$

where

$$S(\mathbf{u})=\frac{1}{|\mathbf{u}|^3}(|\mathbf{u}|^2Id-\mathbf{u}\otimes\mathbf{u}) \tag{2.8}$$

is the Landau tensor and $Id$ is the unit tensor. The parameters $\nu_{ee}$ and $\nu_{ei}$ are positive physical constants.

The electronic density $n_e$ and the electronic current $\mathbf{j}$ write

$$n_e(\mathbf{x},t) = \int_{R^n} f_e(\mathbf{x},\mathbf{v},t)d\mathbf{v}, \qquad \mathbf{j}(\mathbf{x},t) = q_e \int_{\mathbb{R}^n} f_e(\mathbf{x},\mathbf{v},t)\mathbf{v}d\mathbf{v}. \tag{2.9}$$

Since the ions are supposed fixed, the ion density $n_i$ is considered as a known function of space.

## 2.2 Properties of the collision operators.

The electron-electron collision operator satisfies mass, momentum and energy conservation properties

$$\int_{\mathbb{R}^n} C_{ee}(f_e,f_e) \begin{pmatrix} 1 \\ \mathbf{v} \\ \mathbf{v}^2 \end{pmatrix} d\mathbf{v} = 0, \tag{2.10}$$

while the electron-ion collision operator satisfies only mass and energy conservation

$$\int_{\mathbb{R}^n} C_{ei}(f_e) \begin{pmatrix} 1 \\ \mathbf{v}^2 \end{pmatrix} d\mathbf{v} = 0. \tag{2.11}$$

They both dissipate the entropy i.e.

$$\int_{\mathbb{R}^n} C_{ei}(f_e)\ log\, f_e\, d\mathbf{v} \leq 0, \qquad \int_{\mathbb{R}^n} C_{ee}(f_e,f_e)\ log\ \ f_e\ d\mathbf{v} \leq 0, \tag{2.12}$$

which implies that the Boltzmann entropy

$$\mathcal{H}(f_e) = \int_{\mathbb{R}^n} (f_e\ log f_e\ -\ f_e)d\mathbf{v} \tag{2.13}$$

is a Lyapunov function for equation (2.1).

*Properties.*

*(i)* The equilibrium states of the electron-ion collision operator $C_{ei}$ $\Big($i.e. $C_{ei}(f_e) = 0\Big)$ are given by the set of isotropic functions $f_e(\mathbf{v}) = f_e(|\mathbf{v}|)$.

*(ii)* The equilibrium states of the electron-electron collision operator $C_{ee}$ $\Big($i.e. $C_{ee}(f_e,f_e) = 0\Big)$ are given by the Maxwellian distribution functions

$$f_e(\mathbf{v}) = n_e\ \ (\frac{m_e}{2\pi k_B T})^{3/2}\ \ exp\Big(-\frac{m_e(\mathbf{v}-\mathbf{u})^2}{2k_B T}\Big), \tag{2.14}$$

where $k_B$ is the Boltzmann constant, $n_e$ is the electronic density, $T$ is the temperature and $\mathbf{u}$ represents the mean velocity.

*(iii)* The equilibrium states for both collision operators $\Big($i.e. $C_{ee}(f_e,f_e)+C_{ei}(f_e)=0\Big)$ are given by the isotropic Maxwellian distribution function

$$f_e(\mathbf{v}) = n_e \ (\frac{m_e}{2\pi k_B T})^{3/2} \ exp\Big(-\frac{m_e \mathbf{v}^2}{2k_B T}\Big). \tag{2.15}$$

## 2.3 Scaling for the analysis of collisional processes.

For the analysis of collisional processes three important parameters are introduced: the mean free path $\lambda_{ei}$ which represents the average distance travelled by an electron between two collisions, the thermal velocity $v_{th}$ and the electron-ion collision frequency $\nu_{ei}$. They satisfy the relations

$$v_{th} = \sqrt{\frac{k_B T}{m_e}}, \qquad \nu_{e,i} = \frac{v_{th}}{\lambda_{e,i}}. \tag{2.16}$$

These parameters enable us to scale time, space and speed

$$\tilde{t} = \nu_{e,i} t, \quad \tilde{x} = x/\lambda_{e,i}, \quad \tilde{v} = v/v_{th}. \tag{2.17}$$

In the same way, we scale the electric field, the magnetic field and the distribution function

$$\tilde{\mathbf{E}} = \frac{e\mathbf{E}}{m_e v_{th} \nu_{e,i}}, \quad \tilde{\mathbf{B}} = \frac{e\mathbf{B}}{m_e \nu_{e,i}}, \quad \tilde{f} = f_e \frac{v_{th}^3}{n_0}. \tag{2.18}$$

$n_0$ is the initial electronic density.

With these dimensionless quantities the system (2.1) becomes the following system where we have omitted the tildes

$$\begin{cases} \dfrac{\partial f}{\partial t}+\mathbf{v}.\nabla_{\mathbf{x}} f-(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}} f=\dfrac{1}{Z}C_{e,e}(f,f)+C_{e,i}(f)\,, \\ \dfrac{\partial \mathbf{E}}{\partial t}-\dfrac{1}{\beta^2}\nabla_{\mathbf{x}}\times\mathbf{B}=-\dfrac{\mathbf{j}}{\alpha^2}, \\ \dfrac{\partial \mathbf{B}}{\partial t}+\nabla_{\mathbf{x}}\times\mathbf{E}=0, \\ \nabla_{\mathbf{x}}.\mathbf{E}=\dfrac{1}{\alpha^2}(1-n), \\ \nabla_{\mathbf{x}}.\mathbf{B}=0, \end{cases} \tag{2.19}$$

where $\alpha=\frac{v_{e,i}}{\omega_{pe}}$, $\omega_{pe}$ represents the electronic plasma frequency, $\beta=v_{th}/c$, $n=n_e/n_0$ and $Z$ the charge of the ions. In this work $Z$ is taken equal to 1.

The dimensionless collision operators $C_{ee}(f,f)$ and $C_{ei}(f)$ write like in (2.6) for $\nu_{ee}=\nu_{ei}=1$.

## 2.4 The electrostatic case.

In the electrostatic case with only one dimension for space ($x\in\mathbb{R}$) and one for velocity ($v\in\mathbb{R}$), the system (2.19) can be written in the following form

$$\begin{cases} \dfrac{\partial f}{\partial t}+v\partial_x f-E\partial_v f=C_{e,e}(f,f)+C_{e,i}(f), \\ \dfrac{\partial E}{\partial t}=-\dfrac{j}{\alpha^2}, \end{cases} \tag{2.20}$$

where Maxwell-Poisson has to be satisfied at initial time.

**Remark 1.** Notice that the fourth equation of system (2.19), called Maxwell-Gauss equation (or Poisson equation) is not used. Indeed, the second equation of (2.19), called Maxwell-Ampere equation and Poisson equation are equivalent if Poisson equation is verified at initial time.

The limit system $(S_0)$ is obtained when the parameter $\alpha$ tends to 0 and corresponds to the quasi-neutral limit. It can be written in the form

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\partial_x f - E\partial_v f = C_{e,e}(f,f) + C_{e,i}(f), \\ \\ j = 0, \end{cases} \tag{2.21}$$

with $n = 1$ at initial time.

When $\alpha$ tends to zero the Maxwell-Poisson equation degenerates into the algebric equation $n = 1$. This condition has to be satisfied at initial time.

When $\alpha$ is equal to zero we lose the possibility to obtain the electric field from the Maxwell-Ampere equation on collisional time scale. The limit is a singular limit, because the Maxwell-Ampere equation degenerates into an algebric equation.

## 2.5 Reformulation of the Maxwell-Ampere equation in the simplified case.

The aim of this part is to provide a reformulation of the Maxwell-Ampere equation that is equivalent and contains explicitly the quasi-neutral limit as a particular case when $\alpha = 0$ for the electrostatic case with only one dimension for space and one for the velocity. Multiplying the first equation of (2.20) by $v$, integrating in velocity and using the definition of the dimensionless current

$$j = -\int_{\mathbb{R}} f v dv, \tag{2.22}$$

we obtain

$$-\frac{\partial j}{\partial t} + \partial_x \left( \int_{\mathbb{R}} v^2 f dv \right) - E \int_{\mathbb{R}} v \partial_v f dv = \int_{\mathbb{R}} C_{e,i} v dv. \tag{2.23}$$

In (2.10) we have seen that

$$\int_{\mathbb{R}} v\, C_{e,e}(f,f) dv = 0. \tag{2.24}$$

It is important to notice that it is not the case for the electron-ion collision operator $C_{e,i}$ (see 2.11).

The derivation in time of the Maxwell-Ampere equation in the electrostatic case leads to

$$\frac{\partial j}{\partial t} = -\alpha^2 \frac{\partial^2 E}{\partial t^2}. \tag{2.25}$$

By using (2.23), we get

$$\alpha^2\frac{\partial^2 E}{\partial t^2}-E\int_{\mathbb{R}} v\partial_v f dv=-\partial_x(\int_{\mathbb{R}} v^2 f dv)+\int_{\mathbb{R}} C_{e,i}vdv. \tag{2.26}$$

As

$$E\int_{\mathbb{R}} v\partial_v f\, dv=-nE, \tag{2.27}$$

the equation (2.26) becomes

$$\alpha^2\frac{\partial^2 E}{\partial t^2}+nE=-\partial_x(\int_{\mathbb{R}} v^2 f)dv+\int_{\mathbb{R}} C_{e,i}vdv. \tag{2.28}$$

When the parameter $\alpha$ tends to 0, we find the limit problem

$$nE=-\partial_x(\int_{\mathbb{R}} v^2 f dv)+\int_{\mathbb{R}} C_{e,i}vdv. \tag{2.29}$$

So the electrostatic field writes

$$E=\frac{-\partial_x(\int_{\mathbb{R}} v^2 f dv)+\int_{\mathbb{R}} C_{e,i}vdv}{n}. \tag{2.30}$$

In this part we have shown that the Fokker-Planck-Landau-Maxwell system (2.20) is **equivalent** to the **Fokker-Planck-Landau-Maxwell reformulated system**

$$\begin{cases} \dfrac{\partial f}{\partial t}+\partial_x(vf)-\partial_v(Ef)=C_{e,e}(f,f)+C_{e,i}(f), \\ \alpha^2\dfrac{\partial^2 E}{\partial t^2}+nE=-\partial_x(\displaystyle\int_{\mathbb{R}} v^2 f)dv+\int_{\mathbb{R}} C_{e,i}vdv, \end{cases} \tag{2.31}$$

where Maxwell-Poisson has to be satisfied at initial time.

The limit system when $\alpha\to 0$ is the following one

$$\begin{cases} \dfrac{\partial f}{\partial t}+\partial_x(vf)-\partial_v(Ef)=C_{e,e}(f,f)+C_{e,i}(f), \\ E=\dfrac{-\partial_x(\int_{\mathbb{R}} v^2 f dv)+\int_{\mathbb{R}} C_{e,i}vdv}{n}, \end{cases} \tag{2.32}$$

where $n=1$ and $j=0$ at initial time.

The second equation of (2.20) imposes $j=0$ when $\alpha=0$. This condition has to be satisfied at initial time.

## 2.6 Reformulation of the Maxwell-Ampere equation in the general case.

In this part, we generalise the method of the previous part to a non-homogeneous collisional plasma with magnetic field in multi-dimensions.

Multiplying this first equation of (2.19) by $-\mathbf{v}$, integrating in velocity and using the definition of the dimensionless current (2.22) we get

$$-\frac{\partial \mathbf{j}}{\partial t}+div_{\mathbf{x}}(\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})-\int_{\mathbb{R}^n}\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}}f d\mathbf{v}=\int_{\mathbb{R}^n}C_{e,i}(f)\mathbf{v}d\mathbf{v}. \tag{2.33}$$

As

$$\int_{\mathbb{R}^n}(\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}}f\,\mathbf{v}\,d\mathbf{v}=\mathbf{j}\times\mathbf{B}, \tag{2.34}$$

the same development as in the electrostatic case is performed.

The derivation in time of the Maxwell-Ampere equation in the general case leads to

$$\frac{\partial \mathbf{j}}{\partial t}=-\alpha^2\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{\alpha^2}{\beta^2}\big[\nabla_{\mathbf{x}}\times\frac{\partial \mathbf{B}}{\partial t}\big]. \tag{2.35}$$

Finally the following form is obtained

$$\alpha^2\frac{\partial^2\mathbf{E}}{\partial t^2}+n_e\mathbf{E}-\mathbf{j}\times\mathbf{B}=-div_{\mathbf{x}}(\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})+\frac{\alpha^2}{\beta^2}\big[\nabla_{\mathbf{x}}\times\frac{\partial \mathbf{B}}{\partial t}\big]+\int_{\mathbb{R}^n}C_{e,i}(f)\mathbf{v}d\mathbf{v}. \tag{2.36}$$

When $\alpha$ tends to 0 in (2.36) we find the limit problem

$$n_e\mathbf{E}=-div_{\mathbf{x}}(\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})+\int_{\mathbb{R}^n}C_{e,i}(f)\mathbf{v}d\mathbf{v}+\mathbf{j}\times\mathbf{B}. \tag{2.37}$$

So the electrostatic field writes

$$\mathbf{E}=\frac{-div_{\mathbf{x}}(\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})+\int_{\mathbb{R}^n}C_{e,i}(f)\mathbf{v}d\mathbf{v}+\mathbf{j}\times\mathbf{B}}{n_e}. \tag{2.38}$$

In this part we have shown that the Fokker-Planck-Landau-Maxwell system (2.19) is **equivalent** to the **Fokker-Planck-Landau-Maxwell reformulated system**

$$\begin{cases} \dfrac{\partial f}{\partial t}+\mathbf{v}.\nabla_{\mathbf{x}} f-(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}} f=C_{e,e}(f,f)+C_{e,i}(f), \\ \alpha^2\dfrac{\partial^2\mathbf{E}}{\partial t^2}+n_e\mathbf{E}-\mathbf{j}\times\mathbf{B}=-div_{\mathbf{x}}(\displaystyle\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})+\frac{\alpha^2}{\beta^2}\Big[\nabla_{\mathbf{x}}\times\frac{\partial\mathbf{B}}{\partial t}\Big]+\int_{\mathbb{R}^n}C_{e,i}(f)\mathbf{v}d\mathbf{v}, \\ \dfrac{\partial\mathbf{B}}{\partial t}+\nabla_{\mathbf{x}}\times\mathbf{E}=0, \end{cases} \tag{2.39}$$

where Maxwell-Poisson and Maxwell-Thomson have to be satisfied at initial time.

**Remark 2.** The fifth equation of the system (2.19) called Maxwell-Thomson equation is not used. Indeed the third and fifth equation of (2.19) called Maxwell-Faraday equation and Maxwell-Thomson equation are equivalent if Maxwell-Thomson equation is verified at initial time.

The limit system of (2.39) when $\alpha\to 0$ is the following one

$$\begin{cases} \dfrac{\partial f}{\partial t}+\mathbf{v}.\nabla_{\mathbf{x}} f-(\mathbf{E}+\mathbf{v}\times\mathbf{B}).\nabla_{\mathbf{v}} f=C_{e,e}(f,f)+C_{e,i}(f), \\ n_e\mathbf{E}-\mathbf{j}\times\mathbf{B}=-div_{\mathbf{x}}(\displaystyle\int_{\mathbb{R}^n}\mathbf{v}\otimes\mathbf{v}f d\mathbf{v})+\int_{\mathbb{R}^n}C_{e,i}\mathbf{v}d\mathbf{v}, \\ \dfrac{\partial\mathbf{B}}{\partial t}+\nabla_{\mathbf{x}}\times\mathbf{E}=0, \end{cases} \tag{2.40}$$

where $n=1$ and $j=0$ have to be satisfied at initial time.

The second equation of (2.40) is called the Generalised Ohm's law.

In this part, a reformulation of the Maxwell-Ampere equation containing the limit case $\alpha=0$ has been performed. This derivation will enable us to construct an Asymptotic-Preserving numerical scheme for the quasi-neutral regime.

# 3 Discrete model

## 3.1 Limitation of the classical numerical scheme.

A classical numerical scheme for the Maxwell-Ampere equation in the collisional regime writes

$$E_i^{n+1} = E_i^n - \frac{j_i^n \Delta t}{\alpha^2}. \tag{3.1}$$

The stability of this scheme depends directly on the parameter $\alpha$. So, when $\alpha$ tends to 0, (3.1) can not be used to calculate the new electric field $E_i^{n+1}$.

The aim of the following part is to establish a numerical scheme which contains explicitly the quasi-neutral case when $\alpha = 0$. In this way, a new numerical scheme is developed for the reformulated Maxwell-Ampere equation.

## 3.2 Construction of an Asymptotic-Preserving Maxwell-Ampere numerical scheme.

In this part the construction of an Asymptotic-Preserving scheme for the Maxwell-Ampere reformulated equation is explained. In this first part the numerical scheme is derived in the case of a non-homogeneous collisional plasma without magnetic field. The next part extends the method to the non-homogeneous collisional case with electromagnetic fields.

### 3.2.1 Case of a non-homogeneous collisional plasma without magnetic field.

In this part an Asymptotic-Preserving scheme is constructed for the second equation of (2.31).

Let us define the primal mesh $\mathcal{M}$ for the velocity variable $v$, decomposed into a family of rectangles $\mathcal{M}_{p+\frac{1}{2}} = ]v_p, v_{p+1}[ \ \ \forall p \in \{-p_f; p_f\}$ where $v_p = p\Delta v$ and $p \in \mathbb{N}$ represents the number of points which discretize the velocity domain. $\Delta v$ represents the energy discretisation step, which is fixed. Denote by $\mathcal{D}$ its associated dual mesh consisting of cells $\mathcal{D}_p = ]v_{p-\frac{1}{2}}, v_{p+\frac{1}{2}}[$ where $v_{p-\frac{1}{2}} = (p-\frac{1}{2})\Delta v$. In the same way, a primal mesh $\mathcal{N}$ is defined for the space variable $x$, decomposed into a family of rectangles $\mathcal{N}_{i+\frac{1}{2}} = ]x_i, x_{i+1}[ \ \ \forall i \in \{1; l_f\}$ where $x_i = i\Delta x$ and $i \in \mathbb{N}$ represents the number of points which discretize the space domain. $\Delta x$ represents the space discretisation step, which is fixed. We denote by $\mathcal{E}$ its associated dual mesh consisting of cells $\mathcal{E}_i = ]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$ where $x_{i-\frac{1}{2}} = (i-\frac{1}{2})\Delta x$. Let $h_{i,p}$ (resp. $h_{i+\frac{1}{2},p+\frac{1}{2}}$) be an approximation of $h(x_i, v_p)$ (resp $h(x_{i+\frac{1}{2}}, v_{p+\frac{1}{2}})$) for all distribution functions $h$. The velocity grid is chosen large enough to have $f_{i,pf} = f_{i,-pf} = 0 \ \ \forall i \in \{1; l_f\}$ which means that there are no particles with such velocities.

By using a conservative discretisation for the Fokker-Planck-Landau equation we obtain

$$\frac{f_{i,p}^{n+1} - f_{i,p}^n}{\Delta t} + \frac{(vf^n)_{i+\frac{1}{2},p} - (vf^n)_{i-\frac{1}{2},p}}{\Delta x} - \frac{(E^{n+1}f^n)_{i,p+\frac{1}{2}} - (E^{n+1}f^n)_{i,p-\frac{1}{2}}}{\Delta v} = C_{ee,i}^n + C_{ei,i}^n, \tag{3.2}$$

where the computation of the numerical fluxes is given by

$$(vf^n)_{i+\frac{1}{2},p} = v_p\left(\frac{f^n_{i,p}+f^n_{i+1,p}}{2}\right) - \frac{|v_p|}{2}\left(f^n_{i+1,p}-f^n_{i,p}\right), \tag{3.3}$$

$$(E^{n+1}f^n)_{i,p+\frac{1}{2}} = E^{n+1}_i\left(\frac{f^n_{i,p}+f^n_{i,p+1}}{2}\right) - \frac{|E^{n+1}_i|}{2}\left(f^n_{i,p+1}-f^n_{i,p}\right), \tag{3.4}$$

and

$$C^n_{ei,i} = \frac{1}{\Delta v}\left[S_{p+\frac{1}{2}}\frac{f^n_{i,p+1}-f^n_{i,p}}{\Delta v} - S_{p-\frac{1}{2}}\frac{f^n_{i,p}-f^n_{i,p-1}}{\Delta v}\right] \tag{3.5}$$

with

$$S_{p+\frac{1}{2}} = S\left(\frac{v_p+v_{p+1}}{\Delta v}\right). \tag{3.6}$$

The expression of $S$ is given in (2.8). The numerical scheme for the operator $C_{ee,i}$ is not given, because using (2.10) this term cancels in the calculation. It is important to notice that the electrostatic field is chosen implicit. It will be shown that this choice enables the calculation of the electrostatic field when $\alpha \to 0$.

Using the above numerical fluxes, (3.2) reads

$$\begin{aligned}\frac{f^{n+1}_{i,p}-f^n_{i,p}}{\Delta t} \quad & + \frac{v_p\left[f^n_{i+1,p}-f^n_{i-1,p}\right] - |v_p|\left[f^n_{i+1,p}-2f^n_{i,p}+f^n_{i-1,p}\right]}{2\Delta x} \\ & - \frac{E^{n+1}_i\left[f^n_{i,p+1}-f^n_{i,p-1}\right] - |E^{n+1}_i|\left[f^n_{i,p+1}-2f^n_{i,p}+f^n_{i,p-1}\right]}{2\Delta v} \\ & = C^n_{ee,i}+C^n_{ei,i}.\end{aligned}$$

Multiplying the previous equation by $-v_p\Delta v$ and summing in $p$ leads to

$$\begin{aligned}\frac{-\sum_p v_p f^{n+1}_{i,p}\Delta v + \sum_p v_p f^n_{i,p}\Delta v}{\Delta t} \quad & - \frac{\Delta v}{2\Delta x}\sum_p\left[v_p^2\left(f^n_{i+1,p}-f^n_{i-1,p}\right) - v_p|v_p|\left(f^n_{i+1,p}-2f^n_{i,p}+f^n_{i-1,p}\right)\right] \\ & + \frac{1}{2}\sum_p\left[v_p E^{n+1}_i\left(f^n_{i,p+1}-f^n_{i,p-1}\right) - |E^{n+1}_i|v_p\left(f^n_{i,p+1}-2f^n_{i,p}+f^n_{i,p-1}\right)\right] \\ & = -\sum_p C^n_{ei,i}v_p\Delta v.\end{aligned}$$

Then using the discrete definition of the current

$$j_i = -\sum_p v_p f_{i,p} \Delta v \tag{3.7}$$

the computation of the previous equation leads to

$$\begin{aligned}\frac{j_i^{n+1}-j_i^n}{\Delta t} \quad & -\frac{\Delta v}{2\Delta x}\sum_p \left[v_p^2\left(f_{i+1,p}^n - f_{i-1,p}^n\right) - v_p|v_p|\left(f_{i+1,p}^n - 2f_{i,p}^n + f_{i-1,p}^n\right)\right] \\ & + \frac{1}{2}\sum_p \left[v_p E_i^{n+1}\left(f_{i,p+1}^n - f_{i,p-1}^n\right) - |E_i^{n+1}|v_p\left(f_{i,p+1}^n - 2f_{1,i,p}^n + f_{i,p-1}^n\right)\right] \\ & = -\sum_p C_{ei,i}^n v_p \Delta v.\end{aligned}$$

The following scheme for the Maxwell-Ampere equation is used

$$\frac{E_i^{n+1}-E_i^n}{\Delta t} = -\frac{j_i^{n+1}}{\alpha^2}. \tag{3.8}$$

Contrarily to the classical scheme (3.1) the current $j$ in (3.8) is chosen implicit.

By using (3.8), we get

$$\begin{aligned}-\alpha^2\frac{E_i^{n+1}-2E_i^n+E_i^{n-1}}{\Delta t^2} \quad & -\frac{\Delta v}{2\Delta x}\sum_p \left[v_p^2\left(f_{i+1,p}^n - f_{i-1,p}^n\right) - v_p|v_p|\left(f_{i+1,p}^n - 2f_{i,p}^n + f_{i-1,p}^n\right)\right] \\ & + \frac{1}{2}\sum_p \left[v_p E_i^{n+1}\left(f_{i,p+1}^n - f_{i,p-1}^n\right) - |E_i^{n+1}|v_p\left(f_{i,p+1}^n - 2f_{1,i,p}^n + f_{i,p-1}^n\right)\right] \\ & = -\sum_p C_{ei,i}^n v_p \Delta v.\end{aligned}$$

**Remark 3.** It is important to notice that

$$\sum_p |E_i^{n+1}|v_p\left(f_{i,p+1}^n - 2f_{i,p}^n + f_{i,p-1}^n\right) = 0.$$

Indeed a discrete integration by part gives

$$\begin{aligned}\sum_p |E_i^{n+1}| v_p \left(f_{i,p+1}^n - 2f_{i,p}^n + f_{i,p-1}^n\right) &= |E_i^{n+1}| \Big[\sum_p v_p (f_{i,p+1}^n - f_{i,p}^n) - \sum_p v_{p+1}(f_{i,p+1}^n - f_{i,p}^n)\Big], \\ &= |E_i^{n+1}| \Big[\sum_p (v_p - v_{p+1})(f_{i,p+1}^n - f_{i,p}^n)\Big], \\ &= -|E_i^{n+1}| \Delta v \Big[\sum_p (f_{i,p+1}^n - f_{i,p}^n)\Big], \\ &= 0\end{aligned}$$

because of boundary condition $f_{i,pf}^n = f_{i,-pf}^n = 0$. Therefore, no linearisation nor approximation is required to compute $E_i^{n+1}$.

Finally, the Asymptotic-Preserving scheme for the second equation of (2.31) writes

$$\begin{aligned}-\alpha^2 \frac{E_i^{n+1} - 2E_i^n + E_i^{n-1}}{\Delta t^2} &- \frac{\Delta v}{2\Delta x} \sum_p \Big[v_p^2 \left(f_{i+1,p}^n - f_{i-1,p}^n\right) - v_p |v_p| \left(f_{i+1,p}^n - 2f_{i,p}^n + f_{i-1,p}^n\right)\Big] \\ &+ \frac{E_i^{n+1}}{2} \sum_p v_p \left(f_{i,p+1}^n - f_{i,p-1}^n\right) \\ &= -\sum_p C_{ei,i}^n v_p \Delta v\end{aligned}$$

which is the numerical scheme for the reformulated Maxwell-Ampere equation in the case of a inhomogeneous collisional plasma.

In the limit case when $\alpha$ tends to zero, the scheme becomes

$$E_i^{n+1} = \frac{\frac{\Delta v}{\Delta x} \sum_p \Big[v_p^2 \left(f_{i+1,p}^n - f_{i-1,p}^n\right) - v_p |v_p| \left(f_{i+1,p}^n - 2f_{i,p}^n + f_{i-1,p}^n\right)\Big] - 2\sum_p C_{ei,i}^n v_p \Delta v}{\sum_p v_p \left(f_{i,p+1}^n - f_{i,p-1}^n\right)}. \tag{3.9}$$

In the case the expression obtained is well consistent with the limit equation (2.30), this is a key point to obtain the asymptotic preserving property.

### 3.2.2 Generalisation to a non-homogeneous collisional plasma with electromagnetic fields.

In this part we derive the numerical scheme for the reformulated Maxwell-Ampere equation in the simplified case of 1 dimension in space and 3 dimensions in velocity. The

scheme can be extended to the case of 3 dimensions in space. We consider a cartesian case with an electric and a magnetic field of the form

$$E=(E_x(t,x,y),E_y(t,x,y),0),\ \ B=(0,0,B_z(t,x,y)). \tag{3.10}$$

Following the same method as for the electrostatic case, we derive the following numerical scheme for the reformulated Maxwell-Ampere equation

$$\begin{aligned}
-\alpha^2\frac{E_{x,l}^{n+1}-2E_{x,l}^{n}+E_{x,l}^{n-1}}{\Delta t^2} \quad & -\Delta v_x\Delta v_y\Delta v_z\sum_{i,j,k}\left(E_{x,l}^{n+1}+j\Delta v_y B_{z,l}^{n+1}\right)f_{l,i,j,p}^{n}\\
& -\Delta v_x^2\Delta v_z\sum_{i,j,k}v_{x,i}\left(E_{y,l}^{n+1}-i\Delta v_x B_{z,l}^{n+1}\right)f_{l,i,j,p}^{n}\\
& =-\sum_{i,j,k}C_{ei,i}^{n}v_{x,i}\Delta v_x\Delta v_y\Delta v_z+\frac{\Delta v_x\Delta v_y\Delta v_z}{2\Delta x}\sum_{i,j,k}\Big[v_{x,i}^2\left(f_{l+1,i,j,k}^{n}-f_{l-1,i,j,k}^{n}\right)\\
& -v_{x,i}|v_{x,i}|\left(f_{l+1,j,k,p}^{n}-2f_{l,i,j,k}^{n}+f_{l-1,i,j,k}^{n}\right)\Big],
\end{aligned}$$

$$\begin{aligned}
-\alpha^2\frac{E_{y,l}^{n+1}-2E_{y,l}^{n}+E_{y,l}^{n-1}}{\Delta t^2} \quad & -\Delta v_y^2\Delta v_z\sum_{i,j,k}\left(E_{x,l}^{n+1}+j\Delta v_y B_{z,l}^{n+1}\right)f_{l,i,j,p}^{n}\\
& -\Delta v_x\Delta v_y\Delta v_z\sum_{i,j,k}v_{x,i}\left(E_{y,l}^{n+1}-i\Delta v_x B_{z,l}^{n+1}\right)f_{l,i,j,p}^{n}\\
& =-\sum_{i,j,k}C_{ei,i}^{n}v_{y,i}\Delta v_x\Delta v_y\Delta v_z+\frac{\alpha^2}{\beta^2\Delta t}\Big[\frac{B_{z,l+1}^{n+1}-B_{z,l-1}^{n+1}}{2\Delta x}-\frac{B_{z,l+1}^{n}-B_{z,l-1}^{n}}{2\Delta x}\Big]\\
& +\frac{\Delta v_x\Delta v_y\Delta v_z}{2\Delta x}\sum_{i,j,k}\Big[v_{y,i}v_{x,i}\left(f_{l+1,i,j,k}^{n}-f_{l-1,i,j,k}^{n}\right)\\
& -v_{y,i}|v_{x,i}|\left(f_{l+1,i,j,k}^{n}-2f_{l,i,j,k}^{n}+f_{l-1,i,j,k}^{n}\right)\Big],
\end{aligned}$$

where l is the index for space, i the index for the first coordinate in speed, j for the second and k for the third. Also $\Delta t$, $\Delta x$, $\Delta v_x$, $\Delta v_y$, $\Delta v_z$ are respectively the time step, the space step, the velocity step in the first, second and third dimension. In this case there are two equations, we notice they are coupled.

### 3.3 Stability property

The asymptotic-preserving property also requires that the scheme is uniformly stable with respect to the parameter $\alpha$. The rigorous proof of the asymptotic stability property is challenging and in general, the few results presented describe simplified linearised models where a linear stability study is conducted ( [16,17]). In the present case, because of the dependence of the space and velocity variables in addition to the collisional operators such a property is not easily derived and a rigorous stability analysis of the method seems beyond the scope of this paper. However, we can give some elements of proof in a simplified case of a linearised collisionless homogeneous case with one dimension for velocity ($v \in \mathbb{R}$) without magnetic field. The model reads

$$\begin{cases} \dfrac{\partial f}{\partial t} - E\dfrac{\partial f}{\partial v} = 0, \\ \alpha^2 \dfrac{\partial E}{\partial t} = -j. \end{cases}$$

We consider the following linearisation around the equilibrium state given by a Maxwellian distribution function with no electric field

$$f = f^m + f^1, \qquad E = 0 + E^1,$$

with $f^m$ a Maxwellian distribution function. The linearised system reads

$$\begin{cases} \dfrac{\partial f^1}{\partial t} + 2E^1 v f^m = 0, \\ \alpha^2 \dfrac{\partial E^1}{\partial t} = -j^1. \end{cases} \tag{3.11}$$

In the numerical method proposed, the electric field is chosen implicit as well as the electronic current. Then omitting the index 1 for simplicity, the numerical scheme reads

$$\begin{cases} \dfrac{f_p^{n+1} - f_p^n}{\Delta t} + 2E^{n+1} v_p f_p^m = 0, \\ \alpha^2 \dfrac{E^{n+1} - E^n}{\Delta t} = -j^{n+1} = \displaystyle\sum_{p=-p_f}^{p_f} f_p^{n+1} v_p \Delta v. \end{cases} \tag{3.12}$$

The previous system can also be written in the following linear system form

$$\begin{pmatrix} E \\ f_{-p_f} \\ f_{-p_f+1} \\ \vdots \\ f_{p_f-1} \\ f_{p_f} \end{pmatrix}^{n+1} = M^{\alpha} \begin{pmatrix} E \\ f_{-p_f} \\ f_{-p_f+1} \\ \vdots \\ f_{p_f-1} \\ f_{p_f} \end{pmatrix}^{n}$$

where the matrix $M^{\alpha}$ is given by

$$M^{\alpha} = \begin{pmatrix} \frac{A^{\alpha}\alpha^2}{\Delta t} & A^{\alpha}\Delta v v_{-pf} & \cdots & A^{\alpha}\Delta v v_{pf} \\ -\frac{B^{\alpha}_{-p_f}\alpha^2}{\Delta t} & 1-B^{\alpha}_{-p_f}\Delta v v_{-p_f} & \cdots & -B^{\alpha}_{-pf}\Delta v v_{pf} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{B^{\alpha}_{p_f}\alpha^2}{\Delta t} & -B^{\alpha}_{p_f}\Delta v v_{-p_f} & \cdots & 1-B^{\alpha}_{pf}\Delta v v_{pf} \end{pmatrix}$$

with

$$A^{\alpha} = \frac{\Delta t}{\alpha^2 + 2\Delta t^2 \sum_{p=-p_f}^{p_f} f_p^m v_p^2 \Delta v}, \qquad B_p^{\alpha} = \frac{2\Delta t^2 f_p^m v_p}{\alpha^2 + 2\Delta t^2 \sum_{p=-p_f}^{p_f} f_p^m v_p^2 \Delta v}.$$

The eigenvalues of the matrix $M^{\alpha}$ are given by

$$\underbrace{1,1,\ldots,1}_{2p_f-1},\ K+i\sqrt{K-K^2},\ K-i\sqrt{K-K^2},$$

with $K = \dfrac{\alpha^2}{(\alpha^2 + 2\Delta t^2 \sum_{p=-p_f}^{p_f} f_p^m v_p^2 \Delta v)}$.

As $\alpha \in [0,1]$ one remarks that $K \in [0,1]$. It follows that the eigenvalues of $M^{\alpha}$ are in modulus less or equal than 1. The numerical scheme (3.12) for the simplified model (3.11) is then stable for all $\alpha$. One remarks that in spite of the simplicity of the model (3.11), the form of the matrice $M^{\alpha}$ is not trivial and an extension to the general model seems challenging. However, in a more general case, the numerical tests for the wide range of input parameters, witness of the stability of the method.

Kinetic codes are usually numerically expensive and limited to short time scales. Angular moments models can be seen as a compromise between kinetic and fluid models.

# 4 The Asymptotic-Preserving scheme for the $M_1$-Maxwell moments model

This part is devoted to the derivation of an Asymptotic-Preserving scheme for the $M_1$ model associated to the system (2.19). For the sake of clarity, we firstly recall the derivation of the $M_1$ model that is performed in ( [20,29,30]) and next we present the numerical scheme.

## 4.1 Moments models.

If $S^2$ is the unit sphere, $\overrightarrow{\Omega}=\frac{\overrightarrow{v}}{|v|}$ represents the direction of propagation of the particle and $\mu=\Omega_x=cos\,\theta$, $\theta\in[0,\pi]$. In this part, we choose a one dimensional direction of propagation, i.e we take $\mu\in[-1,1]$ as the direction of propagation instead of $\overrightarrow{\Omega}$. By setting, $\zeta=|v|$ the distribution function f writes in spherical coordinates $f(\mu,\zeta,x)$. Hence the three first angular moments of the distribution function are given by

$$f_0(\zeta)=\zeta^2\int_{-1}^{1}f(\mu,\zeta)d\mu,\ f_1(\zeta)=\zeta^2\int_{-1}^{1}f(\mu,\zeta)\mu d\mu,\ f_2(\zeta)=\zeta^2\int_{-1}^{1}f(\mu,\zeta)\mu^2 d\mu.$$

For moment models one fundamental point is the definition of the closure which guarantees that the highest moment writes as a function of the previous ones. This closure corresponds to an approximation of the distribution function from which the moment system is constructed.

## 4.2 Closure for the $M_1$ moment model.

In order to close the problem we need to define $f_2$ as a function of $f_0$ and $f_1$. For the $M_1$ model, the construction of the closure is based on an entropy minimum principle ( [28,32]). Indeed the closure is obtained by solving

$$\min_{g\geq 0}\{\ \mathcal{H}(g)\ /\ \forall\zeta\in\mathbb{R}^+,\int_{-1}^{1}g(\mu,\zeta)d\mu=f_0(\zeta),\ \int_{-1}^{1}g(\mu,\zeta)\mu d\mu=f_1(\zeta)\ \},\qquad(4.1)$$

where $\mathcal{H}$ is the Boltzmann entropy defined in (2.13).

The entropy minimum principle implies that the solution of (4.1) writes ( [20,21,29])

$$f(\mu,\zeta)=\rho(\zeta)exp(-\mu\ a_1(\zeta)),\qquad(4.2)$$

where $\rho(\zeta)$ is a positive scalar, and $a_1(\zeta)$ a real valued scalar.

Referring to ( [20]), $f_2$ writes

$$f_2 = \chi(f_0, f_1) f_0 \quad \text{with} \quad \chi = \frac{|a_1|^2 - 2|a_1| coth(|a_1|) + 2}{|a_1|^2}.$$

### 4.3 Collision operators.

Moments extraction of the electron-electron collision operator (2.6) is complicated due to its nonlinearity. That is why, it is often simplified by considering only the isotropic part of the distribution function ( [8]). Nevertheless, it has been shown ( [29]) that this simplification does not preserve the realisibility domain which guarantees that the solution of the moments system can be expressed as the moments of a nonnegative distribution function. Consequently, the collisional operators that are used ( [29,30]), are constructed according to a linearisation around the equilibrium state of $C_{ei}$. So $C_{ee}$ is approached by $Q_{ee}$ by using a linearisation around the isotropic part of the distribution function.

Hence, $Q_{ee}$ writes

$$Q_{e,e}(f) = \frac{1}{\zeta^2} \partial_\zeta \Big( \zeta \int_0^{+\infty} \tilde{J}(\zeta, \zeta') \Big[ F^0(\zeta') \frac{1}{\zeta} \partial_\zeta f(\zeta) - f(\zeta) \frac{1}{\zeta'} \partial_{\zeta'} F^0(\zeta') \Big] \zeta'^2 d\zeta' \Big) \tag{4.3}$$

with

$$\tilde{J}(\zeta, \zeta') = \frac{2}{3} inf\left(\frac{1}{\zeta^3}, \frac{1}{\zeta'^3}\right) \zeta'^2 \zeta^2 \tag{4.4}$$

and $F_0$ is the isotropic part of the electron distribution function.

For the electron-ion operator, no approximation is performed because this operator is already linear. In the following $Q_{e,i}$ will replace $C_{e,i}$ to be consistent with the notation (4.3) of $Q_{e,e}$.

$$Q_{ei}(f) = \frac{1}{\zeta^3} \frac{\partial}{\partial \mu} \Big( (1 - \mu^2) \frac{\partial f}{\partial \mu} \Big). \tag{4.5}$$

### 4.4 The $M_1$ moment model.

The angular integration ( [30]) leads to

$$\begin{cases} \partial_t f_0 + \nabla_x . (\zeta f_1) - \partial_\zeta (E f_1) = Q_0(f_0), \\ \partial_t f_1 + \nabla_x . (\zeta f_2) - \partial_\zeta (E f_2) + E \dfrac{(f_0 - f_2)}{\zeta} = Q_1(f_1) + Q_0(f_1), \end{cases} \tag{4.6}$$

where the collisional operators $Q_0$ and $Q_1$ are given by

$$Q_0(f_0) = \frac{2}{3}\partial_\zeta\Big(\zeta^2 A(\zeta)\partial_\zeta(\frac{f_0}{\zeta^2}) - \zeta B(\zeta) f_0\Big), \tag{4.7}$$

$$Q_1(f_1) = -\frac{2f_1}{\zeta^3}. \tag{4.8}$$

The coefficients $A(\zeta)$ and $B(\zeta)$ write

$$A(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\omega^3})\omega^2 f_0(\omega) d\omega, \tag{4.9}$$

$$B(\zeta) = \int_0^\infty min(\frac{1}{\zeta^3}, \frac{1}{\omega^3})\omega^3 \partial_\omega(\frac{f_0(\omega)}{\omega^2}) d\omega. \tag{4.10}$$

## 4.5 A numerical scheme for the $M_1$ model.

In this part the reformulation of the Maxwell-Ampere equation for the $M_1$ model is detailed. Considering a conservative scheme for the system (4.6) we write

$$\frac{f_{0,i,p}^{n+1} - f_{0,i,p}^n}{\Delta t} + \frac{(\zeta f_1^n)_{i+\frac{1}{2},p} - (\zeta f_1^n)_{i-\frac{1}{2},p}}{\Delta x} - \frac{(E^{n+1} f_1^n)_{i,p+\frac{1}{2}} - (E^{n+1} f_1^n)_{i,p-\frac{1}{2}}}{\Delta \zeta} = 0, \tag{4.11}$$

$$\begin{aligned} \frac{f_{1,i,p}^{n+1} - f_{1,i,p}^n}{\Delta t} &+ \frac{(\zeta f_2^n)_{i+\frac{1}{2},p} - (\zeta f_2^n)_{i-\frac{1}{2},p}}{\Delta x} - \frac{(E^{n+1} f_2^n)_{i,p+\frac{1}{2}} - (E^{n+1} f_2^n)_{i,p-\frac{1}{2}}}{\Delta \zeta} \\ &+ \frac{E^{n+1}}{\zeta_p}(f_{0,i,p}^n - f_{2,i,p}^n) = Q_{1,i,p}^n + Q_{0,i,p}^n. \end{aligned} \tag{4.12}$$

The discrete collision operators involved in (4.12) are respectively given by

$$Q_{1,i,p}^n = -\frac{2f_{1,i,p}^n}{\zeta_p^3},$$

$$\begin{aligned} Q_{0,i,p}^n = \frac{2}{3\Delta\zeta_p}\Big[&(\zeta_{p+\frac{1}{2}}^2 A(\zeta_{p+\frac{1}{2}})\frac{1}{\Delta\zeta_{p+\frac{1}{2}}}\Big(\frac{f_{1,i,p+1}^n}{\zeta_{p+1}^2} - \frac{f_{1,i,p}^n}{\zeta_p^2}\Big) - \zeta_{p+\frac{1}{2}} B(\zeta_{p+\frac{1}{2}}) f_{1,i,p+\frac{1}{2}}^n) \\ &- (\zeta_{p-\frac{1}{2}}^2 A(\zeta_{p-\frac{1}{2}})\frac{1}{\Delta\zeta_{p-\frac{1}{2}}}\Big(\frac{f_{1,i,p}^n}{\zeta_p^2} - \frac{f_{1,i,p-1}^n}{\zeta_{p-1}^2}\Big) - \zeta_{p-\frac{1}{2}} B(\zeta_{p-\frac{1}{2}}) f_{1,i,p-\frac{1}{2}}^n)\Big]. \end{aligned}$$

Using HLL numerical fluxes in (4.11) and (4.12), it holds that

$$\frac{f_{0,i,p}^{n+1}-f_{0,i,p}^{n}}{\Delta t}+\frac{\zeta_p\left[f_{1,i+1,p}^{n}-f_{1,i-1,p}^{n}\right]-|\zeta_p|\left[f_{0,i+1,p}^{n}-2f_{0,i,p}^{n}+f_{0,i-1,p}^{n}\right]}{2\Delta x}$$
$$-\frac{E_i^{n+1}\left[f_{1,i,p+1}^{n}-f_{1,i,p-1}^{n}\right]-|E_i^{n+1}|\left[f_{0,i,p+1}^{n}-2f_{0,i,p}^{n}+f_{0,i,p-1}^{n}\right]}{2\Delta\zeta}=0 \tag{4.13}$$

and

$$\frac{f_{1,i,p}^{n+1}-f_{1,i,p}^{n}}{\Delta t}+\frac{\zeta_p\left[f_{2,i+1,p}^{n}-f_{2,i-1,p}^{n}\right]-|\zeta_p|\left[f_{1,i+1,p}^{n}-2f_{1,i,p}^{n}+f_{1,i-1,p}^{n}\right]}{2\Delta x}$$
$$-\frac{E_i^{n+1}\left[f_{2,i,p+1}^{n}-f_{2,i,p-1}^{n}\right]-|E_i^{n+1}|\left[f_{1,i,p+1}^{n}-2f_{1,i,p}^{n}+f_{1,i,p-1}^{n}\right]}{2\Delta\zeta} \tag{4.14}$$
$$+\frac{E_i^{n+1}}{\zeta_p}(f_{0,i,p}^{n}-f_{2,i,p}^{n})=Q_{1,i,p}^{n}+Q_{0,i,p}^{n}.$$

Multiplying the previous equation (4.14) by $-\zeta_p\Delta\zeta$ and summing in $p$ leads to

$$\frac{-\sum_p\zeta_p f_{1,i}^{n+1}\Delta\zeta+\sum_p\zeta_p f_{1,i}^{n}\Delta\zeta}{\Delta t}-$$
$$\frac{\Delta\zeta}{2\Delta x}\sum_p\left[\zeta_p^2\left(f_{2,i+1,p}^{n}-f_{2,i-1,p}^{n}\right)-\zeta_p^2\left(f_{1,i+1,p}^{n}-2f_{1,i,p}^{n}+f_{1,i-1,p}^{n}\right)\right]+ \tag{4.15}$$
$$\frac{1}{2}\sum_p\left[\zeta_p E_i^{n+1}\left(f_{2,i,p+1}^{n}-f_{2,i,p-1}^{n}\right)-|E_i^{n+1}|\zeta_p\left(f_{1,i,p+1}^{n}-2f_{1,i,p}^{n}+f_{1,i,p-1}^{n}\right)\right]-$$
$$\sum_p E_i^{n+1}(f_{0,i,p}^{n}-f_{2,i,p}^{n})\Delta\zeta=-\sum_p\zeta_p Q_{1,i,p}^{n}\Delta\zeta.$$

Here again, the term containing the electron-electron collision operator cancels.

We use the definition of the dimensionless current $j$

$$j=-\int_{\mathbb{R}^+}f_1\zeta d\zeta, \tag{4.16}$$

which can be written on the discrete form

$$j_i^n=-\sum_p f_{1,i,p}^{n}\zeta_p\Delta\zeta. \tag{4.17}$$

Therefore the scheme (4.15) becomes

$$\begin{aligned}\frac{j_i^{n+1}-j_i^n}{\Delta t} &- \frac{\Delta\zeta}{2\Delta x}\sum_p\Big[\zeta_p^2\Big(f_{2,i+1,p}^n-f_{2,i-1,p}^n\Big)-\zeta_p^2\Big(f_{1,i+1,p}^n-2f_{1,i,p}^n+f_{1,i-1,p}^n\Big)\Big]\\ &+ \frac{1}{2}\sum_p\Big[\zeta_p E_i^{n+1}\Big(f_{2,i,p+1}^n-f_{2,i,p-1}^n\Big)-|E_i^{n+1}|\zeta_p\Big(f_{1,i,p+1}^n-2f_{1,i,p}^n+f_{1,i,p-1}^n\Big)\Big] \quad (4.18)\\ &- \sum_p E_i^{n+1}(f_{0,i,p}^n-f_{2,i,p}^n)\Delta\zeta=-\sum_p\zeta_p Q_{1,i,p}^n\Delta\zeta.\end{aligned}$$

Using the scheme (3.8), expression (4.18) becomes

$$\begin{aligned}-\alpha^2\frac{E_i^{n+1}-2E_i^n+E_i^{n-1}}{\Delta t^2}- &\quad \frac{\Delta\zeta}{2\Delta x}\sum_p\Big[\zeta_p^2\Big(f_{2,i+1,p}^n-f_{2,i-1,p}^n\Big)-\zeta_p^2\Big(f_{1,i+1,p}^n-2f_{1,i,p}^n+f_{1,i-1,p}^n\Big)\Big]\\ &+\frac{1}{2}\sum_p\Big[\zeta_p E_i^{n+1}\Big(f_{2,i,p+1}^n-f_{2,i,p-1}^n\Big)-|E_i^{n+1}|\zeta_p\Big(f_{1,i,p+1}^n-2f_{1,i,p}^n+f_{1,i,p-1}^n\Big)\Big]\\ &-E_i^{n+1}\sum_p(f_{0,i,p}^n-f_{2,i,p}^n)\Delta\zeta=-\sum_p\zeta_p Q_{1,i,p}^n\Delta\zeta.\end{aligned}$$

Like for the kinetic scheme in (3.2.1), it holds that

$$\sum_p|E_i^{n+1}|\zeta_p\Big(f_{1,i,p+1}^n-2f_{1,i,p}^n+f_{1,i,p-1}^n\Big)=0. \quad (4.19)$$

Therefore the final scheme obtained reads

$$E_i^{n+1}=\frac{-\alpha^2\dfrac{(2E_i^n-E_i^{n-1})}{\Delta t^2}+\beta_1(f_{0,i}^n,f_{1,i}^n)}{-\dfrac{\alpha^2}{\Delta t^2}+\beta_2(f_{0,i}^n,f_{1,i}^n)}, \quad (4.20)$$

where the coefficients $\beta_1$ and $\beta_2$ are given by

$$\beta_1=\frac{\Delta\zeta}{2\Delta x}\sum_p\Big[\zeta_p^2\Big(f_{2,i+1,p}^n-f_{2,i-1,p}^n\Big)-\zeta_p^2\Big(f_{1,i+1,p}^n-2f_{1,i,p}^n+f_{1,i-1,p}^n\Big)\Big]-\sum_p\zeta_p Q_{1,i,p}^n\Delta\zeta,$$

$$\beta_2=\frac{1}{2}\sum_p\Big[\zeta_p\Big(f_{2,i,p+1}^n-f_{2,i,p-1}^n\Big)\Big]-\sum_p(f_{0,p,i}^n-f_{2,p,i}^n)\Delta\zeta.$$

**Remark 4.** The stability of this new scheme does not depend on the parameter $\alpha$. So, the electrostatic field can be obtained even if $\alpha$ becomes equal to zero.

**Remark 5.** Following the same procedure as for the Fokker-Planck-Maxwell system, this reformulation can be generalised for multi-dimension problems with magnetic fields.

# 5 Numerical test cases

This section presents two physically relevant numerical experiments where opposite regimes are considered. The first one studies two counterpropagating beams of electrons. In this case collective electrostatic effects are predominant. The second one deals with the relaxation of a localised temperature profile in the quasi-neutral regime. In this regime collisions between particles dominates.

## 5.1 Two electron beams interaction.

In this part we study the interaction between two electron beams. This collisionless test case enables us to study the regime where electrostatic effects are predominant. Therefore for this test case we have $C_{ee} = C_{ei} = 0$.

Consider two electron beams propagating at velocity $v_0$ and $v_1$. The dispersion relation is given by

$$1 - \frac{1}{(\omega - kv_0)^2} - \frac{1}{(\omega - kv_1)^2} = 0,$$

where $v_0$ and $v_1$ denote the beams velocities.

This configuration can lead to electrostatic instabilities. Indeed, the solutions of the form $Ae^{i\omega t + ikx}$ are unstable when $\omega_I$ the imaginary part of $\omega$ is strictly positive. In the case $v_0 = -v_1$ we can show that the solution is stable if $kv_0 \geq \sqrt{2}$.

This test is problematic for the $M_1$ model. Indeed, if we consider two electron beams propagating with opposite velocities the distribution function is well defined. Nevertheless, the $M_1$ model considers only the angular moments $f_0$ and $f_1$. For the calculation of $f_1$ the two populations contributions cancel and we get $f_1 = 0$. The $M_1$ model sees an isotropic configuration which is not the reality. To overcome this problem we use the superposition principle that is valid because the model is linear. Two particle populations (one per beam) are considered. For each time step the $M_1$ problem is solved for the first population then for the second one. Hence the Maxwell equations are solved taking into account the two distribution functions.

In the case of two streams propagating with opposite velocities $v_d$ and $-v_d$, the initial conditions are

$$f(x,v,t=0)=0.5[(1+Acos(kx))M_{v_d}(v)+(1-Acos(kx))M_{-v_d}(v)],$$

with

$$M_{\pm v_d}(v)=n_e(\frac{m_e}{2\pi k_B T_e})^{3/2}exp\Big(\frac{-m_e(v\mp v_d)^2}{2k_B T_e}\Big).$$

The parameter $A$ is introduced to perturb the initial condition in order to enable the development of the electrostatic instability. The velocity modulus goes from 0 to 12 $v_{th}$ and the space scale from 0 to 25 $\lambda_{De}$. With 100 points for the space grid and 128 points for the velocity modulus grid the results are converged. In Figure 1 the distribution function is represented in the phase space for the initial time and the final time $t=30$ plasma periods. In this example $v_d=4$, $A=0.001$ and periodical boundary conditions are used. On the second plot the interaction between the two streams is observed.

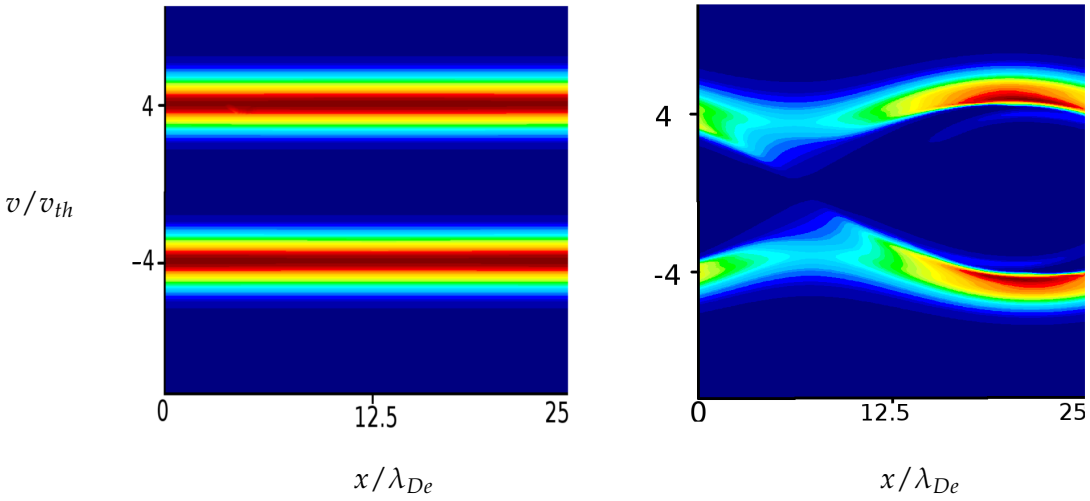


Figure 1: Distribution function as a function of space and velocity at initial time (left) and after 30 plasma periods (right).

Our results have been compared with a kinetic code ( [22]). In Figure 2, the evolution of the electrostatic energy is represented as a function of time for the ($M_1$-AP) code in green and for a kinetic code in red. The first plot shows the results for A=0.001 and the second one for A=0.1. In the case of small perturbations (A=0.001), the $M_1$ model and the kinetic code give analogous results. In the case of strong perturbations (A=0.1), the ($M_1$-AP) code and the kinetic code show some differences after a long time. In the case of a strong perturbation, a non-linear regime is obtained and it is well-known that the $M_1$ model is not accurate enough ( [20]).

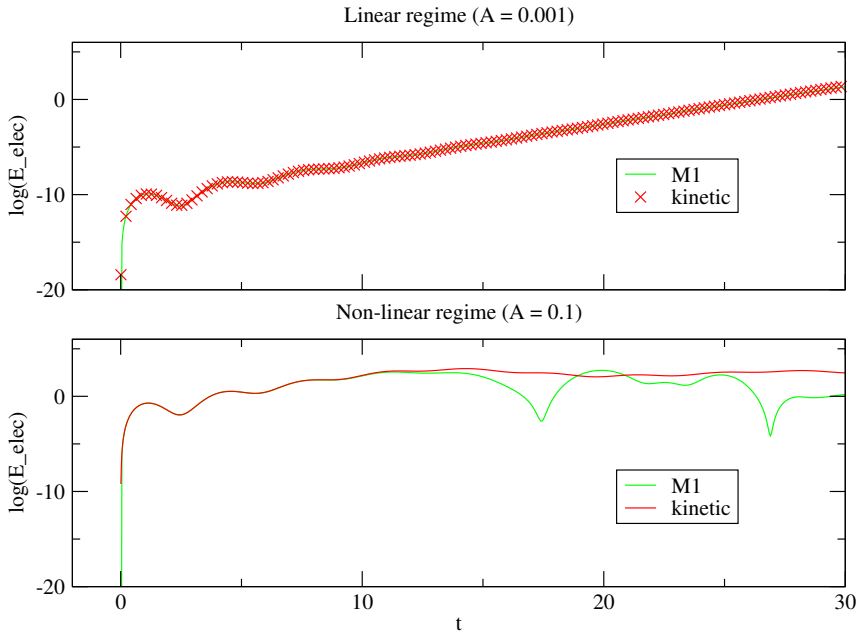


Figure 2: Temporal evolution of the electrostatic energy (dimensionless units) in the linear regime (top) and in the non-linear regime (bottom).

This numerical experiment shows the good behaviour of the ($M_1$-AP) scheme in a regime where electrostatic effects are predominant.

## 5.2 Hot spot relaxation.

We now study the relaxation of a localised temperature perturbation generated, for example, by a short laser pulse. Suppose that the laser impulse duration is shorter than the relaxation time. This phenomenon investigated by many authors ( [1], [4]) corresponds physically to the heating of a plasma during a short time and to the relaxation phenomenon which follows. The important temperature gradients due to the localised heating induce a non-local heat transport. Here, we consider the collisional regime. This configuration is particularly interesting because it enables to study the coupling of the $M_1$ model with the Maxwell-Ampere Asymptotic-Preserving scheme.

Initially the distribution function for electrons is a Maxwellian with a Gaussian temperature profile

$$T_e(x,t=0) = T_0 + T_1 \exp(-\frac{x^2}{D^2}), \tag{5.1}$$

where the hot spot size $D$ is a characteristic scale of inhomogenity. First we make a few remarks on the formulation of the problem related to the ambipolar electric field. In the case of a smooth temperature gradient, the following formula for the electric field is obtained ( [33])

$$\frac{eE}{m_e} = -\frac{\nabla_{\mathbf{x}} \int_{\mathbb{R}^3} F_0 v^7 dv}{6 \int_{\mathbb{R}^3} F_0 v^5 dv} \tag{5.2}$$

where $F_0$ is the isotropic part of the electron distribution function. For a Maxwellian distribution function, this field is expressed through the classical formula

$$eE = -T_e \Big( \frac{\nabla_{\mathbf{x}} n_e}{n_e} + \frac{5}{2} \frac{\nabla_{\mathbf{x}} T_e}{T_e} \Big). \tag{5.3}$$

The local heat flux is given by the Spitzer-Harm formula ( [34])

$$q_{SH} = -\kappa_{SH} \nabla_{\mathbf{x}} T_e \tag{5.4}$$

with conductivity

$$\kappa_{SH} = \frac{128}{3\pi} \frac{Z + 0.24}{Z + 4.2} n_e v_{th} \lambda_{ei}. \tag{5.5}$$

Note that already for $D^{-1}\lambda_{ei} > 0.06/\sqrt{Z}$, the classical transport theory is not applicable.

In a first simulation presented here, we choose typical parameters for ICF studies $T_0 = 1$Kev, $T_1 = 4$Kev and $D = 8{,}44\lambda_{ei}$. There is no electric field at the initial time. We choose specular reflection conditions for boundary conditions. The space scale goes from $-80\lambda_{ei}$ to $80\lambda_{ei}$. The velocity modulus scale goes from 0 to $50\, v_{th}$.

Figure 3 shows the evolution of the temperature and electric field profiles until 30 $\tau_{ei}$. Then at $t = 2\ \tau_{ei}$, we observe that the temperature profile starts to relax to a colder temperature and the electric field which is proportional to the gradient of temperature also decreases. The numerical scheme reproduces the good behaviour of the hot spot relaxation phenomenon.

The results of our $M_1$-Asymptotic-Preserving scheme ($M_1$-AP) have been compared with the ones obtained by a kinetic code ( [31]). In Figure 4, the temperature and the electrostatic field profiles are represented as a function of space for different times. The ($M_1$-AP) results are given in green while the kinetic results are in red. Both results show a good agreement. Small differences are observed concerning the amplitude of the temperature and the electric field. The relaxation phenomenon observed with the ($M_1$-AP) code is faster than the one with the kinetic code.
It is interesting to notice that there is a large difference of calculation time. The simulation with the kinetic code requires the use of 50 processes during several days while

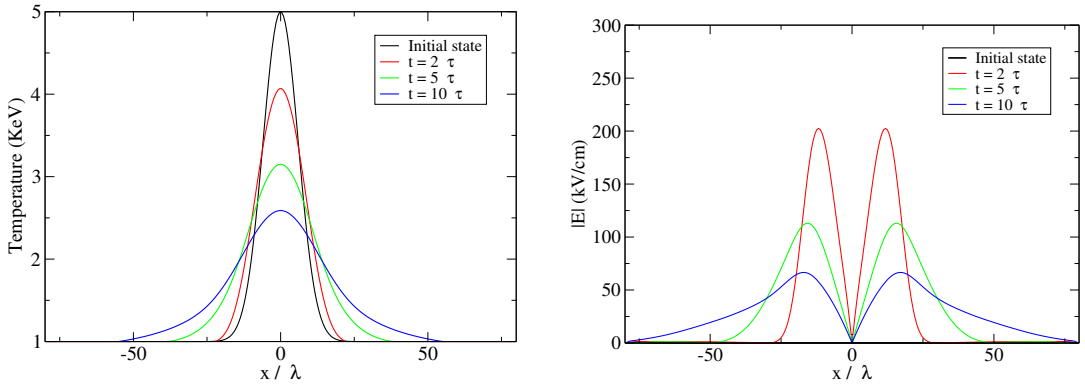


Figure 3: Representation of the temperature and electric field as function of space for different times.

the ($M_1$-AP) code only needs few minutes with one process. Moreover, thanks to the rapidity of the $M_1$ Asymptotic-Preserving code a mesh convergence study has been performed. With 500 points for the space grid and 80 points for the energy grid the results are converged. The time step used is $\Delta t = 10^{-3}\,\tau_{ei}$ in order to respect classical stability conditions.

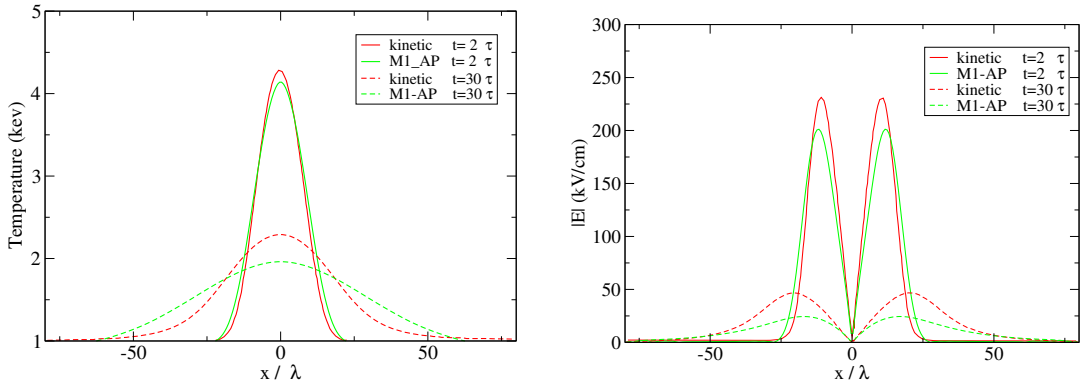


Figure 4: Comparison of the temperature (left) and electric field (right) for a kinetic code [31] and the $M_1$ Asymptotic-Preserving scheme.

**Remark 6.** In this case the parameter $\alpha$ which represents the ratio between the electron-ion collision frequency and the electron plasma frequency is equal to $4.10^{-4}$. In order to avoid a severe constraint on the time step we use the new $M_1$-Asymptotic-Preserving scheme. With the same CFL conditions, the classic Maxwell-Ampere numerical scheme breaks down from the very first iterations.

**Remark 7.** It is important to notice that the Asymptotic-Preserving scheme is stable

even in the case $\alpha = 0$.

In a second stage, a new simulation was performed in order to compare the results obtained using the ($M_1$-AP) scheme with the ones obtained using another kinetic code ( [2]) and a hydrodynamic code based on the classical transport theory ( [33]- [34]). For the simulation presented, we choose the parameters $T_0 = 1$Kev, $T_1 = 2$Kev, $Z = 80$ and $D = 100\lambda_{ei}$. The results are given at time $t = 120\tau_{ei}$. The space scale goes from $-2500\lambda_{ei}$ to $2500\lambda_{ei}$. In Figure 5, the temperature and the heat flux are represented for the three codes. Dimensionless quantities are used here. It appears that the three temperature profiles are very close. The hydrodynamic temperature is slightly smaller than the two others while the ($M_1$-AP) scheme and the kinetic scheme are in very good agreement. The different heat flux profiles are also compared in Figure 5. The ($M_1$-AP) flux and the kinetic flux are close and it appears that the ($M_1$-AP) flux is slightly more spread out. The hydrodynamic flux on the contrary is much larger than the two others and is also more localised. In this regime, one can again observe the good behaviour of the ($M_1$-AP) scheme. This scheme gives close results with the kinetic code while the hydrodynamic approach overestimates the heat flux.

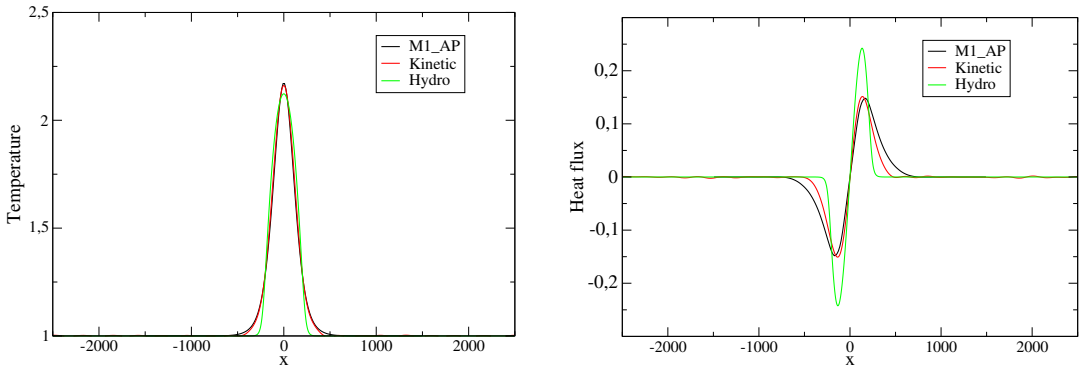


Figure 5: Comparison of the temperature (left) and heat flux (right) for a kinetic code [2], a hydrodynamic code and the $M_1$ Asymptotic-Preserving scheme.

# 6 Conclusion

In this work, we have constructed an Asymptotic-Preserving scheme for the full Fokker-Planck-Landau-Maxwell system which handles the quasi-neutral limit without any contraction of time and space steps. It is important to note that this model is considered with real collisional operators. This fact is important in plasma physics because the model is relevant for Coulombian interactions. We have first established a reformulated Fokker-Planck-Landau-Maxwell system then used it to construct the Asymptotic-Preserving scheme. The method has been extended to the general case of collisional

plasmas in electromagnetic fields for multi-dimensions problems. An $M_1$-Asymptotic-Preserving scheme has been derived. Next, the $M_1$-Asymptotic-Preserving scheme has been implemented and two numerical test cases have been performed. The first one corresponds to a regime where electromagnetic effects are predominant. The second one on the contrary shows the efficiency of the Asymptotic-Preserving scheme in the quasi-neutral regime. The scheme, accurate and fast, works in both regimes.

In this paper and in previous ones ( [29, 30]), the ions have been considered as fixed due to their important masses compared to the one of electrons. The model must be extended to study time intervals during which ion motion can not be neglected anymore. Therefore, we expect to develop in a forthcoming paper a model taking into account ion motion.

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