Open loop stabilization of incompressible Navier-Stokes equations in a 2d channel using power series expansion*

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Abstract

In this article, we discuss the stabilization of incompressible Navier-Stokes equations in a 2d channel around a fluid at rest when the control acts only on the normal component of the upper boundary. In this case, the linearized equations are not controllable nor stabilizable at an exponential rate higher than $\nu\pi^2/L^2$, when the channel is of width L and of length 2π and ν denotes the viscosity parameter. Our main result allows to go above this threshold and reach any exponential decay rate by using the non-linear term to control the directions which are not controllable for the linearized equations. Our approach therefore relies on writing the controlled trajectory as an expansion of order two taking the form $\varepsilon\alpha + \varepsilon^2\beta$ for some $\varepsilon > 0$ small enough. This method is inspired by the previous work [18] by J.-M. Coron and E. Crépeau on the controllability of the Korteweg de Vries equations.

1 Introduction

1.1 Setting and main result

The goal of this article is to discuss the stabilization of an incompressible fluid locally around the rest state in a 2d channel. We set

$$\Omega = \mathbb{T} \times (0, L), \tag{1.1}$$

where $\mathbb{T}$ is the 1d torus, identified with $(0, 2\pi)$ with periodic boundary conditions in the x_1-variable, and $L > 0$, see Figure 1.1. The Navier-Stokes equations read as follows:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \text{div } u = 0, & \text{in } (0,\infty) \times \Omega, \\ u(t, x_1, 0) = (0, 0), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(t, x_1, L) = (0, g(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ u(0, x_1, x_2) = u^0(x_1, x_2), & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here, $u = u(t, x_1, x_2) = (u_1(t, x_1, x_2), u_2(t, x_1, x_2))$ denotes the velocity ($\in \mathbb{R}^2$) of the fluid, $p = p(t, x_1, x_2)$ denotes the pressure and $g(t, x_1)$ is the boundary control. The viscosity coefficient $\nu > 0$ is assumed to be a positive constant.

A simple stationary solution of system (1.2) is given by $(u, p) = (0, 0, 0)$ with $g = 0$, corresponding to a fluid at rest. Our goal is to prove a local stabilization result around this particular state at any exponential decay rate $\omega > 0$. The main difficulty for stabilizing the fluid is that the control function g acts only on the normal component of the velocity. Therefore, due to the incompressibility condition div $u = 0$, we necessarily have

$$\int_{\mathbb{T}} g(t, x_1)\, dx_1 = 0, \quad \forall t \geqslant 0. \tag{1.3}$$

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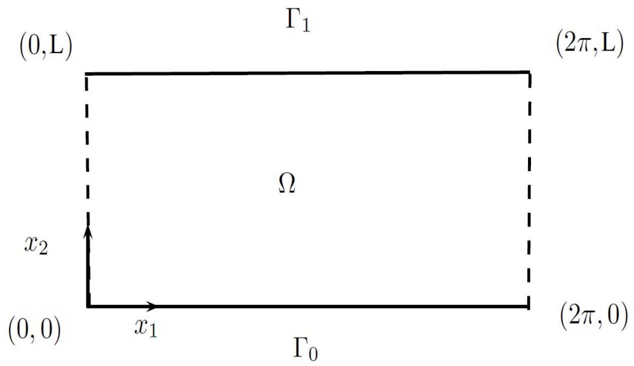

Figure 1: The domain Ω.

In particular, if we consider the linearized equations

$$\begin{cases} \partial_t u - \nu\Delta u + \nabla p = 0, & \text{in } (0,\infty)\times\Omega, \\ \text{div } u = 0, & \text{in } (0,\infty)\times\Omega, \\ u(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ u(t,x_1,L) = (0,g(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ u(0,x_1,x_2) = u^0(x_1,x_2), & \text{in } \Omega, \end{cases} \tag{1.4}$$

and if we expand u into Fourier series

$$u(t,x_1,x_2) = \sum_{k\in\mathbb{Z}} u_k(t,x_2)e^{ikx_1},$$

then the 0-mode u_0 defined by

$$u_0(t,x_2) = \int_{\mathbb{T}} u(t,x_1,x_2)\,dx_1 = \begin{pmatrix} u_{0,1}(t,x_2) \\ u_{0,2}(t,x_2) \end{pmatrix}, \tag{1.5}$$

satisfies the uncontrolled equation

$$\begin{cases} \partial_t u_{0,1} - \nu\partial_{22}u_{0,1} = 0, & \text{in } (0,\infty)\times(0,L), \\ u_{0,1}(t,0) = u_{0,1}(t,L) = 0, & \text{on } (0,\infty), \\ u_{0,1}(0,x_2) = \displaystyle\int_{\mathbb{T}} u^0(x_1,x_2)\cdot e_1 \; dx_1, & \text{in } (0,L), \\ u_{0,2}(t,x_2) = 0, & \text{in } (0,\infty)\times(0,L). \end{cases} \tag{1.6}$$

In particular, this implies that any stabilization strategy based only on the linearized system (1.4) will fail to stabilize the full system (1.2) at a rate higher than $\nu\pi^2/L^2$, corresponding to the first eigenvalue $\lambda_{0,1} = -\nu\pi^2/L^2$ of the operator $\nu\partial_{22}$ with Dirichlet boundary conditions on $(0,L)$.

That leads to a natural restriction on the decay rate of the solutions if we use a strategy based on the linearized system (1.4). We refer to the work [29] for decay results in this spirit for the linearized system (1.4).

Our goal is to show that one can achieve the stabilization of the non-linear system (1.2) at a rate higher than $\nu\pi^2/L^2$ by using the effect of the non-linearity.

Before stating our main result, we introduce some functional spaces adapted to deal with systems (1.2) and (1.4). Namely, we define

$$\mathbf{V}^1(\Omega) = \left\{u = (u_1,u_2) \in H^1(\Omega)\times H^1(\Omega) \,|\, \text{div } u = 0 \text{ in } \Omega\right\}, \tag{1.7}$$

$$\mathbf{V}^1_0(\Omega) = \left\{u \in \mathbf{V}^1(\Omega) \,|\, u(x_1,0) = u(x_1,L) = 0 \text{ for } x_1 \in \mathbb{T}\right\}. \tag{1.8}$$

In these spaces, let us recall that the functions are 2π periodic in the x_1 variable, the periodicity with respect to the x_1-variable being encoded in the condition $x_1 \in \mathbb{T}$.

We are then in position to state our main result.

Theorem 1.1 (Local open loop exponential stabilization)**.** *Let $\omega_0 > 0$. There exist $\gamma > 0$ and $C > 0$, depending on ω_0 such that, for all $u^0 \in \mathbf{V}_0^1(\Omega)$ obeying*

$$\|u^0\|_{\mathbf{V}_0^1(\Omega)} \leqslant \gamma, \tag{1.9}$$

there exists $g \in L^2((0,\infty) \times \mathbb{T})$ satisfying (1.3) *such that the solution (u,p) of system* (1.2) *satisfies*

$$\forall t \geqslant 0, \quad \|u(t)\|_{\mathbf{V}^1(\Omega)} \leqslant Ce^{-\omega_0 t}. \tag{1.10}$$

The strategy used to prove Theorem 1.1 will be exposed in Section 3 and the detailed proof of Theorem 1.1 will be given in Section 5.

Let us emphasize that we go beyond the threshold $\nu\pi^2/L^2$ imposed by the linearized equations (1.4). Therefore, due to the above remarks, Theorem 1.1 cannot be obtained as a consequence of the stabilizability properties of the linearized equations (1.4) only, and the non-linearity in (1.2) shall be used. In order to do that, we perform a power series expansion of the solution u in the spirit of [17, Chap. 8], where such a strategy is explained for a control problem in a finite dimensional setting, and of [18], see also [14, 15], where it is applied to obtain controllability results for the Korteweg de Vries equation. We also refer to [8, 10] for applications of these techniques to the case of Schrödinger equations.

In fact, we will expand the solution u at order 2 in the form $u = \varepsilon\alpha + \varepsilon^2\beta$, where $\varepsilon > 0$ is small, see Section 3 for more details. This will allow us to somehow decouple the dynamics of the linearized system (1.4) satisfied by α to the one of second order satisfied by β, in which the term $\alpha \cdot \nabla\alpha$ can be seen as an indirect control on the dynamics of β. With this respect, we are close to the setting developed recently in [21] in a finite-dimensional context in which a second order expansion was used to propose time-varying feedback laws to stabilize a class of quadratic systems, and [22] where a similar strategy is developed to stabilize the Korteweg de Vries equation in a case in which the linearized equations are not controllable. Still, our approach constructs an open loop control. We thus call the property stated in Theorem 1.1 an *open loop stabilization result*, using the wording of [12, Part I, Chap.1, Definition 2.3]). So far, we do not know if we can construct a time-periodic feedback to obtain stabilization results at any rate for (1.2), but the works [21, 22] might suggest some ideas in this direction.

1.2 Related references

The stabilization of the linearized incompressible Navier-Stokes system in a 2d channel (with periodic conditions with respect to x_1) linearized around a steady-state parabolic laminar flow profile $(\mathscr{P}(x_2), 0)$, with $\mathscr{P}(x_2) = C(x_2^2 - Lx_2)$ with $C \in \mathbb{R}$ (Poiseuille flow) has been studied in [29, 6, 5]. In particular, in [29] the author proved that the linearized equations of (1.2) around the state $(\mathscr{P}(x_2), 0)$ is exponentially stabilizable with some decay rate ω_0, $0 < \omega_0 \leqslant \nu\pi^2/L^2$ by a finite-dimensional feedback control acting on the normal velocity on the upper wall $\{x_2 = L\}$. Similar stability results were obtained when the controls act on the normal components of the velocity on both lower and upper walls $\{x_2 \in \{0, L\}\}$ in [5]. In [6], it was shown that the exponential stability of the linearized system around $(\mathscr{P}(x_2), 0)$ can be achieved with probability 1 using a finite number of Fourier modes and a stochastic boundary feedback controller acting on the normal component of velocity only.

Let us also mention that there are several results on the boundary stabilization of Navier-Stokes equations [7, 28, 1, 4, 37, 33], all of them using the stabilization properties of the linearized equations. Here, we emphasize that when using only one boundary control acting on the normal component of velocity and trying to obtain a decay rate higher than $\nu\pi^2/L^2$, further work is required.

Our approach is actually closely related to the question of local exact controllability to trajectories for the Navier-Stokes equations when considering distributed controls with some vanishing components. On this issue, let us mention [19]. There, the authors consider the two-dimensional Navier-Stokes system in a torus and establish local null controllability with internal controls having one vanishing component. In such a case, the linearized equations around the state 0 are not null-controllable. As in our case, a whole family of solutions cannot be controlled directly. Thus, the authors use the non-linearity to recover the controllability of the non-linear system, in the spirit of the *return method*, see e.g. [17]. Similar ideas were also developed recently in [20] to show local null controllability for the three-dimensional Navier-Stokes equations by using a distributed control with two vanishing components. Let us also mention that, according to [23] and an easy extension argument, Navier-Stokes equations are locally null-controllable when the controls act on the boundary. However, the controls constructed in that way act on both the tangential and normal components of the velocity. The possibility of controlling Navier-Stokes equations with controls acting only on the normal component of the velocity does not seem to follow easily from the construction in [17, 20].

In fact, our result is also related to the boundary controllability results obtained in the works [11, 2] for particular linear parabolic systems with controls acting at the boundary on only some components of the system. Still, as in our case there is an infinite number of directions on which the control does not act (recall (1.6)), our strategy does not follow the ones developed in these articles.

We should also mention that one motivation to study the case of a control acting only on the normal components of the boundary of the fluid comes from stabilization problems for fluid-structure models when the structure is located at the boundary of the fluid and the control acts on the structure. We refer to [33, 35] and [24, 25] for several models of this type. We have also to mention the works [26] which studies a related unique continuation problem for Stokes equations and [30], where a unique continuation result is proved in the case of a channel with Dirichlet boundary conditions on the whole boundary. See also [31], [32] for more refined results of the same type.

1.3 Outline

In Section 2, we recall some results concerning the linearized system (1.4) and the unique continuation properties of the eigenvectors of the Stokes operator, emphasizing the presence of eigenvectors for which unique continuation fails. We also introduce some notations associated to our stabilization problem. In Section 3, we present the general strategy of our approach and describe the main results we need, in particular to control the Navier-Stokes equations in the vector space spanned by the eigenvectors of the 0-mode of (1.4) corresponding to eigenvalues larger than $-\omega_0$. In Section 4, we show how to control the projection on this space. This contains the main technical difficulties of our work. In Section 5, we prove Theorem 1.1 by collecting together all the estimates proved in the previous sections and showing the exponential decay of the norm of the solution at rate ω_0. Section 6 then provides some further comments. The proof of some technical results are given in the appendix.

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2 Preliminaries on the linearized system (1.4)

In this section, we recall some basic facts on the linearized equation (1.4) and give a modal description adapted to our setting.

2.1 Functional framework

In the following, we will often deal with functions u defined in Ω taking values in $\mathbb{R}^2$, hence belonging to functional spaces of the form $(L^2(\Omega))^2$, or $(H^s(\Omega))^2$, $s \in \{1,2\}$. To simplify notations, we will simply denote by $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^s(\Omega)$, the spaces $(L^2(\Omega))^2$ and $(H^s(\Omega))^2$ respectively.

Recall that $\Omega = \mathbb{T} \times (0, L)$ (see (1.1) and Figure 1.1). For convenience, we define $\Gamma_0 = \{(x_1, 0) \,|\, x_1 \in \mathbb{T}\}$, $\Gamma_1 = \{(x_1, L) \,|\, x_1 \in \mathbb{T}\}$ the lower and upper boundaries, and $\Gamma = \Gamma_0 \cup \Gamma_1$. In addition to the spaces $\mathbf{V}^1(\Omega)$, $\mathbf{V}^1_0(\Omega)$ defined in (1.7)–(1.8), we introduce the spaces

$$\mathbf{V}^0(\Omega) = \left\{ u = (u_1, u_2) \in \mathbf{L}^2(\Omega) \,|\, \text{div } u = 0,\ \langle u.n, 1\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0 \right\}, \tag{2.1}$$

$$\mathbf{V}^0_n(\Omega) = \left\{ u \in \mathbf{V}^0(\Omega) \,|\, u \cdot n = 0 \text{ on } \Gamma \right\}. \tag{2.2}$$

We also introduce the Helmholtz operator $\mathbb{P}$ (also called Leray projector) as the orthogonal projection operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{V}^0_n(\Omega)$. This operator $\mathbb{P}$ can be defined as follows

$$\mathbb{P}f = f - \nabla p - \nabla q,$$

where

$$\left\{ \begin{array}{ll} \Delta p = \text{div } f, & \text{in } \Omega, \\ p = 0, & \text{on } \Gamma, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} \Delta q = 0, & \text{in } \Omega, \\ \partial_n q = (f - \nabla p) \cdot n, & \text{on } \Gamma. \end{array} \right.$$

The Stokes operator is then given by

$$A = \nu\mathbb{P}\Delta, \text{ with domain } \mathscr{D}(A) = \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega) \text{ on } \mathbf{V}^0_n(\Omega). \tag{2.3}$$

It is well-known that A is the infinitesimal generator of a strongly continuous analytic semigroup $(e^{tA})_{\{t\geqslant 0\}}$ on $\mathbf{V}^0_n(\Omega)$ since A is maximal dissipative and self-adjoint. Besides, A has a compact resolvent, so the spectrum of A consists of a set of isolated real eigenvalues of finite multiplicity going to $-\infty$.

We now rapidly describe the functional setting adapted to the linearized equations (1.4).

As we mentioned, the control function g has to satisfy condition (1.3). We therefore introduce the set

$$L^2_0(\mathbb{T}) = \left\{ g \in L^2(\mathbb{T}) \,\Big|\, \int_0^{2\pi} g(x_1)dx_1 = 0 \right\}.$$

To put system (1.4) in an abstract form, we introduce the Dirichlet operator $\mathbb{D} \in \mathscr{L}(L^2_0(\mathbb{T}), \mathbf{V}^0(\Omega))$ defined by

$$\mathbb{D}g = w, \text{ with } \begin{cases} -\nu\Delta w + \nabla p = 0, & \text{in } \Omega, \\ \text{div } w = 0, & \text{in } \Omega, \\ w = (0,0), & \text{on } \Gamma_0, \\ w = (0,g), & \text{on } \Gamma_1. \end{cases} \tag{2.4}$$

Then the linearized equations (1.4) can be rewritten in the following abstract form

$$\begin{cases} \mathbb{P}u' = \tilde{A}\mathbb{P}u + (-\tilde{A})\mathbb{P}\mathbb{D}g, & \text{for } t > 0, \\ (I-\mathbb{P})u = (I-\mathbb{P})\mathbb{D}g, & \text{for } t \geqslant 0, \\ \mathbb{P}u(0) = \mathbb{P}u^0, \end{cases} \tag{2.5}$$

where $\tilde{A}$ is the extension to the space $(\mathscr{D}(A))'$ of the unbounded operator A with domain $\mathscr{D}(\tilde{A}) = \mathbf{V}^0_n(\Omega)$ defined by the extrapolation method. Therefore, the control operator in (1.4) reads as

$$B = (-\tilde{A})\mathbb{P}\mathbb{D} : L^2_0(\mathbb{T}) \longrightarrow \mathscr{D}(A)'. \tag{2.6}$$

The study of the Cauchy problem for (1.4) is done in [34]. For $u_0 \in \mathbf{V}^0_n(\Omega)$ and $g \in L^2(0,T;L^2_0(\mathbb{T}))$ equation (1.4) admits a unique weak solution in $L^2(0,T;\mathbf{V}^0(\Omega))$. But we will often use the following regularity results on Stokes equations with source term and non-homogeneous Dirichlet boundary conditions, that can be found in [34], see in particular [34, Theorem 2.3, item (ii)] and [34, Theorem 2.5]. If $u^0 \in \mathbf{V}^1_0(\Omega)$, $f \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $g \in H^1_0(0,T;L^2_0(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$, then the solution u of

$$\begin{cases} \mathbb{P}u' = \tilde{A}\mathbb{P}u + \mathbb{P}f + (-\tilde{A})\mathbb{P}\mathbb{D}g, & \text{for } t > 0, \\ (I-\mathbb{P})u = (I-\mathbb{P})(\mathbb{D}g), & \text{for } t \geqslant 0, \\ \mathbb{P}u(0) = \mathbb{P}u^0, \end{cases} \tag{2.7}$$

or equivalently

$$\begin{cases} \partial_t u - \nu\Delta u + \nabla p = f, & \text{in } (0,\infty)\times\Omega, \\ \text{div } u = 0, & \text{in } (0,\infty)\times\Omega, \\ u(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ u(t,x_1,L) = (0,g(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T}, \\ u(0,x_1,x_2) = u^0(x_1,x_2), & \text{in } \Omega, \end{cases} \tag{2.8}$$

belongs to $H^1(0,T;\mathbf{V}^0(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ and

$$\|u\|_{H^1(0,T;\mathbf{V}^0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))} \leqslant C\left(\left\|u^0\right\|_{\mathbf{V}^1_0(\Omega)} + \|f\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|g\|_{H^1_0(0,T;L^2_0(\mathbb{T}))\cap L^2(0,T;H^2(\mathbb{T}))} \right). \tag{2.9}$$

2.2 Stabilizable eigenvectors

Stabilizability for a given decay rate $\omega > 0$ of the pair (A, B) appearing in (2.3), (2.6) reduces to show the following unique continuation property (see e.g. [3]):

$$\text{If } A^*\Phi = \lambda\Phi \text{ for some Re } \lambda \geqslant -\omega \text{ and if } B^*\Phi = 0, \text{ then } \Phi = 0. \tag{2.10}$$

In (2.3), $A^* = A$ and therefore we have to verify (2.10) for $\lambda \in \mathbb{R}$ and $\lambda \geqslant -\omega$. Note that we already know that such a unique continuation property is violated for the eigenvectors corresponding to the 0-mode (1.5)–(1.6). In Proposition 2.1 afterwards, we will show that it is the only case in which the unique continuation property (2.10) fails.

We start by rewriting the equation $A\Phi = \lambda\Phi$ in its PDE form

$$\begin{cases} \lambda\Phi - \nu\Delta\Phi + \nabla q = 0, & \text{in } \Omega, \\ \text{div } \Phi = 0, & \text{in } \Omega, \\ \Phi = 0, & \text{on } \Gamma. \end{cases} \tag{2.11}$$

The computation of $B^*\Phi$ yields to

$$B^*\Phi(x_1) = q(x_1, L) - \frac{1}{2\pi}\int_{\mathbb{T}} q(x_1, L)dx_1, \quad x_1 \in \mathbb{T}. \tag{2.12}$$

Expanding (Φ, q) into Fourier series

$$(\Phi, q) = (\phi_1, \phi_2, q) \text{ with } \begin{cases} \phi_1(x_1, x_2) = \sum_{k\in\mathbb{Z}} \phi_{1,k}(x_2)e^{ikx_1}, & (x_1, x_2) \in \Omega, \\ \phi_2(x_1, x_2) = \sum_{k\in\mathbb{Z}} \phi_{2,k}(x_2)e^{ikx_1}, & (x_1, x_2) \in \Omega, \\ q(x_1, x_2) = \sum_{k\in\mathbb{Z}} q_k(x_2)e^{ikx_1}, & (x_1, x_2) \in \Omega, \end{cases} \tag{2.13}$$

the eigenvalue problem (2.11) for $(\phi_{1,k}, \phi_{2,k}, q_k)$ reads as follows

$$\begin{cases} (\lambda + \nu k^2)\phi_{1,k}(x_2) - \nu\phi''_{1,k}(x_2) + ikq_k(x_2) = 0, & \text{in } (0, L), \\ (\lambda + \nu k^2)\phi_{2,k}(x_2) - \nu\phi''_{2,k}(x_2) + q'_k(x_2) = 0, & \text{in } (0, L), \\ ik\phi_{1,k}(x_2) + \phi'_{2,k}(x_2) = 0, & \text{in } (0, L), \\ \phi_{1,k}(0) = \phi_{1,k}(L) = \phi_{2,k}(0) = \phi_{2,k}(L) = 0, \end{cases} \tag{2.14}$$

while the observation is

$$B^*\Phi(x_1) = \sum_{k\in\mathbb{Z}\setminus\{0\}} q_k(L)e^{ikx_1}, \quad x_1 \in \mathbb{T}.$$

As explained in the introduction, under this form we immediately check that, the unique continuation cannot hold for the 0-mode. But, $B^*\Phi = 0$ implies that $q_k(L) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Based on that remark, we will prove the following result.

Proposition 2.1. *Assume that Φ satisfies $A\Phi = \lambda\Phi$ for some $\lambda \in \mathbb{R}$ and $B^*\Phi = 0$. Expanding (Φ, q) as in (2.13), for all $k \in \mathbb{Z} \setminus \{0\}$, we have $\phi_{1,k} = \phi_{2,k} = q_k = 0$ everywhere in $(0, L)$.*

Proof. If (Φ, q) written as in (2.13) solves (2.14) and if $B^*\Phi = 0$, then we have $q_k(L) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$. Therefore, for $k \in \mathbb{Z} \setminus \{0\}$, $\phi_{2,k}$ solves

$$\begin{cases} \nu\phi^{(4)}_{2,k}(x_2) - (\lambda + 2\nu k^2)\phi''_{2,k}(x_2) + k^2(\lambda + \nu k^2)\phi_{2,k}(x_2) = 0 \ \text{ in } (0, L), \\ \phi_{2,k}(0) = \phi_{2,k}(L) = \phi'_{2,k}(0) = \phi'_{2,k}(L) = \phi'''_{2,k}(L) = 0. \end{cases} \tag{2.15}$$

Obviously, if $\lambda = 0$, multiplying (2.15) by $\phi_{2,k}$, integrating over $(0, L)$ and using the boundary conditions, we get $\phi_{2,k} = 0$. Similarly, if $\lambda = -\nu k^2$, multiplying (2.15) by $\phi_{2,k}$, integrating in space and using the boundary conditions, we get $\phi'_{2,k} = 0$. Hence $\phi_{2,k} = 0$ thanks to the boundary conditions.

We therefore focus on the case $\lambda \notin \{0, -\nu k^2\}$, and we look for a solution θ of the following problem:

$$\begin{cases} \nu\theta^{(4)}(x_2) - (\lambda + 2\nu k^2)\theta''(x_2) + k^2(\lambda + \nu k^2)\theta(x_2) = 0 & \text{in } (0, L), \\ \theta(0) = \theta'(0) = 0, \end{cases} \quad \text{with } \theta'(L) \neq 0. \tag{2.16}$$

As (2.16) is an ODE with constant coefficients, with roots $\pm k$, $\pm\sqrt{k^2 + \lambda/\nu}$ (all distinct since $\lambda \notin \{0, -\nu k^2\}$), θ has to be of the form

$$\theta(x_2) = C_1 e^{kx_2} + C_2 e^{-kx_2} + C_3 e^{(\sqrt{k^2+\frac{\lambda}{\nu}})x_2} + C_4 e^{(-\sqrt{k^2+\frac{\lambda}{\nu}})x_2}, \quad x_2 \in (0, L),$$

for C_1, C_2, C_3, C_4 suitable constants. We then choose θ of the form

$$\theta(x_2) = \sqrt{k^2 + \frac{\lambda}{\nu}} \sinh(kx_2) - k \sinh\left(\sqrt{k^2 + \frac{\lambda}{\nu}} x_2\right), \quad x_2 \in (0, L),$$

so that it satisfies the ODE $(2.16)_{(1)}$ in $(0, L)$, the boundary condition $\theta(0) = \theta'(0) = 0$ and

$$\theta'(L) = k\sqrt{k^2 + \frac{\lambda}{\nu}} \left(\cosh(kL) - \cosh\left(\sqrt{k^2 + \frac{\lambda}{\nu}} L\right)\right).$$

As $\lambda \notin \{0, -\nu k^2\}$, $\theta'(L) \neq 0$ (since $\lambda \in \mathbb{R} \setminus \{0, -\nu k^2\}$, $\sqrt{k^2 + \lambda/\nu} \in (\mathbb{R}_+ \setminus \{0, k\}) \cup i\mathbb{R}$).

Using this function θ as a test function, we multiply (2.15) by θ and integrate, and we get $\phi''_{2,k}(L) = 0$. Therefore, $\phi_{2,k}$ satisfies $\phi_{2,k}(L) = \phi'_{2,k}(L) = \phi''_{2,k}(L) = \phi'''_{2,k}(L) = 0$ and satisfies the linear ODE (2.15) without source term. Hence $\phi_{2,k} = 0$ everywhere and $\phi_{1,k}$ and q_k also identically vanish in $(0, L)$ thanks to (2.14). □

2.3 Projection

According to Proposition 2.1, a part of the spectrum of the operator A is detectable through B^*. Given $\omega_0 > 0$, we introduce

$$\omega > \omega_0,$$

and decompose the functional space $\mathbf{V}^0_n(\Omega)$ into the following vector spaces:

- The stable part corresponding to the vector space spanned by the eigenfunctions of A with eigenvalues smaller than $-\omega$.
- The unstable undetectable part corresponding to the vector space spanned by the eigenfunctions of A in the 0-mode with eigenvalues larger than or equal to $-\omega$.
- The unstable detectable part corresponding to the vector space spanned by the eigenfunctions of A which do not belong to the 0-mode with eigenvalues larger than or equal to $-\omega$.

Note that the eigenvalues of A may be multiple so we need to be slightly more precise than these vague statements.

The *stable* part is given by

$$\mathbf{Z}_s = \text{Span}\,\{\Phi \,|\, A\Phi = \lambda\Phi, \text{ with } \lambda < -\omega\},$$

while the *unstable* part is given by

$$\mathbf{Z}_u = \text{Span}\,\{\Phi \,|\, A\Phi = \lambda\Phi, \text{ with } \lambda \geqslant -\omega\}.$$

Now, the spectrum of the 0-mode is given by the sequence of eigenvectors

$$\Psi_{0,\ell}(x_1, x_2) = \sqrt{\frac{1}{\pi L}} \begin{pmatrix} \sin\left(\dfrac{\ell\pi x_2}{L}\right) \\ 0 \end{pmatrix}, \text{ corresponding to the eigenvalue } \lambda_{0,\ell} = -\frac{\nu\ell^2\pi^2}{L^2}, \tag{2.17}$$

indexed by $\ell \in \mathbb{N}^*$.

Therefore we introduce the spaces

$$\begin{aligned} \mathbf{Z}_{uu} &= \mathrm{Span}\,\{\Psi_{0,\ell} \text{ with } \lambda_{0,\ell} \geqslant -\omega\}, && (2.18)\\ \mathbf{Z}_{ud} &= \mathbf{Z}_{uu}^{\perp_{\mathbf{L}^2(\Omega)}} \cap \mathbf{Z}_u. && (2.19) \end{aligned}$$

corresponding respectively to the *unstable undetectable* and the *unstable detectable* parts of the spectrum.

In particular we have

$$\mathbf{V}^0_n(\Omega) = \mathbf{Z}_s \oplus \mathbf{Z}_{ud} \oplus \mathbf{Z}_{uu}.$$

We introduce the orthogonal projections $\mathbb{P}_s$, $\mathbb{P}_u$, $\mathbb{P}_{ud}$ and $\mathbb{P}_{uu}$ in $\mathbf{L}^2(\Omega)$ on $\mathbf{Z}_s$, $\mathbf{Z}_u$, $\mathbf{Z}_{ud}$ and $\mathbf{Z}_{uu}$ respectively.

3 Strategy

As said in the introduction, our approach is based on a power series method. To be more precise, we assume that the controlled solution u and its control g in (1.2) can be expanded as

$$u = \varepsilon\alpha + \varepsilon^2\beta, \quad p = \varepsilon p_1 + \varepsilon^2 p_2, \quad g = \varepsilon g_1 + \varepsilon^2 g_2, \tag{3.1}$$

for some $\varepsilon > 0$ small enough, where (α, β), (p_1, p_2), (g_1, g_2) are all of order 1. This allows us to look for (g_1, g_2) such that the solution (α, β) of

$$\begin{cases} \partial_t\alpha - \nu\Delta\alpha + \nabla p_1 = 0, & \text{in } (0,\infty)\times\Omega,\\ \mathrm{div}\ \alpha = 0, & \text{in } (0,\infty)\times\Omega,\\ \alpha(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T},\\ \alpha(t,x_1,L) = (0, g_1(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T},\\ \alpha(0,x_1,x_2) = \alpha^0(x_1,x_2), & \text{in } \Omega, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \partial_t\beta - \nu\Delta\beta + \nabla p_2 = -(\alpha+\varepsilon\beta)\cdot\nabla(\alpha+\varepsilon\beta), & \text{in } (0,\infty)\times\Omega,\\ \mathrm{div}\ \beta = 0, & \text{in } (0,\infty)\times\Omega,\\ \beta(t,x_1,0) = (0,0), & \text{on } (0,\infty)\times\mathbb{T},\\ \beta(t,x_1,L) = (0, g_2(t,x_1)), & \text{on } (0,\infty)\times\mathbb{T},\\ \beta(0,x_1,x_2) = \beta^0(x_1,x_2), & \text{in } \Omega, \end{cases} \tag{3.3}$$

is stable and decays exponentially at rate $-\omega_0$.

As the control function g_1 cannot act on the 0-mode of α (recall (1.6)), we will put the component of u on $\mathbf{Z}_{uu}$ in the β-part. Our construction will therefore use the non-linear term $\alpha\cdot\nabla\alpha$ in (3.3) to indirectly control the projection of β on $\mathbf{Z}_{uu}$.

From now onwards we assume that the initial conditions (α^0, β^0) satisfy

$$\left\|\alpha^0\right\|^2_{\mathbf{V}^1_0(\Omega)} + \left\|\beta^0\right\|_{\mathbf{V}^1_0(\Omega)} \leqslant 1, \quad \text{with } \mathbb{P}_{uu}\alpha^0 = 0. \tag{3.4}$$

We then construct the open loop controlled trajectory u by an iterative process. We thus fix $T > 0$ (for instance $T = 1$) and we introduce the time intervals $(nT, (n+1)T)$.

Initialization: the time interval $(0, T)$. During the first time interval, we look for a control function g_1 such that the projection on $\mathbf{Z}_u$ of the solution α of (3.2) at time T vanishes. This can indeed be done:

Proposition 3.1. *Given $\alpha^0 \in \mathbf{V}^1_0(\Omega)$ satisfying $\mathbb{P}_{uu}\alpha^0 = 0$, there exists a control function $g_1 \in H^1_0(0,T; L^2_0(\mathbb{T})) \cap L^2(0,T; H^2(\mathbb{T}))$ such that the solution α of* (3.2) *on* $(0,T)$ *satisfies the controllability requirement*

$$\mathbb{P}_u\alpha(T) = 0. \tag{3.5}$$

We can further impose the following estimates:

$$\|\alpha(T)\|_{\mathbf{V}^1_0(\Omega)} + \|\alpha\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} + \|g_1\|_{H^1_0(0,T;L^2_0(\mathbb{T}))\cap L^2(0,T;H^2(\mathbb{T}))} \leqslant C\left\|\alpha^0\right\|_{\mathbf{V}^1_0(\Omega)}. \tag{3.6}$$

Actually, Proposition 3.1 is a rather easy consequence of Proposition 2.1, the fact that $\mathbf{Z}_u$ is finite dimensional and the fact that $\mathbb{P}_{uu}\alpha^0 = 0$. The detailed proof of a slightly more general result stated in Proposition A.1 (yielding Proposition 3.1 as an immediate corollary) is given in Appendix A.

The control g_2 is simply taken to be 0 on the time interval $(0, T)$ and β is the corresponding solution of (3.3) on the time interval $(0, T)$.

The iterative process. Let $n \in \mathbb{N} \setminus \{0\}$ and assume that we have constructed α on $(0, nT)$ such that

$$\alpha(nT) \in \mathbf{V}^1_0(\Omega), \quad \beta(nT) \in \mathbf{V}^1_0(\Omega), \tag{3.7}$$

and

$$\mathbb{P}_u \alpha(nT) = 0. \tag{3.8}$$

Obviously, one could maintain that last condition (3.8) of vanishing projection on $\mathbf{Z}_u$ for $t \geqslant nT$ simply by taking $g_1(t) = 0$ for $t \geqslant nT$. But we will need to use the non-linear term $\alpha \cdot \nabla \alpha$ in the equation (3.3) of β to control the projection on $\mathbf{Z}_{uu}$ of β at time $(n+1)T$, so we shall not take $g_1 = 0$ on $(nT, (n+1)T)$.

Instead, we shall build the control function g_1 on the time interval $(nT, (n+1)T)$ by using following result:

Theorem 3.2. *Let $\tilde\beta^0 \in \mathbf{Z}_{uu}$ and $f \in L^2(0, T; \mathbf{H}^2(\Omega)) \cap H^1(0, T; \mathbf{V}^0(\Omega))$. Then there exists a control function $\tilde g_1 \in H^1_0(0, T; H^2(\mathbb{T}) \cap L^2_0(\mathbb{T}))$ such that the solution $\tilde\alpha$ of*

$$\begin{cases} \partial_t \tilde\alpha - \nu\Delta\tilde\alpha + \nabla\tilde p_1 = 0, & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \tilde\alpha = 0, & \text{in } (0, T) \times \Omega, \\ \tilde\alpha(t, x_1, 0) = (0, 0), & \text{on } (0, T) \times \mathbb{T}, \\ \tilde\alpha(t, x_1, L) = (0, \tilde g_1(t, x_1)), & \text{on } (0, T) \times \mathbb{T}, \\ \tilde\alpha(0, x_1, x_2) = 0, & \text{in } \Omega, \end{cases} \tag{3.9}$$

satisfies

$$\tilde\alpha(T) = 0 \text{ in } \Omega, \tag{3.10}$$

and such that the solution $\tilde\beta$ of

$$\begin{cases} \partial_t \tilde\beta - \nu\Delta\tilde\beta + \nabla\tilde p_2 = -(f + \tilde\alpha) \cdot \nabla(f + \tilde\alpha), & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \tilde\beta = 0, & \text{in } (0, T) \times \Omega, \\ \tilde\beta(t, x_1, 0) = \tilde\beta(t, x_1, L) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\ \tilde\beta(0, x_1, x_2) = \tilde\beta^0(x_1, x_2), & \text{in } \Omega, \end{cases} \tag{3.11}$$

satisfies

$$\mathbb{P}_{uu}\tilde\beta(T) = 0. \tag{3.12}$$

We can further impose the following estimates:

$$\|\tilde\alpha\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + \left\|\tilde\beta\right\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + \|\tilde g_1\|^2_{H^1_0(0,T;H^2(\mathbb{T}))} \\ \leqslant C \left(\|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{V}^0(\Omega))} + \left\|\tilde\beta^0\right\|_{\mathbf{V}^1_0(\Omega)} \right). \tag{3.13}$$

Theorem 3.2, whose proof is postponed to Section 4, is the delicate point of our approach, as it shows that the non-linearity can be used to control the projection on the unstable undetectable space. With this respect, Theorem 3.2 should be compared with [18, Proposition 8] and [15, Proposition 3.1] in the context of Korteweg de Vries equations, or [10, Proposition 8] in the context of Schrödinger equations.

We now describe how Theorem 3.2 is applied in our argument. On the interval $(nT, (n+1)T)$, we introduce the solution $\widehat\alpha$ of

$$\begin{cases} \partial_t \widehat\alpha - \nu\Delta\widehat\alpha + \nabla\widehat p_1 = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \operatorname{div} \widehat\alpha = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \widehat\alpha(t, x_1, 0) = \widehat\alpha(t, x_1, L) = (0, 0), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \widehat\alpha(nT, x_1, x_2) = \alpha(nT, x_1, x_2), & \text{in } \Omega. \end{cases} \tag{3.14}$$

Note that $\mathbb{P}_u \widehat{\alpha}((n+1)T) = 0$ since $\mathbb{P}_u \alpha(nT) = 0$.

We then use Theorem 3.2 with

$$\tilde{\beta}^0 = \mathbb{P}_{uu}\beta(nT) \quad \text{and} \quad f(t) = \widehat{\alpha}(t - nT),$$

and let $\tilde{\alpha}_n$, $\tilde{g}_{1,n}$ and $\tilde{\beta}_n$ be the functions given by Theorem 3.2, and for $t \in (nT, (n+1)T)$ we set

$$\tilde{\alpha}(t) = \tilde{\alpha}_n(t - nT), \quad g_1(t) = \tilde{g}_{1,n}(t - nT), \quad \tilde{\beta}(t) = \tilde{\beta}_n(t - nT),$$

so that we have

$$\tilde{\alpha}((n+1)T) = 0, \quad \text{and} \quad \mathbb{P}_{uu}\tilde{\beta}((n+1)T) = 0.$$

We then choose the control function g_2 on $(nT, (n+1)T)$ given by Proposition A.1 such that the solution $\widehat{\beta}$ of

$$\begin{cases} \partial_t \widehat{\beta} - \nu \Delta \widehat{\beta} + \nabla \widehat{p}_2 = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \operatorname{div} \widehat{\beta} = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \widehat{\beta}(t, x_1, 0) = (0, 0), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \widehat{\beta}(t, x_1, L) = (0, g_2(t, x_1)), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \widehat{\beta}(nT, x_1, x_2) = (I - \mathbb{P}_{uu})\beta(nT, x_1, x_2), & \text{in } \Omega, \end{cases} \tag{3.15}$$

satisfies

$$\mathbb{P}_{ud}\widehat{\beta}((n+1)T) = -\mathbb{P}_{ud}\tilde{\beta}((n+1)T) = -\mathbb{P}_u\tilde{\beta}((n+1)T). \tag{3.16}$$

This can be done thanks to Proposition A.1, which moreover shows that $\mathbb{P}_{uu}\widehat{\beta}((n+1)T) = 0$ since $\mathbb{P}_{uu}\widehat{\beta}(nT) = 0$. So

$$\mathbb{P}_u\widehat{\beta}((n+1)T) = -\mathbb{P}_u\tilde{\beta}((n+1)T), \tag{3.17}$$

or equivalently

$$\mathbb{P}_u\left(\widehat{\beta}((n+1)T) + \tilde{\beta}((n+1)T)\right) = 0.$$

The above construction provides g_1 and g_2 on $(nT, (n+1)T)$. The functions α and β are then simply given by equations (3.2)–(3.3) during the time interval $(nT, (n+1)T)$.

By construction, on the time interval $(nT, (n+1)T)$, we have the identity

$$\alpha = \widehat{\alpha} + \tilde{\alpha}, \tag{3.18}$$

implying in particular that $\alpha((n+1)T) = \widehat{\alpha}((n+1)T) + \tilde{\alpha}((n+1)T) = \widehat{\alpha}((n+1)T)$, so $\mathbb{P}_u\alpha((n+1)T) = \mathbb{P}_u\widehat{\alpha}((n+1)T)$. Therefore condition (3.8) at time $(n+1)T$ holds, i.e. $\mathbb{P}_u\alpha((n+1)T) = 0$.

On the other hand, β should be close to $\tilde{\beta} + \widehat{\beta}$ in the time interval $(nT, (n+1)T)$ up to terms of the order of ε. Indeed,

$$\beta_\varepsilon = \beta - (\tilde{\beta} + \widehat{\beta}), \tag{3.19}$$

satisfies

$$\begin{cases} \partial_t \beta_\varepsilon + \varepsilon^2 \beta_\varepsilon \nabla \beta_\varepsilon - \nu \Delta \beta_\varepsilon + \nabla q_2 = -f_\varepsilon, & \text{in } (nT, (n+1)T) \times \Omega, \\ \operatorname{div} \beta_\varepsilon = 0, & \text{in } (nT, (n+1)T) \times \Omega, \\ \beta_\varepsilon(t, x_1, 0) = \beta_\varepsilon(t, x_1, L) = (0, 0), & \text{on } (nT, (n+1)T) \times \mathbb{T}, \\ \beta_\varepsilon(nT, x_1, x_2) = 0, & \text{in } \Omega, \end{cases} \tag{3.20}$$

with

$$\begin{aligned} f_\varepsilon &= \varepsilon(\beta_\varepsilon + \tilde{\beta} + \widehat{\beta}) \cdot \nabla \alpha + \varepsilon \alpha \cdot \nabla(\beta_\varepsilon + \tilde{\beta} + \widehat{\beta}) + \varepsilon^2 \left((\beta_\varepsilon + \tilde{\beta} + \widehat{\beta}) \cdot \nabla(\beta_\varepsilon + \tilde{\beta} + \widehat{\beta}) - \beta_\varepsilon \nabla \beta_\varepsilon \right) \\ &= \operatorname{div}\left(\varepsilon(\tilde{\beta} + \widehat{\beta}) \otimes \alpha + \varepsilon \alpha \otimes (\tilde{\beta} + \widehat{\beta}) + \varepsilon^2 (\tilde{\beta} + \widehat{\beta}) \otimes (\tilde{\beta} + \widehat{\beta}) \right) \\ &\quad + \operatorname{div}\left(\varepsilon \beta_\varepsilon \otimes (\alpha + \varepsilon(\tilde{\beta} + \widehat{\beta})) + \varepsilon(\alpha + \varepsilon(\tilde{\beta} + \widehat{\beta})) \otimes \beta_\varepsilon \right). \end{aligned} \tag{3.21}$$

In Section 4, we shall then present a detailed proof of Theorem 3.2, which is the main point in our argument. Section 5 will then put together all the required estimates to show the exponential decay of the solution u at a rate ω_0 and conclude Theorem 1.1.

Let us finally mention that all the above functions α, $\tilde{\alpha}$, $\widehat{\alpha}$, $\tilde{\beta}$ and $\widehat{\beta}$ are defined independently on ε on each time interval of the form $(nT, (n+1)T)$. Still, these functions depend on $\mathbb{P}_u\alpha(nT)$ and $\mathbb{P}_u\beta(nT)$, this latter one involving ε through β_ε in (3.19). However, we will not make explicit this dependence in ε in the functions α, $\tilde{\alpha}$, $\widehat{\alpha}$, $\tilde{\beta}$ and $\widehat{\beta}$ for simplicity of notations.

4 Proof of Theorem 3.2

To simplify notations, in this section we omit the superscript $\tilde{}$ in Theorem 3.2.

The proof of Theorem 3.2 will require several preliminary steps.

We will begin by showing that the 1-modes of the solutions α of (3.9) are exactly controllable to trajectories, see Section 4.1.

Based on this result, given $\ell \in \mathbb{N}$, we will construct two independent null-controlled solutions α_ℓ^a and α_ℓ^b of (3.9)–(3.10) such that for

$$\alpha = a\alpha_\ell^a + b\alpha_\ell^b, \quad (a,b) \in \mathbb{R}^2,$$

the corresponding solution β of

$$\begin{cases} \partial_t \beta - \nu\Delta\beta + \nabla p_2 = -\alpha \cdot \nabla\alpha, & \text{in } (0,T)\times\Omega, \\ \text{div } \beta = 0, & \text{in } (0,T)\times\Omega, \\ \beta(t,x_1,0) = \beta(t,x_1,L) = (0,0), & \text{on } (0,\infty)\times\mathbb{T}, \\ \beta(0,x_1,x_2) = 0, & \text{in } \Omega, \end{cases} \tag{4.1}$$

satisfies

$$\langle \beta(T), \Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)} = ab,$$

see Section 4.2.

In Section 4.3, we then deduce that, given any $\beta^1 \in \mathbf{Z}_{uu}$, one can find a controlled solution α solving (3.9)–(3.10) such that the corresponding solution β of (4.1) satisfies $\mathbb{P}_{uu}\beta(T) = \beta^1$.

Once this will be done, the proof of Theorem 3.2 will follow easily, see Section 4.4.

4.1 Null controllability of the 1-modes of (3.9)

In this section, we are only interested in the 1-modes of the solutions α of (3.9). This means that we restrict ourselves to functions (α, p) such that

$$\begin{aligned} \alpha^0(x_1,x_2) &= \alpha^{0,c}(x_2)\cos(x_1) + \alpha^{0,s}(x_2)\sin(x_1), \\ \alpha(t,x_1,x_2) &= \alpha^c(t,x_2)\cos(x_1) + \alpha^s(t,x_2)\sin(x_1), \\ p(t,x_1,x_2) &= p^c(t,x_2)\cos(x_1) + p^s(t,x_2)\sin(x_1), \end{aligned} \tag{4.2}$$

with control functions g chosen as

$$g(t,x_1) = g^c(t)\cos(x_1) + g^s(t)\sin(x_1). \tag{4.3}$$

Easy computations show that (α, p) in (4.2) is a solution of (3.9) with control function g of the form (4.3) if and only if $(\alpha_1^c, \alpha_2^s, p^s)$ solves

$$\begin{cases} \partial_t\alpha_1^c + \nu\alpha_1^c - \nu\partial_{22}\alpha_1^c + p^s = 0, & \text{in } (0,T)\times(0,L), \\ \partial_t\alpha_2^s + \nu\alpha_2^s - \nu\partial_{22}\alpha_2^s + \partial_2 p^s = 0, & \text{in } (0,T)\times(0,L), \\ -\alpha_1^c + \partial_2\alpha_2^s = 0, & \text{in } (0,T)\times(0,L), \\ \alpha_1^c(t,0) = \alpha_1^c(t,L) = \alpha_2^s(t,0) = 0, & \text{in } (0,T), \\ \alpha_2^s(t,L) = g^s(t), & \text{in } (0,T), \\ (\alpha_1^c(0,x_2), \alpha_2^s(0,x_2)) = (\alpha_1^{0,c}(x_2), \alpha_2^{0,s}(x_2)), & \text{in } (0,L), \end{cases} \tag{4.4}$$

and $(\alpha_1^s, \alpha_2^c, p^c)$ solves

$$\begin{cases} \partial_t\alpha_1^s + \nu\alpha_1^s - \nu\partial_{22}\alpha_1^s - p^c = 0, & \text{in } (0,T)\times(0,L), \\ \partial_t\alpha_2^c + \nu\alpha_2^c - \nu\partial_{22}\alpha_2^c + \partial_2 p^c = 0, & \text{in } (0,T)\times(0,L), \\ \alpha_1^s + \partial_2\alpha_2^c = 0, & \text{in } (0,T)\times(0,L), \\ \alpha_1^s(t,0) = \alpha_1^s(t,L) = \alpha_2^c(t,0) = 0, & \text{in } (0,T), \\ \alpha_2^c(t,L) = g^c(t), & \text{in } (0,T), \\ (\alpha_1^s(0,x_2), \alpha_2^c(0,x_2)) = (\alpha_1^{0,s}(x_2), \alpha_2^{0,c}(x_2)), & \text{in } (0,L). \end{cases} \tag{4.5}$$

Of course, the two systems (4.4) and (4.5) behave similarly since one can be obtained from the other by the transformation $(\alpha_1, \alpha_2, p) \to (-\alpha_1, \alpha_2, p)$. We therefore focus on the controllability property of

$$\begin{cases} \partial_t \alpha_1 + \nu \alpha_1 - \nu \partial_{22} \alpha_1 - p = 0, & \text{in } (0,T) \times (0,L), \\ \partial_t \alpha_2 + \nu \alpha_2 - \nu \partial_{22} \alpha_2 + \partial_2 p = 0, & \text{in } (0,T) \times (0,L), \\ \alpha_1 + \partial_2 \alpha_2 = 0, & \text{in } (0,T) \times (0,L), \\ \alpha_1(t,0) = \alpha_1(t,L) = \alpha_2(t,0) = 0, & \text{in } (0,T), \\ \alpha_2(t,L) = g(t), & \text{in } (0,T), \\ (\alpha_1(0,x_2), \alpha_2(0,x_2)) = (\alpha_1^0(x_2), \alpha_2^0(x_2)), & \text{in } (0,L). \end{cases} \tag{4.6}$$

We now introduce the functional spaces

$$\begin{aligned} \boldsymbol{\mathcal{V}}^0(0,L) &= \{(\alpha_1,\alpha_2) \in \mathbf{L}^2(0,L) |\, \alpha_1 + \partial_2 \alpha_2 = 0 \text{ in } (0,L)\}, \\ \boldsymbol{\mathcal{V}}_n^0(0,L) &= \{(\alpha_1,\alpha_2) \in \mathbf{L}^2(0,L) |\, \alpha_1 + \partial_2 \alpha_2 = 0 \text{ in } (0,L),\, \alpha_2(0) = \alpha_2(L) = 0\}, \\ \boldsymbol{\mathcal{V}}_0^1(0,L) &= \{(\alpha_1,\alpha_2) \in \mathbf{H}_0^1(0,L) |\, \alpha_1 + \partial_2 \alpha_2 = 0 \text{ in } (0,L)\}, \\ \boldsymbol{\mathcal{V}}^1(0,L) &= \{(\alpha_1,\alpha_2) \in \mathbf{H}^1(0,L) |\, \alpha_1 + \partial_2 \alpha_2 = 0 \text{ in } (0,L),\, \alpha_1(0) = \alpha_1(L) = \alpha_2(0) = 0\}, \end{aligned}$$

which are the natural functional spaces to work with when considering (4.6), corresponding for $\boldsymbol{\mathcal{V}}^0(0,L)$, $\boldsymbol{\mathcal{V}}_n^0(0,L)$ and $\boldsymbol{\mathcal{V}}_0^1(0,L)$ to the projection on the 1-modes of $\mathbf{V}^0(\Omega)$, $\mathbf{V}_0^1(\Omega)$ and $\mathbf{V}_0^1(\Omega)$ respectively.

We will not recall the whole Cauchy theory for (4.6), which can be deduced easily from the results in [34] by projecting on the first Fourier mode the Stokes equation (1.4). We shall in particular use extensively the following result: if $(\alpha_1^0, \alpha_2^0) \in \boldsymbol{\mathcal{V}}^1(0,L)$ and $g \in H^1(0,T)$ with $\alpha_2^0(L) = g(0)$, then the solution α of (4.6) belongs to $L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\boldsymbol{\mathcal{V}}^0(0,L))$ and, similarly as in (2.9), we get an estimate of the form

$$\|\alpha\|_{L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\boldsymbol{\mathcal{V}}^0(0,L))} \leqslant C \left(\left\| (\alpha_1^0, \alpha_2^0) \right\|_{\boldsymbol{\mathcal{V}}^1(0,L)} + \|g\|_{H^1(0,T)} \right). \tag{4.7}$$

The goal of Section 4.1 is to prove the following lemma:

Lemma 4.1. *System* (4.6) *is null controllable in any time $T > 0$ with controls in $H_0^1(0,T)$. To be more precise, for any $(\alpha_1^0, \alpha_2^0) \in \boldsymbol{\mathcal{V}}_0^1(0,L)$, there exists a control function $g \in H_0^1(0,T)$ such that the solution (α_1, α_2) of* (4.6) *satisfies*

$$(\alpha_1(T,x_2), \alpha_2(T,x_2)) = (0,0) \text{ in } (0,L). \tag{4.8}$$

Besides, the controlled trajectory α lies in $L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\boldsymbol{\mathcal{V}}^0(0,L))$.

Before going into the proof of Lemma 4.1, let us mention that we prove null controllability with controls in $H_0^1(0,T)$. This regularity is needed to obtain the regularity of the controlled trajectory.

Let us also note that, as pointed out in the recent work [16], similar arguments as the one used for establishing Lemma 4.1 can be developed to show that for all $k \in \mathbb{N} \setminus \{0\}$, the k-mode of the equation (1.4) is null-controllable with controls in $L^2(0,T)$. The work [16] also shows that this family of equations is null-controllable uniformly with respect to the Fourier parameter $k \in \mathbb{N} \setminus \{0\}$ through a deeper spectral analysis as the one we propose here. Still, we have chosen to present a detailed proof of Lemma 4.1 to underline how controls in $H_0^1(0,T)$ can be constructed and to introduce several spectral computations that will be useful afterwards.

An easy argument also shows the following corollary, which is the result we will actually use in the proof of Theorem 3.2:

Corollary 4.2. *Given $(\alpha_1^0, \alpha_2^0) \in \boldsymbol{\mathcal{V}}^1(0,L)$ and a trajectory $\overline{\alpha} \in L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\boldsymbol{\mathcal{V}}^0(0,L))$ satisfying* (4.6) *with control function $\overline{g} \in H^1(0,T)$, there exists a control function $g \in H^1(0,T)$ with $g(0) = \alpha_2^0(L)$ and $g(T) = \overline{g}(T)$ such that the solution $\alpha = (\alpha_1, \alpha_2)$ of* (4.6) *satisfies*

$$\alpha(T) = \overline{\alpha}(T) \text{ in } (0,L). \tag{4.9}$$

and $\alpha \in L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\boldsymbol{\mathcal{V}}^0(0,L))$.

Proof of Corollary 4.2. We work on $\tilde{\alpha} = \alpha - \overline{\alpha}$. The control problem is then equivalent to find a null-control $\tilde{g}$ for $\tilde{\alpha}$ solution of (4.6) with initial condition $\tilde{\alpha}^0 \in \mathcal{V}^1(0,L)$ and satisfying the null-controllability requirement $\tilde{\alpha}(T) = 0$. We therefore start by choosing a smooth function $\tilde{g}$ on $(0,T/2)$ such that $\tilde{g}(0) = \tilde{\alpha}_2^0(L)$ and vanishing at time $T/2$. We then have $\tilde{\alpha}(T/2) \in \mathcal{V}_0^1(0,L)$, and we can apply Lemma 4.1 to construct $\tilde{g} \in H_0^1(T/2,T)$ such that $\tilde{\alpha}(T) = 0$, i.e. (4.9). The regularity result on α easily follows. □

Proof of Lemma 4.1. We start with the remark that by a scaling argument in time, we can restrict ourselves to $\nu = 1$ without loss of generality.

As usual, our strategy is based on the observability of the adjoint equation of (4.6). But as we want to use $H^1_0(0,T)$ controls, we start by extending system (4.6) with an integrator. Namely, instead of considering (4.6), we consider

$$\begin{cases} \partial_t \alpha_1 + \alpha_1 - \partial_{22}\alpha_1 - p = 0, & \text{in } (0,T)\times(0,L), \\ \partial_t \alpha_2 + \alpha_2 - \partial_{22}\alpha_2 + \partial_2 p = 0, & \text{in } (0,T)\times(0,L), \\ \alpha_1 + \partial_2\alpha_2 = 0, & \text{in } (0,T)\times(0,L), \\ \alpha_1(t,0) = \alpha_1(t,L) = \alpha_2(t,0) = 0, & \text{in } (0,T), \\ \alpha_2(t,L) = g(t), & \text{in } (0,T), \\ g'(t) = h(t), & \text{in } (0,T), \\ (\alpha_1(0,x_2), \alpha_2(0,x_2), g(0)) = (\alpha_1^0(x_2), \alpha_2^0(x_2), 0), & \text{in } (0,L), \end{cases} \tag{4.10}$$

where the control function h will be looked for in $L^2(0,T)$ and the control objective is

$$(\alpha_1(T,x_2), \alpha_2(T,x_2)) = (0,0) \text{ in } (0,L) \text{ and } g(T) = 0.$$

The adjoint state (w_1, w_2, ξ) then satisfies the following equation:

$$\begin{cases} -\partial_t w_1 + w_1 - \partial_{22} w_1 - q = 0, & \text{in } (0,T)\times(0,L), \\ -\partial_t w_2 + w_2 - \partial_{22} w_2 + \partial_2 q = 0, & \text{in } (0,T)\times(0,L), \\ w_1 + \partial_2 w_2 = 0, & \text{in } (0,T)\times(0,L), \\ w_1(t,0) = w_1(t,L) = w_2(t,0) = w_2(t,L) = 0, & \text{in } (0,T), \\ -\xi'(t) = q(t,L), & \text{in } (0,T), \\ (w_1(T,x_2), w_2(T,x_2), \xi(T)) = (w_1^T(x_2), w_2^T(x_2), \xi^T), & \text{in } (0,L). \end{cases} \tag{4.11}$$

The observability property for (4.11) corresponding by duality to the null-controllability of (4.10) with controls $h \in L^2(0,T)$ is the following: There exists a constant $C > 0$ such that all solutions (w_1, w_2, ξ) of (4.11) satisfy

$$\|(w_1(0), w_2(0), \xi(0))\|_{\boldsymbol{\mathcal{V}}^0(0,L)\times\mathbb{R}} \leqslant C \|\xi\|_{L^2(0,T)}. \tag{4.12}$$

System (4.11) is triangular: the Stokes part $(4.11)_{(1,2,3,4)}$ can be solved independently, and corresponds to the projection on the 1-modes of the Stokes operator A in (2.3), and the ODE $(4.11)_{(5)}$ can be solved a posteriori.

Our primary goal is therefore to check the following observability property: There exists a constant $C > 0$ such that all solutions (w_1, w_2, ξ) of (4.11) satisfy

$$\|(w_1(0), w_2(0))\|_{\boldsymbol{\mathcal{V}}^0(0,L)} \leqslant C \|\xi\|_{L^2(0,T)}. \tag{4.13}$$

This *a priori* weaker observability result will be shown using a spectral approach and a suitable Müntz-Szász lemma.

We thus consider the spectrum of the operator

$$\mathcal{A}_1 = \mathbb{P}_1(\partial_{22} - I), \text{with domain } \mathscr{D}(\mathcal{A}_1) = \boldsymbol{\mathcal{V}}^1_0(0,L) \cap \mathbf{H}^2(0,L) \text{ on } \boldsymbol{\mathcal{V}}^0_n(0,L), \tag{4.14}$$

where $\mathbb{P}_1$ is the orthogonal projection from $\mathbf{L}^2(0,L)$ to $\boldsymbol{\mathcal{V}}^0_n(0,L)$. As $\mathcal{A}_1$ is negative self-adjoint with compact resolvent (this follows from the fact that it corresponds to the projection on the 1-mode of the operator A), there is an orthonormal basis of $\boldsymbol{\mathcal{V}}^0_n(0,L)$ formed by eigenfunctions $(\Phi_{1,j})_{j\in\mathbb{N}}$ of $\mathcal{A}_1$ corresponding to real eigenvalues $(\lambda_{1,j})_{j\in\mathbb{N}}$ going to $-\infty$. For generic eigenvalue λ of $\mathcal{A}_1$ of the form $\lambda = \lambda_{1,j}$ for some $j \in \mathbb{N}$, we will also use the notation Φ_λ to denote $\Phi_{1,j}$.

We then study the eigenvalue problem

$$\begin{cases} \lambda\phi_1 + \phi_1 - \partial_{22}\phi_1 - q = 0, & \text{in } (0,L), \\ \lambda\phi_2 + \phi_2 - \partial_{22}\phi_2 + \partial_2 q = 0, & \text{in } (0,L), \\ \phi_1 + \partial_2\phi_2 = 0, & \text{in } (0,L), \\ \phi_1(0) = \phi_1(L) = \phi_2(0) = \phi_2(L) = 0, & \end{cases} \tag{4.15}$$

for which an easy energy identity shows that $\lambda < -1$ is needed to get a non-trivial solution. Besides, Proposition 2.1 yields that each eigenvalue is simple: otherwise, one could combine them in a non-trivial way in order to construct an eigenvector for which the corresponding observation would vanish, i.e. $\phi_2'''(L) = 0$.

Arguing as in (2.15), ϕ_2 solves

$$\phi_2^{(4)} - (\lambda+2)\phi_2'' + (\lambda+1)\phi_2 = 0 \text{ in } (0,L), \quad \phi_2(0) = \phi_2(L) = \phi_2'(0) = \phi_2'(L) = 0,$$

so that ϕ_2 can be expanded as

$$\phi(x_2) = C_1 e^{x_2} + C_2 e^{-x_2} + C_3 e^{\mu x_2} + C_4 e^{-\mu x_2}, \qquad \text{with } \mu = \sqrt{1+\lambda}.$$

Here, the complex square root denotes the one of non-negative imaginary part. Thus, the eigenvalues of (4.15) are given by the equation

$$\det M(\mu) = 0 \text{ where } M(\mu) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \mu & -\mu \\ e^L & e^{-L} & e^{\mu L} & e^{-\mu L} \\ e^L & -e^{-L} & \mu e^{\mu L} & -\mu e^{-\mu L} \end{pmatrix}.$$

Recalling $\lambda < -1$, we necessarily have $\mu = \sqrt{1+\lambda} \in \mathbf{i}\mathbb{R}$, and we write $\mu = \mathbf{i}\tilde{\mu}$ with $\tilde{\mu} \in \mathbb{R}$. The equation $\det M(\mu) = 0$ then reads in $\tilde{\mu}$ as follows:

$$[\sinh(L)\sin(\tilde{\mu}L)]\tilde{\mu}^2 - 2[1-\cosh(L)\cos(\tilde{\mu}L)]\tilde{\mu} - \sinh(L)\sin(\tilde{\mu}L) = 0. \tag{4.16}$$

Therefore, the solution of that equation for large $\tilde{\mu}$ should satisfy $\sin(\tilde{\mu}L) \simeq 0$. To be more precise, comparing the solutions of (4.16) with the roots of $[\sinh(L)\sin(\tilde{\mu}L)]\tilde{\mu}^2$, by Rouché's theorem, we get that there exists $\ell_0 > 0$ such that:

- for all $\ell \geqslant \ell_0$, there exists a unique solution $\tilde{\mu}_\ell$ in the ball $B(\ell\pi/L, \pi/4)$, which in fact lies in the interval $(\ell\pi/L - \pi/4, \ell\pi/L + \pi/4)$ as we know that the eigenvalues λ correspond to real $\tilde{\mu}$. The solution $\tilde{\mu}_\ell$ of (4.16) corresponds to $\lambda_{1,\ell} = -\tilde{\mu}_\ell^2 - 1$.
- there is no solution $\tilde{\mu}$ of (4.16) between $\tilde{\mu}_\ell$ and $\tilde{\mu}_{\ell+1}$ for $\ell \geqslant \ell_0$.
- $\tilde{\mu}_\ell - \ell\pi/L \to 0$ as $\ell \to \infty$.

For $\ell \geqslant \ell_0$, The corresponding eigenfunction $(\phi_{1,\ell}, \phi_{2,\ell})$ of (4.15) is such that $\phi_{2,\ell}$ takes the form

$$\phi_{2,\ell}(x_2) = C_1(\mu_\ell)e^{x_2} + C_2(\mu_\ell)e^{-x_2} + C_3(\mu_\ell)e^{\mu_\ell x_2} + C_4(\mu_\ell)e^{-\mu_\ell x_2}.$$

Using $M(\tilde{\mu}_\ell)(\mathrm{Adj}M(\tilde{\mu}_\ell)) = \det M(\tilde{\mu}_\ell)I = 0$, where $\mathrm{Adj}M(\tilde{\mu}_\ell)$ is the transpose of the cofactor matrix (i.e. the adjugate) of $M(\tilde{\mu}_\ell)$, and the fact that the eigenvalue λ_ℓ is simple (thus implying that $\mathrm{Adj}M(\tilde{\mu}_\ell)$ is non-trivial), one set of suitable coefficients for $\phi_{2,\ell}$ is

$$\begin{aligned}
C_1(\mu_\ell) &= \mu_\ell^2\left(e^{-(\mu_\ell+1)L} - e^{(\mu_\ell-1)L}\right) + \mu_\ell\left(2 - e^{-(\mu_\ell+1)L} - e^{(\mu_\ell-1)L}\right), \\
C_2(\mu_\ell) &= \mu_\ell^2\left(e^{(\mu_\ell+1)L} - e^{-(\mu_\ell-1)L}\right) + \mu_\ell\left(2 - e^{(\mu_\ell+1)L} - e^{-(\mu_\ell-1)L}\right), \\
C_3(\mu_\ell) &= \mu_\ell\left(2 - e^{-(\mu_\ell+1)L} - e^{-(\mu_\ell-1)L}\right) + \left(e^{-(\mu_\ell+1)L} - e^{-(\mu_\ell-1)L}\right), \\
C_4(\mu_\ell) &= \mu_\ell\left(2 - e^{(\mu_\ell+1)L} - e^{(\mu_\ell-1)L}\right) + \left(e^{(\mu_\ell+1)L} - e^{(\mu_\ell-1)L}\right).
\end{aligned}$$

Recalling $\mu_\ell \in \mathbf{i}\mathbb{R}$, we get from these expressions

$$\|\phi_{2,\ell}\|_{H_0^1(0,L)} \leqslant K|\mu_\ell|^2$$

and, following,

$$\|(\phi_{1,\ell}, \phi_{2,\ell})\|_{\mathbf{L}^2(0,L)} \leqslant K|\mu_\ell|^2,$$

while

$$\begin{aligned}
\phi_{2,\ell}'''(L) &= C_1(\mu_\ell)e^{L} - C_2(\mu_\ell)e^{-L} + C_3(\mu_\ell)(\mu_\ell)^3 e^{\mu_\ell L} - C_4(\mu_\ell)(\mu_\ell)^3 e^{-\mu_\ell L} \\
&= 2(\mu_\ell^2 - 1)\left(\mu_\ell^2(e^{\mu_\ell L} - e^{-\mu_\ell L}) - \mu_\ell(e^{L} - e^{-L})\right) \\
&= -4\mathbf{i}(\mu_\ell^2 - 1)(\tilde\mu_\ell^2 \sin(\tilde\mu_\ell L) + \tilde\mu_\ell \sinh(L)) \\
&= -4\mathbf{i}(\mu_\ell^2 - 1)\left(\tilde\mu_\ell\left(\frac{2}{\sinh(L)}(1-\cosh(L)\cos(\tilde\mu_\ell L)) + \sinh(L)\right) + \sin(\tilde\mu_\ell L)\right),
\end{aligned}$$

where we have used (4.16) to write:

$$\tilde\mu_\ell^2 \sin(\tilde\mu_\ell L) = \frac{2\tilde\mu_\ell}{\sinh(L)}(1-\cosh(L)\cos(\tilde\mu_\ell L)) + \sin(\tilde\mu_\ell L).$$

Therefore, as $\tilde\mu_\ell - \ell\pi/L \to 0$ as $\ell \to \infty$, we deduce the existence of ℓ_1 such that for $\ell \geqslant \ell_1$,

$$|\phi_{2,\ell}'''(L)| \geqslant c|\tilde\mu_\ell|^3 = c|\mu_\ell|^3,$$

for some $c > 0$ independent of ℓ.

We therefore have shown that all the eigenvalues $\lambda_{1,\ell}$ for $\ell \geqslant \ell_1$ and corresponding normalized eigenvector $(\phi_{1,\ell}, \phi_{2,\ell})$ satisfy

$$\lambda_{1,\ell} \leqslant -c\frac{\ell^2\pi^2}{L^2}, \quad \lambda_{1,\ell} - \lambda_{1,\ell+1} \geqslant c, \quad |\phi_{2,\ell}'''(L)| \geqslant c.$$

for some $c > 0$ independent of $\ell \geqslant \ell_1$.

The other eigenvalues λ correspond to $\tilde\mu = -\mathbf{i}\sqrt{\lambda+1}$ bounded by $\ell_1\pi/L$, and therefore are in finite number: we write them $(\lambda_{1,j})_{j\in J}$ where J is finite, each one of multiplicity one. Indeed, this is a straightforward a consequence of Proposition 2.1 as the corresponding eigenvectors $(\phi_{1,j}, \phi_{2,j})$ of $\lambda_{1,j}$ all satisfy $\phi_{2,j}'''(L) \neq 0$ from Proposition 2.1.

Therefore we get the following properties:

$$\sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \frac{1}{|\lambda|} < \infty, \qquad \inf_{\ell}\{\lambda_{1,\ell} - \lambda_{1,\ell+1}\} > 0, \tag{4.17}$$

and there exists $c > 0$ such that each normalized eigenvector $(\phi_{1,\lambda}, \phi_{2,\lambda})$ of $\mathcal{A}_1$ corresponding to a pressure q_λ,

$$|\phi_{2,\lambda}'''(L)| = |q_\lambda(L)| \geqslant c. \tag{4.18}$$

Let us now consider a solution (w_1, w_2, ξ) of (4.11). According to the spectral theory of $\mathcal{A}_1$, (w_1, w_2) can be expanded as

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} a_\lambda e^{\lambda(T-t)}\Phi_\lambda. \tag{4.19}$$

In particular, we can compute

$$\xi(t) = \left(\xi(T) - \sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \frac{a_\lambda q_\lambda(L)}{\lambda}\right) + \sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \frac{a_\lambda q_\lambda(L)}{\lambda} e^{\lambda(T-t)}. \tag{4.20}$$

Therefore, using (4.17), Müntz-Szász Lemma (stated under the present form in [27, Proposition 3.2]) applies:

$$e^{-T}\left(\xi(T) - \sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \frac{a_\lambda q_\lambda(L)}{\lambda}\right)^2 + \sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \left|\frac{a_\lambda q_\lambda(L)}{\lambda}\right|^2 e^{(\lambda-1)T} \leqslant C\int_{T/2}^{T} |e^{t-T}\xi(t)|^2\, dt. \tag{4.21}$$

This obviously implies

$$\sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} \left| \frac{a_\lambda q_\lambda(L)}{\lambda} \right|^2 e^{\lambda T} \leqslant C \int_0^T |\xi(t)|^2 \, dt. \tag{4.22}$$

Using (4.18), we derive

$$\sum_{\lambda \text{ eigenvalue of } \mathcal{A}_1} |a_\lambda|^2 e^{2\lambda T} \leqslant C \int_0^T |\xi(t)|^2 \, dt.$$

With the orthogonality of the eigenvectors of $(\Phi_{1,\ell})_{\ell\in\mathbb{N}}$ of $\mathcal{A}_1$ in (4.14) and the expansion (4.19), we deduce (4.13) for any arbitrary $T > 0$.

The proof of (4.12) can then be done by a simple contradiction argument (note that it can also be deduced from (4.20) and (4.21) above). Indeed, assume that we get a sequence (w_1^n, w_2^n, ξ^n) of solutions of (4.11) such that

$$\lim_{n\to\infty} \|\xi^n\|_{L^2(0,T)} = 0, \quad \|(w_1^n(0), w_2^n(0), \xi^n(0))\|_{\boldsymbol{\mathcal{V}}^0(0,L)\times\mathbb{R}} = 1. \tag{4.23}$$

From (4.13), we immediately have $\|(w_1^n(0), w_2^n(0))\|_{\boldsymbol{\mathcal{V}}^0(0,L)} \to 0$ as $n \to \infty$. Besides, from (4.13) on the time interval $(T/2, T)$, $\|(w_1^n(T/2), w_2^n(T/2))\|_{\boldsymbol{\mathcal{V}}^0(0,L)}$ converges to 0 as $n \to \infty$. As the semigroup generated by $\mathcal{A}_1$ is analytic, $\|(w_1^n(T/4), w_2^n(T/4))\|_{\mathscr{D}(\mathcal{A}_1^2)}$ goes to 0, and $\|q^n(t,1)\|_{L^2(0,T/4)}$ converges to 0 as $n \to \infty$. Following, ξ^n also strongly converges to 0 in $H^1(0,T/4)$ as $n \to \infty$ and consequently $\xi^n(0)$ converges to 0 as $n \to \infty$, contradicting (4.23) and therefore concluding the proof of (4.12). □

Remark 4.3. *The proof given above can be easily adapted to show the following result: For any $T > 0$, there exists $C > 0$, such that any solution (w_1, w_2) of*

$$\begin{cases} -\partial_t w_1 + w_1 - \partial_{22} w_1 - q = 0, & \text{in } (0,T)\times(0,L), \\ -\partial_t w_2 + w_2 - \partial_{22} w_2 + \partial_2 q = 0, & \text{in } (0,T)\times(0,L), \\ w_1 + \partial_2 w_2 = 0, & \text{in } (0,T)\times(0,L), \\ w_1(t,0) = w_1(t,L) = w_2(t,0) = w_2(t,L) = 0, & \text{in } (0,T), \\ (w_1(T,x_2), w_2(T,x_2)) = (w_1^T(x_2), w_2^T(x_2)), & \text{in } (0,L) \end{cases} \tag{4.24}$$

satisfies

$$\|(w_1(0), w_2(0))\|_{\boldsymbol{\mathcal{V}}^0(0,L)} \leqslant C \|q\|_{L^2(0,T)}. \tag{4.25}$$

4.2 Entering one missing direction

The goal of this subsection is to show the following theorem:

Theorem 4.4. *Let $\ell \in \mathbb{N}^*$ and $T > 0$.*

There exists two control functions g_ℓ^a and g_ℓ^b in $H_0^1(0,T; H^2(\mathbb{T}) \cap L_0^2(\mathbb{T}))$ such that for all a and b in $\mathbb{R}$, if we denote by α the solution of (3.9) with control function $g = a g_\ell^a + b g_\ell^b$, $\alpha(T) = 0$ and the solution β of (4.1) satisfies

$$\langle \beta(T), \Psi_{0,\ell} \rangle_{\mathbf{L}^2(\Omega)} = ab. \tag{4.26}$$

We can further impose that α and β belong to $L^2(0,T; \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{V}^0(\Omega))$ with respective norms bounded by $|a| + |b|$ and $(|a| + |b|)^2$.

Besides, for $\ell_1 \neq \ell$, there exists a coefficient $\gamma_{\ell,\ell_1} \in \mathbb{R}$ such that

$$\langle \beta(T), \Psi_{0,\ell_1} \rangle_{\mathbf{L}^2(\Omega)} = ab \, \gamma_{\ell_1,\ell}. \tag{4.27}$$

The next paragraphs aim at proving Theorem 4.4. Basically, Theorem 4.4 states that we can enter in the undetectable direction $\Psi_{0,\ell}$.

Remark 4.5. *Let us point out the recent work [9] which, roughly speaking, states that considering an order two expansion of a dynamical system is in general a bad idea for small time local controllability when considering scalar control inputs. Theorem 4.4 is not in contradiction with this result as in our case, for all $t > 0$, the control g belongs to a two-dimensional vector space.*

In the rest of this section, when no confusion arises, we will forget the index ℓ to lighten the notations.

4.2.1 Proof of Theorem 4.4, part 1: Computation of the component of β on $\Psi_{0,\ell}$

To begin with, for β solution of (4.1), we compute

$$\beta_{uu}(t) := \sqrt{\pi L}\langle \beta(t), \Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)} = \int_\Omega \beta_1(t,x_1,x_2)\sin\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2.$$

Using (4.1), we obtain

$$\begin{aligned}\frac{d}{dt}\beta_{uu}(t) &= \int_\Omega \partial_t\beta_1(t,x_1,x_2)\sin\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2\\ &= \int_\Omega (\nu\Delta\beta_1 - \partial_1 p_2 - \alpha_1\partial_1\alpha_1 - \alpha_2\partial_2\alpha_1)\sin\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2\\ &= -\frac{\nu\ell^2\pi^2}{L^2}\beta_{uu}(t) + \frac{\ell\pi}{L}\int_\Omega \alpha_1(t,x_1,x_2)\alpha_2(t,x_1,x_2)\cos\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2. \qquad (4.28)\end{aligned}$$

As $\beta^0 = 0$, we get

$$e^{\nu\ell^2\pi^2T/L^2}\beta_{uu}(T) = \frac{\ell\pi}{L}\int_0^T\int_\Omega e^{\nu\ell^2\pi^2t/L^2}\alpha_1(t,x_1,x_2)\alpha_2(t,x_1,x_2)\cos\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2dt.$$

If we choose the control function g in (3.9) of the form (4.3) for g^c and g^s two null-controls of (4.5) and (4.4) respectively, the solution α of (3.9) writes as in (4.2) and

$$\begin{aligned}\frac{L}{\pi\ell}e^{\nu\ell^2\pi^2T/L^2}\beta_{uu}(T) &= \int_0^T\int_\Omega e^{\nu\ell^2\pi^2t/L^2}\alpha_1(t,x_1,x_2)\alpha_2(t,x_1,x_2)\cos\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2dt\\ &= \int_0^T\int_0^L\int_0^{2\pi} e^{\nu\ell^2\pi^2t/L^2}(\alpha_1^c(t,x_2)\cos(x_1) + \alpha_1^s(t,x_2)\sin(x_1))\\ &\qquad\times(\alpha_2^c(t,x_2)\cos(x_1) + \alpha_2^s(t,x_2)\sin(x_1))\cos\left(\frac{\ell\pi x_2}{L}\right) dx_1dx_2dt\\ &= \pi\int_0^T\int_0^L e^{\nu\ell^2\pi^2t/L^2}[\alpha_1^c(t,x_2)\alpha_2^c(t,x_2) + \alpha_1^s(t,x_2)\alpha_2^s(t,x_2)]\cos\left(\frac{\ell\pi x_2}{L}\right) dx_2dt\\ &= \pi\int_0^T \left\langle \begin{pmatrix}\alpha_1^c(t)\\ \alpha_2^s(t)\end{pmatrix}, F(t)\right\rangle_{\mathbf{L}^2(0,L)} dt,\\ &\qquad \text{where } F(t,x_2) = \cos\left(\frac{\ell\pi x_2}{L}\right)e^{\nu\pi^2\ell^2t/L^2}\begin{pmatrix}\alpha_2^c(t,x_2)\\ \alpha_1^s(t,x_2)\end{pmatrix}. \qquad (4.29)\end{aligned}$$

It is therefore convenient to introduce the adjoint equation of (4.6), namely

$$\begin{cases} -\partial_t Z + \nu Z - \nu\partial_{22}Z + \begin{pmatrix} q\\ \partial_2 q\end{pmatrix} = F(t,x_2), & \text{in } (0,T)\times(0,L),\\ -Z_1 + \partial_2 Z_2 = 0, & \text{in } (0,T)\times(0,L),\\ Z(t,0) = Z(t,L) = (0,0), & \text{in } (0,T),\\ Z(T,x_2) = 0, & \text{in } (0,L).\end{cases} \qquad (4.30)$$

We then easily get that

$$\frac{L}{\ell\pi^2}e^{\nu\ell^2\pi^2T/L^2}\sqrt{\pi L}\langle\beta(t),\Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)} = \frac{L}{\ell\pi^2}e^{\nu\ell^2\pi^2T/L^2}\beta_{uu}(T) = \int_0^T g^s(t)q(t,L)\,dt. \qquad (4.31)$$

Note that F depends only on α_2^c and α_1^s which are prescribed by g^c (recall (4.5)). Following, Z in (4.30) and q in (4.31) only depends on g^c. Therefore, the projection of β on $\Psi_{0,\ell}$ is actually given by a quadratic form in (g^s, g^c) which is linear in each variable, g^s and g^c respectively. In particular, taking $g^s(t) = ag^s_{\mathrm{ref}}(t)$ and $g^c(t) = bg^c_{\mathrm{ref}}(t)$ for some controls of reference g^s_{ref}, g^c_{ref} and a and b in $\mathbb{R}$, from the above computation we have

$$\langle\beta(t),\Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)} = ab\frac{\ell\pi^2}{\sqrt{\pi L}L}e^{-\nu\ell^2\pi^2T/L^2}\int_0^T g^s_{\mathrm{ref}}(t)q_{\mathrm{ref}}(t,L)\,dt, \qquad (4.32)$$

where β solves (4.1) with α solving (3.9) with control $g(t,x_1) = a g^s_{\text{ref}}(t)\sin(x_1) + b g^c_{\text{ref}}(t)\cos(x_1)$, and q_{ref} denotes the pressure obtained by solving (4.30) with source term

$$F_{\text{ref}}(t,x_2) = \cos\left(\frac{\ell\pi x_2}{L}\right) e^{\nu\pi^2\ell^2 t/L^2} \begin{pmatrix} \alpha^c_{2,\text{ref}}(t,x_2) \\ \alpha^s_{1,\text{ref}}(t,x_2) \end{pmatrix}, \tag{4.33}$$

where $(\alpha^s_{1,\text{ref}}, \alpha^c_{2,\text{ref}})$ is the solution of (4.5) with control g^c_{ref}.

Thus, our goal is to now design two control functions of reference g^c_{ref} and g^s_{ref} such that, with the above notations,

$$\frac{\ell\pi^2}{\sqrt{\pi L}L} e^{-\nu\ell^2\pi^2 T/L^2} \int_0^T g^s_{\text{ref}}(t) q_{\text{ref}}(t,L)\,dt = 1.$$

This will be precisely the goal of Section 4.2.2. Of course, these controls will *a priori* depend on ℓ.

To end this section, let us point out that the identity (4.32) does not use the specific choice of ℓ, so for $\ell_1 \in \mathbb{N}$, we will also have

$$\langle \beta(t), \Psi_{0,\ell_1}\rangle_{\mathbf{L}^2(\Omega)} = ab\frac{\ell_1\pi^2}{\sqrt{\pi L}L} e^{-\nu\ell_1^2\pi^2 T/L^2} \int_0^T g^s_{\text{ref}}(t) q_{\text{ref},\ell_1}(t,L)\,dt,$$

where q_{ref,ℓ_1} is the pressure obtained by solving (4.30) with source term

$$F_{\text{ref},\ell_1}(t,x_2) = \cos\left(\frac{\ell_1\pi x_2}{L}\right) e^{\nu\pi^2\ell_1^2 t/L^2} \begin{pmatrix} \alpha^c_{2,\text{ref}}(t,x_2) \\ \alpha^s_{1,\text{ref}}(t,x_2) \end{pmatrix}.$$

This immediately implies (4.27).

4.2.2 End of the proof of Theorem 4.4 up to a technical lemma

According to the above computations, the main point in the following will be to prove the following Lemma:

Lemma 4.6. *There exists $g^c_{ref} \in H^1_0(0,T)$ such that the corresponding solution $(\alpha^s_{1,ref}, \alpha^c_{2,ref})$ starting from $(\alpha^{0,s}_{1,ref}, \alpha^{0,c}_{2,ref}) = (0,0)$ of (4.5) is null-controlled at time $t = T$, and the corresponding solution (Z_{ref}, q_{ref}) of (4.30) with source term F_{ref} in (4.33) satisfies*

$$\int_0^T |q_{ref}(t,L)|^2\,dt \neq 0. \tag{4.34}$$

We can furthermore impose $g^c_{ref} = 0$ on $(3T/4, T)$, $(\alpha^s_{1,ref}, \alpha^c_{2,ref}) = 0$ on $(3T/4,T)$ and $(Z_{ref}, q_{ref}) = 0$ on $(3T/4,T)$.

We postpone the proof of Lemma 4.6 to Section 4.2.3 and conclude the proof of Theorem 4.4 assuming Lemma 4.6.

Let g^c_{ref} be as in Lemma 4.6 with $g^c_{\text{ref}} = 0$ on $(3T/4,T)$, $(\alpha^s_{1,\text{ref}}, \alpha^c_{2,\text{ref}}) = 0$ on $(3T/4,T)$ and $(Z_{\text{ref}}, q_{\text{ref}}) = 0$ on $(3T/4,T)$. We then choose $g^s_{\text{ref}} \in H^1_0(0,3T/4)$ such that

$$\int_0^{3T/4} g^s_{\text{ref}}(t) q_{\text{ref}}(t,L)\,dt = \frac{\sqrt{\pi L}L}{\ell\pi^2} e^{\nu\ell^2\pi^2 T/L^2}.$$

The solution $(\alpha^c_{1,\text{ref}}, \alpha^s_{2,\text{ref}})$ of (4.4) starting from $(\alpha^{0,c}_{1,\text{ref}}, \alpha^{0,s}_{2,\text{ref}}) = (0,0)$ with control function g^s_{ref} on $(0,3T/4)$ then reaches some unknown state $(\alpha^c_{1,\text{ref}}(3T/4), \alpha^s_{2,\text{ref}}(3T/4)) \in \mathcal{V}^1_0(0,1)$. Using Lemma 4.1, we can find $g^s_{\text{ref}} \in H^1_0(3T/4,T)$ such that at time T, $(\alpha^c_{1,\text{ref}}(T), \alpha^s_{2,\text{ref}}(T)) = (0,0)$. Besides, as $q_{\text{ref}} = 0$ on the time interval $(3T/4,T)$, we have the following identity:

$$\int_0^T g^s_{\text{ref}}(t) q_{\text{ref}}(t,L)\,dt = \frac{\sqrt{\pi L}L}{\ell\pi^2} e^{\nu\ell^2\pi^2 T/L^2}.$$

It follows that if we take control functions g of the form

$$g(t,x_1) = a g^s_{\text{ref}}(t)\sin(x_1) + b g^c_{\text{ref}}(t)\sin(x_1),$$

the solution α of (3.9) writes

$$\alpha = a \begin{pmatrix} \alpha^c_{1,\mathrm{ref}}(t,x_2)\cos(x_1) \\ \alpha^s_{2,\mathrm{ref}}(t,x_2)\sin(x_1) \end{pmatrix} + b \begin{pmatrix} \alpha^s_{1,\mathrm{ref}}(t,x_2)\sin(x_1) \\ \alpha^c_{2,\mathrm{ref}}(t,x_2)\cos(x_1) \end{pmatrix},$$

and satisfies (3.10), while from (4.32) the solution β of (4.1) satisfies (4.26). This concludes the proof of Theorem 4.4 up to Lemma 4.6 which is proved afterwards.

4.2.3 Proof of Lemma 4.6 up to a technical result

Below, we omit the index "ref" for simplifying notations.

The proof of Lemma 4.6 is based on a 4-steps construction of the controlled function g^c and the corresponding controlled trajectory (α^s_1, α^c_2) solution of (4.5). The main point is to introduce an intermediate time interval in which the solution α will have a prescribed form and for which the corresponding pressure q cannot vanish on the boundary.

Namely, we decompose $(0,T)$ into the intervals $(0,T/4)$, $(T/4,T/2)$, $(T/2,3T/4)$ and $(3T/4,T)$.

For $\mu \in \mathbb{R}$ that will be suitably chosen later, we introduce the solution $(\alpha^*(x_2), p^*(x_2))$ of the stationary Stokes equation:

$$\begin{cases} \mu\alpha^*_1 + \nu\alpha^*_1 - \nu\partial_{22}\alpha^*_1 - p^* = 0, & \text{in } (0,L), \\ \mu\alpha^*_2 + \nu\alpha^*_2 - \nu\partial_{22}\alpha^*_2 + \partial_2 p^* = 0, & \text{in } (0,L), \\ \alpha^*_1 + \partial_2\alpha^*_2 = 0, & \text{in } (0,L), \\ \alpha^*_1(0) = \alpha^*_1(L) = \alpha^*_2(0) = 0, \quad \alpha^*_2(L) = 1. & \end{cases} \tag{4.35}$$

Such solution exists provided μ/ν does not belong to the spectrum of $\mathcal{A}_1$ in (4.14). Then

$$\overline{\alpha}(t,x_2) = e^{\mu t}(\alpha^*_1(x_2), \alpha^*_2(x_2)), \quad \overline{g}(t) = e^{\mu t},$$

solves (4.5).

We then construct the control function g^c and (α^s_1, α^c_2) solution of (4.5) as follows:

- During the time interval $(0,T/4)$, $g^c \in H^1(0,T/4)$ is chosen as a control function satisfying $g^c(0) = 0$ and $g^c(T/4) = \overline{g}(T/4)$ such that the solution (α^s_1, α^c_2) of (4.5) starting at $(0,0)$ reaches $\overline{\alpha}(T/4)$ at time $T/4$. This can be done thanks to Corollary 4.2.
- During the time interval $(T/4,T/2)$, $g^c(t) = e^{\mu t}$ so that the controlled trajectory (α^s_1, α^c_2) of (4.5) satisfies $(\alpha^s_1, \alpha^c_2)(t) = \overline{\alpha}(t)$ for all $t \in (T/4,T/2)$.
- During the time interval $(T/2,3T/4)$, $g^c \in H^1(T/2,3T/4)$ is chosen such that the controlled trajectory (α^s_1, α^c_2) of (4.5) starting from $\overline{\alpha}(T/2)$ at time $T/2$ reaches the state 0 at time $3T/4$. This can be done thanks to Corollary 4.2 with the additional conditions $g^c(T/2) = \overline{g}(T/2)$, $g^c(3T/4) = 0$.
- During the time interval $(3T/4,T)$, the control g^c is set to 0, and the corresponding controlled trajectory (α^s_1, α^c_2) of (4.5) vanishes identically.

This whole construction depends on the parameter μ introduced in (4.35). Our next goal is to show that one can choose μ such that, if (Z,q) denotes the solution of (4.30), then $q(t,L)$ does not identically vanish on $(T/4,T/2)$. We perform a contradiction argument and assume that

$$q(t,L) = 0 \text{ in } (T/4,T/2). \tag{4.36}$$

On $(T/4,T/2)$, Z satisfies the equation

$$\begin{cases} -\partial_t Z + \nu Z - \nu\partial_{22}Z + \begin{pmatrix} q \\ \partial_2 q \end{pmatrix} = e^{(\mu+\nu\ell^2\pi^2/L^2)t}\cos\left(\dfrac{\ell\pi x_2}{L}\right)\begin{pmatrix} \alpha^*_2(x_2) \\ \alpha^*_1(x_2) \end{pmatrix}, & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right)\times(0,L), \\ -Z_1 + \partial_2 Z_2 = 0, & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right)\times(0,L), \\ Z(t,0) = Z(t,L) = (0,0), & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right). \end{cases} \tag{4.37}$$

Therefore, setting

$$Z^*(t,x_2) = e^{-(\mu+\nu\ell^2\pi^2/L^2)t} Z(t,x_2) \text{ and } q^*(t,x_2) = e^{-(\mu+\nu\ell^2\pi^2/L^2)t} q(t,x_2), \tag{4.38}$$

and differentiating the equation satisfied by Z^* in time, we get

$$\begin{cases} -\partial_t\partial_t Z^* - \left(\mu + \dfrac{\nu\ell^2\pi^2}{L^2}\right)\partial_t Z^* + \nu\partial_t Z^* - \nu\partial_{22}\partial_t Z^* + \begin{pmatrix} \partial_t q^* \\ \partial_2\partial_t q^* \end{pmatrix} = 0, & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right)\times(0,L), \\ -\partial_t Z_1^* + \partial_2\partial_t Z_2^* = 0, & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right)\times(0,L), \\ \partial_t Z^*(t,0) = \partial_t Z^*(t,L) = (0,0), & \text{in } \left(\dfrac{T}{4},\dfrac{T}{2}\right), \end{cases} \tag{4.39}$$

while

$$\partial_t q^*(t,L) = 0 \text{ in } \left(\frac{T}{4},\frac{T}{2}\right). \tag{4.40}$$

Applying then the observability inequality (4.25) to $e^{(\mu+\nu\ell^2\pi^2/L^2)t}\partial_t Z^*$ on the time interval $(T/4,T/2)$, we deduce from (4.40) that

$$(\partial_t Z^*, \partial_t q^*) = (0,0) \text{ in } \left(\frac{T}{4},\frac{T}{2}\right), \text{ i.e. } (Z^*(t,x_2), q^*(t,x_2)) = (Z^*(x_2), q^*(x_2)) \text{ in } \left(\frac{T}{4},\frac{T}{2}\right)\times(0,L).$$

so that the equation of Z^* is

$$\begin{cases} -\left(\mu + \dfrac{\nu\ell^2\pi^2}{L^2}\right) Z^* + \nu Z^* - \nu\partial_{22} Z^* + \begin{pmatrix} q^* \\ \partial_2 q^* \end{pmatrix} = \cos\left(\dfrac{\ell\pi x_2}{L}\right)\begin{pmatrix} \alpha_2^*(x_2) \\ \alpha_1^*(x_2) \end{pmatrix}, & \text{in } (0,L), \\ -Z_1^* + \partial_2 Z_2^* = 0, & \text{in } (0,L), \\ Z^*(0) = Z^*(L) = (0,0). \end{cases} \tag{4.41}$$

and condition (4.36) reads

$$q^*(L) = 0. \tag{4.42}$$

This defines an application $\mu \to \alpha^*$ solution of (4.35) for μ/ν not in the spectrum of $\mathcal{A}_1$ and $\mu \to Z^*$ by (4.41) and $\mu \to q^*(L)$ by (4.42) when μ/ν and $-\mu/\nu - \ell^2\pi^2/L^2$ do not belong to the spectrum of $\mathcal{A}_1$. If we are able to find $\mu \in \mathbb{R}$ for which $q^*(L) \neq 0$, we get a contradiction with (4.36) and conclude the proof of Lemma 4.6. We claim that this can be done:

Lemma 4.7. *There exists $\mu \in \mathbb{R}$ such that μ/ν and $-\mu/\nu - \pi^2\ell^2/L^2$ do not belong to the spectrum of $\mathcal{A}_1$ and such that solving* (4.35) *and* (4.41) *yields $q^*(L) \neq 0$.*

The proof of Lemma 4.7 relies on explicit lengthy computations given below, and concludes the proof of Lemma 4.6 by yielding a choice of μ such that (4.42) is violated, so that (4.36) cannot hold.

4.2.4 Proof of Lemma 4.7

For convenience, we introduce W^* solution of

$$\begin{cases} -\left(\mu + \dfrac{\nu\ell^2\pi^2}{L^2}\right) W^* + \nu W^* - \nu\partial_{22} W^* + \begin{pmatrix} r^* \\ \partial_2 r^* \end{pmatrix} = 0, & \text{in } (0,L), \\ -W_1^* + \partial_2 W_2^* = 0, & \text{in } (0,L), \\ W^*(0) = (0,0),\ W^*(L) = (0,1), \end{cases} \tag{4.43}$$

so that multiplying (4.41) by W^*, we get

$$q^*(L) = \int_0^L \cos\left(\frac{\ell\pi x_2}{L}\right)\Big(\alpha_1^*(x_2)W_2^*(x_2) + \alpha_2^*(x_2)W_1^*(x_2)\Big)dx_2. \tag{4.44}$$

Therefore, to compute $q^*(L)$ corresponding to $\mu \in \mathbb{R}$ given, we solve (4.35) and (4.43), which can be done similarly, and we then compute the quantity

$$\int_0^L \cos\left(\frac{\pi x_2}{L}\right)\Big(\alpha_1^*(x_2)W_2^*(x_2) + \alpha_2^*(x_2)W_1^*(x_2)\Big)dx_2.$$

Note that, replacing μ by μ/ν if necessary, we can assume $\nu = 1$ in (4.35) and in (4.43) without loss of generality.

Computation of α^*. In order to compute α^*, we start by computing the equation satisfied by α_2^*:

$$\begin{cases} (\mu + 1 - \partial_{22})(1 - \partial_{22})\alpha_2^* = 0, & \text{in } (0, L), \\ \alpha_1^* + \partial_2 \alpha_2^* = 0, & \text{in } (0, L), \\ \alpha_2^*(0) = \partial_2 \alpha_2^*(0) = \partial_2 \alpha_2^*(L) = 0, \quad \alpha_2^*(L) = 1. \end{cases} \tag{4.45}$$

If $\mu \notin \{0, -1\}$, setting $k = \sqrt{\mu + 1}$, α_2^* can be expanded as

$$\alpha_2^*(x_2) = C_1 e^{x_2} + C_2 e^{-x_2} + C_3 e^{k x_2} + C_4 e^{-k x_2},$$

while

$$\alpha_1^*(x_2) = -C_1 e^{x_2} + C_2 e^{-x_2} - C_3 k e^{k x_2} + C_4 k e^{-k x_2},$$

where the coefficients C_1, C_2, C_3, C_4 are determined by the equation

$$M_{\alpha^*}(k) \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ with } M_{\alpha^*}(k) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & k & -k \\ e^L & e^{-L} & e^{kL} & e^{-kL} \\ e^L & -e^{-L} & k e^{kL} & -k e^{-kL} \end{pmatrix},$$

which is invertible if and only if $\mu \notin \{0, -1\}$ and μ does not belong to the spectrum of $\mathcal{A}_1$. One may thus write

$$\mathrm{Det}(M_{\alpha^*}(k)) \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \mathrm{Adj} M_{\alpha^*}(k) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

i.e. the third line of the cofactor matrix.

Computation of W^*. This can be done as for α^* by replacing μ by $-(\mu + \ell^2\pi^2/L^2)$. Therefore, setting $\rho = \sqrt{1 - (\mu + \ell^2\pi^2/L^2)}$ and assuming that $-(\mu + \pi^2\ell^2/L^2) \notin \{0, -1\}$ and that it does not belong to the spectrum of $\mathcal{A}_1$, we get

$$\begin{aligned} W_2^*(x_2) &= D_1 e^{x_2} + D_2 e^{-x_2} + D_3 e^{\rho x_2} + D_4 e^{-\rho x_2}, \\ W_1^*(x_2) &= D_1 e^{x_2} - D_2 e^{-x_2} + D_3 \rho e^{\rho x_2} - D_4 \rho e^{-\rho x_2}, \end{aligned}$$

where the coefficients D_1, D_2, D_3, D_4 are given by

$$M_{W^*}(\rho) \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ with } M_{W^*} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \rho & -\rho \\ e^L & e^{-L} & e^{\rho L} & e^{-\rho L} \\ e^L & -e^{-L} & \rho e^{\rho L} & -\rho e^{-\rho L} \end{pmatrix}.$$

Similarly as before, one gets:

$$\mathrm{Det}(M_{W^*}(\rho)) \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} = \mathrm{Adj} M_{W^*}(\rho) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

Computation of $q^*(L)$. Based on (4.44) and the above formula for the computation of α^* and W^*, we can compute $q^*(L)$ explicitly. For that let us denote

$$k = \sqrt{\mu + 1}, \quad \rho = \sqrt{-(\mu + \frac{\pi^2\ell^2}{L^2}) + 1}. \tag{4.46}$$

We then compute

$$\tilde{q}(k, \rho) = (Q \mathrm{Adj} M_{\alpha^*}(k) b) \cdot (\mathrm{Adj} M_{W^*}(\rho) b), \text{ where } b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where Q is the matrix given by

$$Q = \begin{pmatrix} 0 & 0 & (\rho-1)I(\rho+1,\ell) & -(\rho+1)I(1-\rho,\ell) \\ 0 & 0 & (\rho+1)I(\rho-1,\ell) & (1-\rho)I(-1-\rho,\ell) \\ (1-k)I(k+1,\ell) & -(k+1)I(k-1,\ell) & (\rho-k)I(k+\rho,\ell) & -(k+\rho)I(k-\rho,\ell) \\ (k+1)I(-k+1,\ell) & (k-1)I(-k-1,\ell) & (k+\rho)I(-k+\rho,\ell) & (k-\rho)I(-k-\rho,\ell) \end{pmatrix},$$

$$\text{where } I(a,\ell) = \int_0^L e^{ax_2}\cos\left(\frac{\ell\pi x_2}{L}\right)dx_2 = \frac{a}{a^2+\ell^2\pi^2/L^2}((-1)^\ell\exp(aL)-1).$$

In particular, for k and ρ as in (4.46), we have the identity:

$$\tilde{q}(k,\rho) = \mathrm{Det}(M_{\alpha^*}(k))\mathrm{Det}(M_{W^*}(\rho))q^*(L). \tag{4.47}$$

Computations performed in *Maxima*[1] yield the following formula for $\tilde{q}(k,\rho)$:

$$\tilde{q}(k,\rho) = J_1(k,\rho)+J_2(k,\rho)+J_3(k,\rho)+J_4(k,\rho), \tag{4.48}$$

where

$$\begin{aligned} J_1(k,\rho) \;=\; & (2\rho e^{-\rho L}-\rho e^{L}+e^{L}-\rho e^{-L}-e^{-L})\times \\ & \Bigg(\frac{(\rho-k)(\rho+k)}{\pi^2\ell^2/L^2+(\rho+k)^2}(2ke^{-kL}-ke^{L}+e^{L}-ke^{-L}-e^{-L})((-1)^\ell e^{(\rho+k)L}-1) \\ & +\frac{(-\rho-k)(k-\rho)}{\pi^2\ell^2/L^2+(k-\rho)^2}(2ke^{kL}-ke^{L}-e^{L}-ke^{-L}+e^{-L})((-1)^\ell e^{(k-\rho)L}-1) \\ & +\frac{(1-k)(k+1)}{\pi^2\ell^2/L^2+(k+1)^2}(k^2e^{kL}-ke^{kL}-k^2e^{-kL}-ke^{-kL}+2ke^{-L})((-1)^\ell e^{(k+1)L}-1) \\ & +\frac{(-k-1)(k-1)}{\pi^2\ell^2/L^2+(k-1)^2}(-k^2e^{kL}-ke^{kL}+k^2e^{-kL}-ke^{-kL}+2ke^{-L})((-1)^\ell e^{(k-1)L}-1)\Bigg), \end{aligned}$$

$$\begin{aligned} J_2(k,\rho) \;=\; & (2\rho e^{\rho L}-\rho e^{L}-e^{L}-\rho e^{-L}+e^{-L})\times \\ & \Bigg(\frac{(\rho-k)(\rho+k)}{\pi^2\ell^2/L^2+(\rho-k)^2}(2ke^{-kL}-ke^{L}+e^{L}-ke^{-L}-e^{-L})((-1)^\ell e^{(\rho-k)L}-1) \\ & +\frac{(-\rho-k)(k-\rho)}{\pi^2\ell^2/L^2+(-k-\rho)^2}(2ke^{kL}-ke^{L}-e^{L}-ke^{-L}+e^{-L})((-1)^\ell e^{(-k-\rho)L}-1) \\ & +\frac{(1-k)(k+1)}{\pi^2\ell^2/L^2+(1-k)^2}(k^2e^{kL}-ke^{kL}-k^2e^{-kL}-ke^{-kL}+2ke^{-L})((-1)^\ell e^{(1-k)L}-1) \\ & +\frac{(-k-1)(k-1)}{\pi^2\ell^2/L^2+(-k-1)^2}(-k^2e^{kL}-ke^{kL}+k^2e^{-kL}-ke^{-kL}+2ke^{-L})((-1)^\ell e^{(-k-1)L}-1)\Bigg), \end{aligned}$$

$$\begin{aligned} J_3(k,\rho) \;=\; & (\rho^2e^{\rho L}-\rho e^{\rho L}-\rho^2e^{-\rho L}-\rho e^{-\rho L}+2\rho e^{-L})\times \\ & \Bigg(\frac{(\rho-1)(\rho+1)}{\pi^2\ell^2/L^2+(\rho+1)^2}(2ke^{-kL}-ke^{L}+e^{L}-ke^{-L}-e^{-L})((-1)^\ell e^{(\rho+1)L}-1) \\ & +\frac{(-\rho-1)(1-\rho)}{\pi^2\ell^2/L^2+(1-\rho)^2}(2ke^{kL}-ke^{L}-e^{L}-ke^{-L}+e^{-L})((-1)^\ell e^{(1-\rho)L}-1)\Bigg), \end{aligned}$$

and

$$\begin{aligned} J_4(k,\rho) \;=\; & (-\rho^2e^{\rho L}-\rho e^{\rho L}+\rho^2e^{-\rho L}-\rho e^{-\rho L}+2\rho e^{L})\times \\ & \Bigg(\frac{(\rho-1)(\rho+1)}{\pi^2\ell^2/L^2+(\rho-1)^2}(2ke^{-kL}-ke^{L}+e^{L}-ke^{-L}-e^{-L})((-1)^\ell e^{(\rho-1)L}-1) \\ & +\frac{(-\rho-1)(1-\rho)}{\pi^2\ell^2/L^2+(-1-\rho)^2}(2ke^{kL}-ke^{L}-e^{L}-ke^{-L}+e^{-L})((-1)^\ell e^{(-1-\rho)L}-1)\Bigg). \end{aligned}$$

[1] The corresponding file is available on S.E.'s webpage.

We then take an increasing sequence $\mu_n \in \mathbb{R}_+^*$ indexed by $n \in \mathbb{N}^*$ going to infinity so that

$$\rho_n = \sqrt{-(\mu_n + \frac{\pi^2\ell^2}{L^2}) + 1},$$

satisfies

$$\exp(\rho_n L) = 1.$$

We then take $k_n = \sqrt{\mu_n + 1} \in \mathbb{R}$, so that we have $k_n \simeq -\mathbf{i}\rho_n$ and $k_n \to \infty$ in the limit $n \to \infty$. Using the Landau notation $o_n(1)$ for a quantity which goes to 0 as $n \to \infty$, this allows to show that

$$\begin{aligned} J_1(k_n, \rho_n) &= 2\rho_n(1-\cosh(L))(1+o_n(1)) \times \\ & \Big(\frac{(\mathbf{i}-1)}{(\mathbf{i}+1)}(-2k_n \cosh(L))(-1)^\ell e^{k_n L} + \frac{(-\mathbf{i}-1)}{(1-\mathbf{i})}(2k_n e^{k_n L})(-1)^\ell e^{k_n L} \\ & +(-1)(k_n^2 e^{k_n L})(-1)^\ell e^{k_n L} e^L + (-1)(-k_n^2 e^{k_n L})(-1)^\ell e^{k_n L} e^{-L}\Big) \\ &= (-1)^{\ell+1} 2\mathbf{i} k_n^3 e^{2k_n L}(1-\cosh(L))\sinh(L)(1+o_n(1)). \end{aligned}$$

Besides, one can easily show that

$$|J_2(k_n, \rho_n)| + |J_3(k_n, \rho_n)| + |J_4(k_n, \rho_n)| \leqslant k_n^3 e^{2k_n L} o_n(1). \tag{4.49}$$

This implies in particular that the sequence $(|\tilde{q}(k_n, \rho_n)|)$ tends to $+\infty$ as $n \to \infty$. Now, we consider the function

$$\tilde{Q} : \mu \in \mathbb{R}_+^* \mapsto \tilde{q}\left(\sqrt{\mu+1}, \sqrt{-(\mu + \frac{\pi^2\ell^2}{L^2}) + 1}\right).$$

It is clear from (4.48) and the formulas giving $J_1(k,\rho)$, $J_2(k,\rho)$, $J_3(k,\rho)$, $J_4(k,\rho)$ that the function $\tilde{Q}$ is analytic in a neighborhood of infinity of the form $[\mu_0, \infty)$. Besides, we know that $\tilde{Q}$ has isolated zeros as $\tilde{Q}(\mu_n)$ with $\mu_n \to \infty$ given above goes to infinity as $n \to \infty$.

Furthermore, $k \mapsto \mathrm{Det}(M_{\alpha^*}(k))$ is an holomorphic function of k which does not vanish identically (its roots correspond to the spectrum of the operator $\mathcal{A}_1$, see the proof of Lemma 4.1), so its roots are isolated. Therefore, the roots of the function

$$\mu \mapsto \mathrm{Det}\left(M_{\alpha^*}\left(\sqrt{\mu+1}\right)\right)$$

are isolated. Similarly, the roots of the function

$$\mu \mapsto \mathrm{Det}\left(M_{W^*}\left(\sqrt{-(\mu + \frac{\pi^2\ell^2}{L^2}) + 1}\right)\right)$$

are isolated.

Therefore, there exists $\mu \in [\mu_0, \infty)$ such that

$$\begin{aligned} &\tilde{q}\left(\sqrt{\mu+1}, \sqrt{-(\mu + \frac{\pi^2\ell^2}{L^2}) + 1}\right) \neq 0, \\ &\mathrm{Det}\left(M_{\alpha^*}\left(\sqrt{\mu+1}\right)\right) \neq 0, \\ &\mathrm{Det}\left(M_{W^*}\left(\sqrt{-(\mu + \frac{\pi^2\ell^2}{L^2}) + 1}\right)\right) \neq 0, \end{aligned}$$

so that we can conclude $q^*(L) \neq 0$ with this choice of μ from the identity (4.47).

4.3 Controllability of β in $\mathbf{Z}_{uu}$

Theorem 4.8. *Let $T > 0$.*

There exists a finite-dimensional vector space $\mathbf{G} \subset H^1_0(0,T; H^2(\mathbb{T}) \cap L^2_0(\mathbb{T}))$ such that for any $\beta^1 \in \mathbf{Z}_{uu}$, there exists a control function $g \in \mathbf{G}$ such that if we denote by α the solution of

$$\begin{cases} \partial_t \alpha - \nu \Delta \alpha + \nabla p_1 = 0, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \alpha = 0, & \text{in } (0,T) \times \Omega, \\ \alpha(t, x_1, 0) = (0,0), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(t, x_1, L) = (0, g(t, x_1)), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(0, x_1, x_2) = 0, & \text{in } \Omega, \end{cases} \tag{4.50}$$

we have $\alpha(T) = 0$ in Ω, and the solution β of (4.1) satisfies

$$\mathbb{P}_{uu} \beta(T) = \beta^1. \tag{4.51}$$

We can further impose that the control function can be chosen such that the map $\mathcal{G} : \beta^1 \in \mathbf{Z}_{uu} \mapsto g \in \mathbf{G}$ is continuous, and the corresponding solutions α of (4.50) with $g = \mathcal{G}(\beta^1)$ and β of (4.1) belong to $L^2(0,T; \mathbf{H}^2(\Omega)) \cap H^1(0,T; \mathbf{V}^0(\Omega))$ with norms bounded as follows:

$$\|\alpha\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + \|\beta\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + \|g\|^2_{H^1_0(0,T;H^2(\mathbb{T}))} \leqslant C \left\|\beta^1\right\|_{\mathbf{V}^1_0(\Omega)}. \tag{4.52}$$

Proof. The basic idea of the proof of Theorem 4.8 is to combine Theorem 4.4 with the fact that, when time evolves and the control is shut down, the projections $t \mapsto \langle \beta(t), \Psi_{0,\ell} \rangle_{\mathbf{L}^2(\Omega)}$ evolve differently. We shall thus combine periods of time in which the control is active with periods of time during which all the controls are switched off.

Let ℓ_0 be the largest integer such that $\nu \ell_0^2 \pi^2 / L^2 \leqslant \omega$, i.e. the dimension of the space $\mathbf{Z}_{uu}$. Set then $T^* = T/(2\ell_0 - 1)$. Applying then Theorem 4.4 in time T^* for $\ell \in \{1, \cdots, \ell_0\}$, for each $\ell \in \{1, \cdots, \ell_0\}$, there exist control functions g^a_ℓ and g^b_ℓ such that for all a_ℓ and b_ℓ in $\mathbb{R}$, the solution α_ℓ of (4.50) with control function $g = a_\ell g^a_\ell + b_\ell g^b_\ell$ satisfies $\alpha_\ell(T^*) = 0$ and the solution β_ℓ of (4.1) satisfies

$$\mathbb{P}_{uu} \beta_\ell(T^*) = a_\ell b_\ell \sum_{j=1}^{\ell_0} \gamma_{j,\ell} \Psi_{0,j}, \quad \text{with } \gamma_{\ell,\ell} = 1. \tag{4.53}$$

We then set $T_k = (2k-1)T^*$ for $k \in \{1, \cdots, \ell_0\}$. For $k \in \{2, \cdots, \ell_0\}$, we introduce time parameters $\tau_k \in [0, T^*]$, to be chosen later. The strategy to choose the control is then as follows:

- For $t \in [0, T_1]$, we choose $g(t, x_1) = a_1 g^a_1(t, x_1) + b_1 g^b_1(t, x_1)$ for $x_1 \in \mathbb{T}$;
- For $k \in \{2, \cdots, \ell_0\}$ and $t \in [T_k - T^* - \tau_k, T_k - \tau_k]$, we choose the control function $g(t, x_1) = a_k g^a_k(t - (T_k - T^* - \tau_k), x_1) + b_k g^b_k(t - (T_k - T^* - \tau_k), x_1)$ for $x_1 \in \mathbb{T}$;
- Otherwise, i.e. for $t \in \mathscr{O} = [0,T] \setminus ([0, T_1] \cup_{k=2}^{\ell_0} [T_k - T^* - \tau_k, T_k - \tau_k])$, we set $g(t, x_1) = 0$ for $x_1 \in \mathbb{T}$.

It is easy to check that, if α denotes the solution α of (4.50) with the above controls, $\alpha(T) = 0$ in Ω and the solution β of (4.1) will satisfy, for all $\ell \in \{1, \cdots, \ell_0\}$,

$$\langle \beta(T), \Psi_{0,\ell} \rangle_{\mathbf{L}^2(\Omega)} = \sum_{k=1}^{\ell_0} \gamma_{\ell,k} a_k b_k e^{-\frac{\nu \pi^2 \ell^2}{L^2}(T - (T_k - \tau_k))}, \tag{4.54}$$

where for simplicity of notations we have introduced the parameter τ_1, that we immediately fix by $\tau_1 = 0$. In other words, we get the following matrix identity:

$$\begin{pmatrix} \langle \beta(T), \Psi_{0,1} \rangle_{\mathbf{L}^2(\Omega)} \\ \langle \beta(T), \Psi_{0,2} \rangle_{\mathbf{L}^2(\Omega)} \\ \vdots \\ \langle \beta(T), \Psi_{0,\ell_0} \rangle_{\mathbf{L}^2(\Omega)} \end{pmatrix} = M_\tau \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \\ \vdots \\ a_{\ell_0} b_{\ell_0} \end{pmatrix} \quad \text{where } (M_\tau)_{\ell,k} = \gamma_{\ell,k} e^{-\frac{\nu \pi^2 \ell^2}{L^2}(T - (T_k - \tau_k))}. \tag{4.55}$$

Note now that the matrix M_τ depends on the choice of the free parameters $\tau = (\tau_k)_{k\in\{2,\cdots,\ell_0\}}$. We shall therefore show that there exists suitable choices of coefficients $\tau_k \in [0, T^*]$ such that the matrix M_τ is invertible. In order to prove this, for all $k \in \{1, \cdots, \ell_0\}$, we introduce the matrix $M_\tau^{(k)}$ formed by the first k lines and columns of M_τ. It is clear that each matrix $M_\tau^{(k)}$ depends only on the choice of the parameters $(\tau_2, \cdots, \tau_k)$. We shall then show by induction that for all $k \in \{1, \cdots, \ell_0\}$, there exists a choice of parameters $(\tau_2, \cdots, \tau_k)$ such that the matrix $M_\tau^{(k)}$ is invertible.

For $k = 1$, $M_\tau^{(1)} = (\gamma_{1,1} e^{-\frac{\nu\pi^2}{L^2}(T-T_1)}) = (e^{-\frac{\nu\pi^2}{L^2}(T-T_1)})$, which is obviously invertible.

Then we assume that for some $k \in \{1, \cdots, \ell_0 - 1\}$, we have found a choice of parameters $(\tau_2, \cdots, \tau_k)$ such that the matrix $M_\tau^{(k)}$ is invertible. We then compute the determinant of the matrix $M_\tau^{(k+1)}$ by developing the last column and using that $\gamma_{k+1,k+1} = 1$:

$$\begin{aligned}\mathrm{Det}(M_\tau^{(k+1)}) = \mathrm{Det}(M_\tau^{(k)}) e^{-\frac{\nu\pi^2(k+1)^2}{L^2}(T-(T_{k+1}-\tau_{k+1}))} \\ + \sum_{j=1}^{k} (-1)^{j+k+1} \mathrm{Det}((M_\tau^{(k+1)})_{j,k+1}) \gamma_{j,k+1} e^{-\frac{\nu\pi^2 j^2}{L^2}(T-(T_{k+1}-\tau_{k+1}))},\end{aligned} \tag{4.56}$$

where $(M_\tau^{(k+1)})_{j,k+1}$ is the minor of the matrix $M_\tau^{(k+1)}$ obtained after having removed the j-th line and the k+1-th column. One then remarks that the matrices $(M_\tau^{(k+1)})_{j,k+1}$ do not depend on τ_{k+1}. Therefore, using that $\mathrm{Det}(M_\tau^{(k)}) \neq 0$ from the induction assumption and that any finite family of real exponentials of the form $(t \mapsto e^{\mu_j t})$ is linearly independent on $\mathbb{R}$ as soon as the family of the μ_j are all distinct, the function $\tau_{k+1} \mapsto \mathrm{Det}(M_\tau^{(k+1)})$ is not identically zero. Besides, as the function $\tau_{k+1} \mapsto \mathrm{Det}(M_\tau^{(k+1)})$ is analytic, there exists $\tau_{k+1} \in [0, T^*]$ such that $\mathrm{Det}(M_\tau^{(k+1)}) \neq 0$.

By recursion, we deduce that there exists a choice of parameters $\tau = (\tau_k)_{k\in\{1,\cdots,\ell_0\}}$ with $\tau_1 = 0$ and $\tau_k \in [0, T^*]$ for all $k \in \{2, \cdots, \ell_0\}$ such that the matrix M_τ is invertible. From now on, we take a choice of parameters $\tau = (\tau_k)_{k\in\{1,\cdots,\ell_0\}}$ with $\tau_1 = 0$ and $\tau_k \in [0, T^*]$ for all $k \in \{2, \cdots, \ell_0\}$ such that the matrix M_τ is invertible.

We can then conclude the proof of Theorem 4.8 as follows. For $\beta^1 \in \mathbf{Z}_{uu}$, we compute

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{\ell_0} \end{pmatrix} = M_\tau^{-1} \begin{pmatrix} \langle \beta^1, \Psi_{0,1} \rangle_{\mathbf{L}^2(\Omega)} \\ \langle \beta^1, \Psi_{0,2} \rangle_{\mathbf{L}^2(\Omega)} \\ \vdots \\ \langle \beta^1, \Psi_{0,\ell_0} \rangle_{\mathbf{L}^2(\Omega)} \end{pmatrix}, \tag{4.57}$$

and

$$\forall \ell \in \{1, \cdots, \ell_0\}, \quad a_\ell = \mathrm{Sign}(A_\ell)\sqrt{|A_\ell|}, \quad b_\ell = \sqrt{|A_\ell|}, \tag{4.58}$$

where Sign is the sign function (with the convention $\mathrm{Sign}(0) = 0$), and we choose the control function as

$$g(t, x_1) = \sum_{k=1}^{\ell_0} \left(a_k g_k^a(t - (T_k - T^* - \tau_k), x_1) + b_k g_k^b(t - (T_k - T^* - \tau_k), x_1) \right) \mathbf{1}_{t-(T_k-T^*-\tau_k)\in[0,T^*]}. \tag{4.59}$$

First, it is clear that the control function g constructed above belongs to a finite-dimensional space $\mathbf{G}$ spanned by the functions $g_k^a(t - (T_k - T^* - \tau_k), x_1)\mathbf{1}_{t-(T_k-T^*-\tau_k)\in[0,T^*]}$, and $g_k^b(t - (T_k - T^* - \tau_k), x_1)\mathbf{1}_{t-(T_k-T^*-\tau_k)\in[0,T^*]}$. Besides, as each of these functions belong to $H_0^1(0, T; H^2(\mathbb{T}) \cap L_0^2(\mathbb{T}))$, $\mathbf{G} \subset H_0^1(0, T; H^2(\mathbb{T}) \cap L_0^2(\mathbb{T}))$.

By construction, the control function g in (4.59) is such that if we denote by α the solution of (4.50) with control function g, we have $\alpha(T) = 0$, and the solution β of (4.1) satisfies (4.51). Besides, we easily check that the map $\beta^1 \in \mathbf{Z}_{uu} \mapsto g \in \mathbf{G}$ given by the above construction is continuous.

Using the explicit construction above, we immediately have that

$$\|g\|_{H_0^1(0,T;H^2(\mathbb{T}))} \leqslant C\sqrt{\|\beta^1\|_{\mathbf{V}_0^1(\Omega)}}.$$

Starting from this estimate, we easily derive (4.52). This concludes the proof of Theorem 4.8. □

4.4 Proof of Theorem 3.2

Let $\beta^0 \in \mathbf{Z}_{uu}$ and $f \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{V}^0(\Omega))$.

Our goal is to perform a fixed point argument. We thus introduce the mapping

$$\mathscr{G} : g \in \mathbf{G} \mapsto \mathcal{G}(-\mathbb{P}_{uu}(\beta(T))) \in \mathbf{G}, \tag{4.60}$$

where $\mathbf{G}$ is the finite dimensional subspace of $H_0^1(0,T;H^2(\mathbb{T}) \cap L_0^2(\mathbb{T}))$ given by Theorem 4.8 and $\mathcal{G}$ is the mapping defined in Theorem 4.8, and β is the solution of

$$\begin{cases} \partial_t \beta - \nu \Delta \beta + \nabla p_2 = -(f \cdot \nabla f) - (\alpha \cdot \nabla f + f \cdot \nabla \alpha), & \text{in } (0,T) \times \Omega, \\ \text{div } \beta = 0, & \text{in } (0,T) \times \Omega, \\ \beta(t, x_1, 0) = \beta(t, x_1, L) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ \beta(0, x_1, x_2) = \beta^0(x_1, x_2), & \text{in } \Omega, \end{cases} \tag{4.61}$$

with α the solution of

$$\begin{cases} \partial_t \alpha - \nu \Delta \alpha + \nabla p_1 = 0, & \text{in } (0,T) \times \Omega, \\ \text{div } \alpha = 0, & \text{in } (0,T) \times \Omega, \\ \alpha(t, x_1, 0) = (0,0), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(t, x_1, L) = (0, g(t, x_1)), & \text{on } (0,T) \times \mathbb{T}, \\ \alpha(0, x_1, x_2) = 0, & \text{in } \Omega. \end{cases} \tag{4.62}$$

If we find a control function $g \in \mathbf{G}$ such that $\mathscr{G}(g) = g$, then by construction the solution α of (4.62) is such that the solution $\check{\beta}$ of

$$\begin{cases} \partial_t \check{\beta} - \nu \Delta \check{\beta} + \nabla \check{p}_2 = -(\alpha \cdot \nabla\ \alpha), & \text{in } (0,T) \times \Omega, \\ \text{div } \check{\beta} = 0, & \text{in } (0,T) \times \Omega, \\ \check{\beta}(t, x_1, 0) = \check{\beta}(t, x_1, L) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ \check{\beta}(0, x_1, x_2) = 0, & \text{in } \Omega, \end{cases}$$

satisfies $\mathbb{P}_{uu}(\check{\beta}(T)) = -\mathbb{P}_{uu}(\beta(T))$ so that the function $\tilde{\beta} = \check{\beta} + \beta$ solves (3.11) with $\tilde{\alpha} = \alpha$ and satisfies (3.12). We have thus reduced the proof of Theorem 3.2 to the proof of the existence of a fixed point to the map $\mathscr{G}$ in (4.60) and to estimates on such a fixed point.

It is clear that the map $\mathscr{G}$ in (4.60) is continuous (we do not need to make precise with which topology $\mathbf{G}$ is endowed as it is a finite dimensional vector space). Besides, the solution α of (4.62) satisfies,

$$\|\alpha\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant C \|g\|_{H_0^1(0,T;H^2(\mathbb{T}))},$$

while the solution β of (4.61) satisfies:

$$\begin{aligned} \|\mathbb{P}_{uu}\beta(T)\|_{\mathbf{V}_0^1(\Omega)} + \|\beta\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} &\leqslant C \|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} \\ &+ C \|f\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} \|\alpha\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + C \left\|\beta^0\right\|_{\mathbf{V}_0^1(\Omega)}, \end{aligned}$$

Using then the estimate (4.52) in Theorem 4.8, we thus deduce that

$$\begin{aligned} &\|\mathscr{G} g\|^2_{H_0^1(0,T;H^2(\mathbb{T}))} \\ &\leqslant C \|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + C \|f\|_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} \|g\|_{H_0^1(0,T;H^2(\mathbb{T}))} + C \left\|\beta^0\right\|_{\mathbf{V}_0^1(\Omega)} \\ &\leqslant C \|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2} \|g\|^2_{H_0^1(0,T;H^2(\mathbb{T}))} + C \left\|\beta^0\right\|_{\mathbf{V}_0^1(\Omega)}, \end{aligned}$$

Therefore, setting

$$\frac{R^2}{2} = C \|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + C \left\|\beta^0\right\|_{\mathbf{V}_0^1(\Omega)},$$

the compact convex set

$$\mathbf{G}_R = \mathbf{G} \cap \{\|g\|_{H_0^1(0,T;H^2(\mathbb{T}))} \leqslant R\}$$

is stable by the map $\mathscr{G}$. Consequently, by Brouwer fixed point theorem, the map $\mathscr{G}$ has a fixed point g in $\mathbf{G}_R$. According to the above choice of R, this fixed point automatically satisfies

$$\|g\|^2_{H_0^1(0,T;H^2(\mathbb{T}))} \leqslant 2C \|f\|^2_{L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))} + 2C \left\|\beta^0\right\|_{\mathbf{V}_0^1(\Omega)},$$

which entails the estimate (3.13).

5 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. In order to do it, we follow the strategy exposed in Section 3 and give estimates at each steps, allowing to conclude the exponential decay at rate ω_0 of the norm of the solution $u = \varepsilon\alpha + \varepsilon^2\beta$ of (1.2).

Section 5.1, respectively 5.2, gives the estimates satisfied by α and β on the time interval $(0,T)$, respectively $(nT,(n+1)T)$.

Section 5.3 then deduce from these estimates the decay of α and β at times nT for $n \in \mathbb{N}$ and ε small enough. Section 5.4 then improves the decay estimates on the function α, allowing to conclude Theorem 1.1 in Section 5.5.

5.1 The initialization step

Recall that (α^0, β^0) are chosen such that (3.4) holds, assumption that we recall here for convenience:

$$\left\|\alpha^0\right\|^2_{\mathbf{V}^1_0(\Omega)} + \left\|\beta^0\right\|_{\mathbf{V}^1_0(\Omega)} \leqslant 1, \quad \text{with } \mathbb{P}_{uu}\alpha^0 = 0. \tag{5.1}$$

Our goal is then to show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0,\varepsilon_0]$,

$$\|\alpha(T)\|^2_{\mathbf{V}^1_0(\Omega)} + \|\beta(T)\|_{\mathbf{V}^1_0(\Omega)} \leqslant C(\left\|\alpha^0\right\|^2_{\mathbf{V}^1_0(\Omega)} + \left\|\beta^0\right\|_{\mathbf{V}^1_0(\Omega)}). \tag{5.2}$$

Estimates (3.6) already gives the estimate we need on α. We complete them with an estimate on β deduced from Corollary B.2. In order to use Corollary B.2, we remark that β solves (B.1) with F as in (B.5) with the choice

$$F_0 = \alpha \otimes \alpha \text{ and } F_1 = \alpha,$$

as we can write

$$\alpha \cdot \nabla\alpha + \varepsilon\beta \cdot \nabla\alpha + \varepsilon\alpha \cdot \nabla\beta = \operatorname{div}(\alpha \otimes \alpha + \varepsilon\beta \otimes \alpha + \varepsilon\alpha \otimes \beta) = \operatorname{div}(F_0 + \varepsilon(F_1 \otimes \beta + \beta \otimes F_1)).$$

Now, we have

$$\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leqslant C\,\|\alpha\|^2_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant C\left\|\alpha^0\right\|^2_{\mathbf{V}^1_0(\Omega)} \leqslant C,$$

where the last estimates comes from (5.1), and

$$\|F_1\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant C\left\|\alpha^0\right\|_{\mathbf{V}^1_0(\Omega)} \leqslant C$$

from (3.6) and (3.4).

Consequently, according to Corollary B.2, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0,\varepsilon_0]$, the solution β of (3.3) in $(0,T)$ belongs to $L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))$ and the following estimate holds:

$$\|\beta(T)\|_{\mathbf{V}^1_0(\Omega)} + \|\beta\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant C\left(\left\|\alpha^0\right\|^2_{\mathbf{V}^1_0(\Omega)} + \left\|\beta^0\right\|_{\mathbf{V}^1_0(\Omega)}\right). \tag{5.3}$$

This completes the proof of (5.2).

5.2 The iterative process

Let $n \in \mathbb{N}^*$.

We first remark that all the construction on the time interval $(nT,(n+1)T)$ depends on the quantities $(\alpha(nT), \mathbb{P}_s\beta(nT), \mathbb{P}_u\beta(nT))$. In order to be able to show the decay of the solution, we shall therefore get an estimate on $(\alpha((n+1)T), \mathbb{P}_s\beta((n+1)T), \mathbb{P}_u\beta((n+1)T))$.

To begin with, we call $D_n \geqslant 0$ a suitable combination of the norms of $\alpha(nT)$ and $\beta(nT)$, namely:

$$D_n = \|\alpha(nT)\|^2_{\mathbf{V}^1_0(\Omega)} + \|\beta(nT)\|_{\mathbf{V}^1_0(\Omega)}. \tag{5.4}$$

In the proof below, C will be used to denote several constants, which may change from line to line but are all independent of the iteration n ($n \in \mathbb{N}$) and of the parameter ε ($\varepsilon \in [0,1]$).

Straightforward estimates on $\widehat{\alpha}$ solving (3.14) yields

$$\|\widehat{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C \|\alpha(nT)\|_{\mathbf{V}_0^1(\Omega)}. \tag{5.5}$$

Besides, as $\alpha(nT)$ satisfies (3.8) and $\widehat{\alpha}(nT) = \alpha(nT)$, we have

$$\|\widehat{\alpha}((n+1)T)\|_{\mathbf{V}_0^1(\Omega)} \leqslant e^{-\omega T} \|\widehat{\alpha}(nT)\|_{\mathbf{V}_0^1(\Omega)} \leqslant e^{-\omega T} \|\alpha(nT)\|_{\mathbf{V}_0^1(\Omega)}. \tag{5.6}$$

As $\tilde{\alpha}((n+1)T) = 0$ by construction and $\alpha = \tilde{\alpha} + \widehat{\alpha}$ (recall (3.18)), we obtain

$$\|\alpha((n+1)T)\|_{\mathbf{V}_0^1(\Omega)} = \|\widehat{\alpha}((n+1)T)\|_{\mathbf{V}_0^1(\Omega)} \leqslant e^{-\omega T} \|\alpha(nT)\|_{\mathbf{V}_0^1(\Omega)}. \tag{5.7}$$

Based on Theorem 3.2, we deduce the following estimate on $(\tilde{\alpha}, \tilde{\beta})$:

$$\begin{aligned}
&\|\tilde{\alpha}\|^2_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} + \left\|\tilde{\beta}\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \\
&\leqslant C\left(\|\widehat{\alpha}\|^2_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{V}^0(\Omega))} + \|\mathbb{P}_{uu}\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right) \\
&\leqslant C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right).
\end{aligned} \tag{5.8}$$

This estimate in particular implies

$$\left\|\tilde{\beta}((n+1)T)\right\|_{\mathbf{V}_0^1(\Omega)} \leqslant C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right), \tag{5.9}$$

and, as $\alpha = \tilde{\alpha} + \widehat{\alpha}$,

$$\|\alpha\|^2_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\mathbb{P}_{uu}\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right) \tag{5.10}$$

From Proposition A.1, the solution $\widehat{\beta}$ of the control problem (3.15)–(3.16) satisfies:

$$\begin{aligned}
\left\|\widehat{\beta}\right\|_{L^2(nT,(n+1)T;H^2(\Omega))\cap H^1(nT,(n+1)T;L^2(\Omega))} &\leqslant C\left(\|(I-\mathbb{P}_{uu})\beta(nT)\|_{\mathbf{V}_0^1(\Omega)} + \left\|\mathbb{P}_{ud}(\tilde{\beta}((n+1)T))\right\|_{\mathbf{V}_0^1(\Omega)}\right) \\
&\leqslant C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right).
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
\left\|\mathbb{P}_s\widehat{\beta}((n+1)T)\right\|_{\mathbf{V}_0^1(\Omega)} &\leqslant e^{-\omega T}\|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}_0^1(\Omega)} + C\left\|\mathbb{P}_u\widehat{\beta}(nT)\right\|_{\mathbf{V}_0^1(\Omega)} + C\left\|\mathbb{P}_{ud}(\tilde{\beta}((n+1)T))\right\|_{\mathbf{V}_0^1(\Omega)} \\
&\leqslant e^{-\omega T}\|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}_0^1(\Omega)} + C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right).
\end{aligned} \tag{5.12}$$

We then have to estimate the solution β_ε of (3.20) with the source term f_ε defined in (3.21). Note that β_ε solves equation (B.1) translated on the time interval $(nT,(n+1)T)$ with F as in (B.5) and

$$F_0 = \varepsilon(\tilde{\beta}+\widehat{\beta})\otimes\alpha + \varepsilon\alpha\otimes(\tilde{\beta}+\widehat{\beta}) + \varepsilon^2(\tilde{\beta}+\widehat{\beta})\otimes(\tilde{\beta}+\widehat{\beta}), \tag{5.13}$$

$$F_1 = \alpha + \varepsilon(\tilde{\beta}+\widehat{\beta}), \tag{5.14}$$

In order to apply Corollary B.2, we shall thus use the bound (5.10) on α and the following bound on $\tilde{\beta}+\widehat{\beta}$ which can be deduced from (5.8) and (5.11):

$$\left\|\tilde{\beta}+\widehat{\beta}\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C\left(\|\alpha(nT)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta(nT)\|_{\mathbf{V}_0^1(\Omega)}\right) \leqslant CD_n. \tag{5.15}$$

This allows to obtain the following estimate on F_0 in (5.13):

$$\begin{aligned}
&\|F_0\|_{L^2(nT,(n+1)T;\mathbf{H}^1(\Omega))} \\
&\leqslant C\varepsilon\left\|\tilde{\beta}+\widehat{\beta}\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))}\|\alpha\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \\
&\qquad + C\varepsilon^2\left\|\tilde{\beta}+\widehat{\beta}\right\|^2_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \\
&\leqslant C\rho_{\varepsilon,n}\left\|\tilde{\beta}+\widehat{\beta}\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \\
&\leqslant C\rho_{\varepsilon,n}D_n,
\end{aligned} \tag{5.16}$$

where we have set

$$\rho_{\varepsilon,n} = \varepsilon \left\|\alpha\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} + \varepsilon^2 \left\|\beta+\tilde{\beta}\right\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))}, \quad (5.17)$$

which, from (5.10) and (5.15), satisfies

$$\rho_{\varepsilon,n} \leqslant C(\varepsilon\sqrt{D_n} + \varepsilon^2 D_n). \quad (5.18)$$

An estimate on F_1 in (5.14) can be deduced directly from (5.10) and (5.15):

$$\|F_1\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C(\sqrt{D_n} + \varepsilon D_n).$$

In particular, recalling that $\varepsilon \in [0,1]$, we have

$$\|F_0\|_{L^2(nT,(n+1)T;\mathbf{H}^1(\Omega))} + \|F_1\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C(\sqrt{D_n} + D_n).$$

so that Corollary B.2 applies: There exists $\tilde{\varepsilon}_n = e_*(C(\sqrt{D_n} + D_n)) > 0$ such that for all $\varepsilon \in [0, \tilde{\varepsilon}_n]$,

$$\|\beta_\varepsilon\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C\rho_{\varepsilon,n} D_n \leqslant C\left(\varepsilon\sqrt{D_n} + \varepsilon^2 D_n\right) D_n, \quad (5.19)$$

where we have used the estimates (5.16)–(5.18).

Setting $\varepsilon_n = \min\{\tilde{\varepsilon}_n, 1/D_n\}$, for all $\varepsilon \in [0,\varepsilon_n]$ we have

$$\|\beta_\varepsilon\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C\sqrt{\varepsilon} D_n. \quad (5.20)$$

Since $\beta = \beta_\varepsilon + (\tilde{\beta} + \widehat{\beta})$ and $\mathbb{P}_u[\widehat{\beta}((n+1)T) + \tilde{\beta}((n+1)T)] = 0$, combining the estimates (5.9), (5.12) and (5.20), we get

$$\begin{aligned} \|\mathbb{P}_s\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)} &\leqslant (e^{-\omega T} + C\sqrt{\varepsilon}) \|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} + C\|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} + C\|\alpha(nT)\|^2_{\mathbf{V}^1_0(\Omega)}, \\ \|\mathbb{P}_u\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)} &\leqslant C\sqrt{\varepsilon}\|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} + C\sqrt{\varepsilon}\|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} + C\sqrt{\varepsilon}\|\alpha(nT)\|^2_{\mathbf{V}^1_0(\Omega)}. \end{aligned} \quad (5.21)$$

for all $\varepsilon \in [0,\varepsilon_n]$.

At this stage, it is important to notice that the estimates (5.21) are valid for $\varepsilon \in [0,\varepsilon_n]$ in which ε_n is given by

$$\varepsilon_n = e_0(D_n), \qquad \text{where } e_0(D) = \min\left\{e_*(C(\sqrt{D} + D)), \frac{1}{D}\right\}, \quad (5.22)$$

and e_* is the decreasing function given by Corollary B.2. Accordingly, e_0 is a decreasing function of D.

Thus, the next step is to check that $\inf_n \varepsilon_n > 0$.

5.3 The decay of α and β at times nT

From (5.7), we immediately have that for all $n \in \mathbb{N}^*$,

$$\|\alpha(nT)\|_{\mathbf{V}^1_0(\Omega)} \leqslant e^{-\omega(n-1)T}\|\alpha(T)\|_{\mathbf{V}^1_0(\Omega)}. \quad (5.23)$$

To show the decay of $(\|\mathbb{P}_s\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)}, \|\mathbb{P}_u\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)})$, we rewrite (5.21) as a vector inequality: for all $n \in \mathbb{N}$, if $\varepsilon \in [0,\varepsilon_n]$,

$$\begin{pmatrix} \|\mathbb{P}_s\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)} \\ \|\mathbb{P}_u\beta((n+1)T)\|_{\mathbf{V}^1_0(\Omega)} \end{pmatrix} \leqslant K_\varepsilon \begin{pmatrix} \|\mathbb{P}_s\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} \\ \|\mathbb{P}_u\beta(nT)\|_{\mathbf{V}^1_0(\Omega)} \end{pmatrix} + C\|\alpha(nT)\|^2_{\mathbf{V}^1_0(\Omega)} \begin{pmatrix} 1 \\ \sqrt{\varepsilon} \end{pmatrix},$$

$$\text{where } K_\varepsilon = \begin{pmatrix} e^{-\omega T} + C\sqrt{\varepsilon} & C \\ C\sqrt{\varepsilon} & C\sqrt{\varepsilon} \end{pmatrix}, \quad (5.24)$$

and the sign $\leqslant$ has to understood component-wise. As K_ε has non-negative entries, we immediately have, for all $n \in \mathbb{N}^*$ and $\forall \varepsilon \in [0, \inf_{j \leqslant n-1} \varepsilon_j]$,

$$\begin{pmatrix} \|\mathbb{P}_s \beta(nT)\|_{\mathbf{V}_0^1(\Omega)} \\ \|\mathbb{P}_u \beta(nT)\|_{\mathbf{V}_0^1(\Omega)} \end{pmatrix} \leqslant K_\varepsilon^{n-1} \begin{pmatrix} \|\mathbb{P}_s \beta(T)\|_{\mathbf{V}_0^1(\Omega)} \\ \|\mathbb{P}_u \beta(T)\|_{\mathbf{V}_0^1(\Omega)} \end{pmatrix} + \sum_{j=1}^{n-1} C \|\alpha(jT)\|^2_{\mathbf{V}_0^1(\Omega)} K_\varepsilon^{n-1-j} \begin{pmatrix} 1 \\ \sqrt{\varepsilon} \end{pmatrix}. \quad (5.25)$$

Now remark that the matrix K_0 obviously has two eigenvalues, 0 and $e^{-\omega T}$. We thus choose $\omega_1 \in (\omega_0, \omega)$, and as the spectrum of a matrix depends continuously of its coefficients, we can find $\varepsilon_* \in (0, 1]$ such that for all $\varepsilon \in [0, \varepsilon_*]$, K_ε has two distinct eigenvalues and both eigenvalues of K_ε are smaller than $e^{-\omega_1 T}$. By diagonalizing K_ε and using continuity with respect to ε , we get the existence of a constant C such that

$$\|K_\varepsilon^n\|_{\mathscr{L}(\mathbb{R}^2)} \leqslant C e^{-n\omega_1 T} \text{ for all } n \in \mathbb{N} \text{ and } \varepsilon \in [0, \varepsilon_*].$$

Using this estimate and (5.23) in (5.25), we obtain, for all $n \in \mathbb{N}^*$ and $\varepsilon \in [0, \min\{\varepsilon_*, \inf_{j \leqslant n-1} \varepsilon_j\}]$,

$$\begin{aligned} \|\beta(nT)\|_{\mathbf{V}_0^1(\Omega)} &\leqslant C e^{-(n-1)\omega_1 T} \|\beta(T)\|_{\mathbf{V}_0^1(\Omega)} + C \|\alpha(T)\|^2_{\mathbf{V}_0^1(\Omega)} \sum_{j=1}^{n-1} e^{-2\omega(j-1)T} e^{-(n-1-j)\omega_1 T} \\ &\leqslant C e^{-(n-1)\omega_1 T} \left(\|\alpha(T)\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta(T)\|_{\mathbf{V}_0^1(\Omega)} \right). \end{aligned} \quad (5.26)$$

Combining the estimates (5.23) and (5.26), we get that for all $n \in \mathbb{N}^*$ and $\varepsilon \in [0, \min\{\varepsilon_*, \inf_{j \leqslant n-1} \varepsilon_j\}]$,

$$D_n \leqslant \left(e^{-2(n-1)\omega_1 T} + C e^{-n\omega_1 T} \right) D_1. \quad (5.27)$$

In particular, taking N such that $e^{-2(N-1)\omega_1 T} + C e^{-N\omega_1 T} \leqslant 1$ and $\varepsilon \in [0, \min\{\varepsilon_*, \inf_{j \leqslant N-1} \varepsilon_j\}]$, from (5.27), $D_N \leqslant D_1$. Consequently, as $\varepsilon_N = e_0(D_N)$ where e_0 is a decaying function (recall (5.22)), we have $\varepsilon_N \geqslant \varepsilon_1$, so that from (5.27), we also have $D_{N+1} \leqslant D_1$ for all $\varepsilon \in [0, \min\{\varepsilon_*, \inf_{j \leqslant N-1} \varepsilon_j\}]$. An easy induction argument then shows that for all $n \geqslant N$ and $\varepsilon \in [0, \min\{\varepsilon_*, \inf_{j \leqslant N-1} \varepsilon_j\}]$, $D_n \leqslant D_1$ and $\varepsilon_n \geqslant \varepsilon_1$.

We thus set

$$\varepsilon = \min \left\{ \varepsilon_*, \inf_{j \leqslant N-1} \varepsilon_j \right\}, \quad (5.28)$$

which is obviously positive and for which, from the above discussion, the estimate (5.26) holds for all $n \in \mathbb{N}^*$.

Combining (3.6), (5.3), (5.23) and (5.26), we thus obtain a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|\alpha(nT)\|_{\mathbf{V}_0^1(\Omega)} \leqslant C e^{-\omega_1 nT} \|\alpha^0\|_{\mathbf{V}_0^1(\Omega)}, \quad (5.29)$$

$$\|\beta(nT)\|_{\mathbf{V}_0^1(\Omega)} \leqslant C e^{-\omega_1 nT} \left(\|\alpha^0\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta^0\|_{\mathbf{V}_0^1(\Omega)} \right), \quad (5.30)$$

which imply in particular that

$$D_n \leqslant C e^{-\omega_1 nT} D_0. \quad (5.31)$$

The estimates in Sections 5.1–5.2 easily yields that, for all $t \geqslant 0$,

$$\|\alpha(t)\|^2_{\mathbf{V}^1(\Omega)} \leqslant C e^{-\omega_1 t} \left(\|\alpha^0\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta^0\|_{\mathbf{V}_0^1(\Omega)} \right), \quad (5.32)$$

$$\|\beta(t)\|_{\mathbf{V}^1(\Omega)} \leqslant C e^{-\omega_1 t} \left(\|\alpha^0\|^2_{\mathbf{V}_0^1(\Omega)} + \|\beta^0\|_{\mathbf{V}_0^1(\Omega)} \right). \quad (5.33)$$

The decay of β is the one we are looking for, but the decay of α stated in (5.32) is slower than the one stated in Theorem 1.1, except at times nT. The goal of the next section is to show that estimate (5.32) can be improved to obtain a time decay on α as $\exp(-\omega_1 t)$.

5.4 Improving the decay of the norm α

In this section, we do a bootstrap argument to improve the estimate (5.32) on the decay of the norm of α.

In order to improve the estimate (5.32), we recall that $\alpha = \widehat{\alpha} + \tilde{\alpha}$. But from (5.5) and (5.29), it is clear that $\widehat{\alpha}$ satisfies, for all $t \geqslant 0$,

$$\|\widehat{\alpha}(t)\|_{\mathbf{V}^1(\Omega)} \leqslant Ce^{-\omega_1 t} \left\|\alpha^0\right\|_{\mathbf{V}_0^1(\Omega)}. \tag{5.34}$$

We shall thus focus on the decay of the norm of $\tilde{\alpha}$. In order to do this, for $n \in \mathbb{N}^*$, using (5.33) and (5.34), we bound $\rho_{\varepsilon,n}$ in (5.17) by

$$\rho_{\varepsilon,n} \leqslant C \|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} + Ce^{-\omega_1 nT}.$$

Consequently, the estimate (5.19) yields, for all $n \in \mathbb{N}^*$,

$$\|\beta_\varepsilon\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C(\|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} + e^{-\omega_1 nT})e^{-\omega_1 nT} D_0.$$

Now, as $\mathbb{P}_u\beta((n+1)T) = \mathbb{P}_u\beta_\varepsilon((n+1)T)$, we have, for all $n \in \mathbb{N}^*$,

$$\|\mathbb{P}_u\beta((n+1)T)\|_{\mathbf{V}_0^1(\Omega)} \leqslant C(\|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} + e^{-\omega_1 nT})e^{-\omega_1 nT} D_0.$$

Consequently, from (5.29), the estimate (5.8) yields, for all $n \in \mathbb{N}^*$,

$$\|\tilde{\alpha}\|^2_{L^2((n+1)T,(n+2)T;\mathbf{H}^2(\Omega))\cap H^1((n+1)T,(n+2)T;\mathbf{L}^2(\Omega))} \leqslant Ce^{-2\omega_1(n+1)T} D_0 + C\|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} e^{-\omega_1 nT} D_0. \tag{5.35}$$

Now, let us show that for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $n \in \mathbb{N}^*$,

$$\|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant C_k e^{-(1-2^{-k})\omega_1 nT} D_0^{1/2}. \tag{5.36}$$

We prove it by induction. For $k = 1$, this is simply estimate (5.32). Now, if we assume that (5.36) holds for some $k \in \mathbb{N}^*$, from (5.35), we have for all $n \in \mathbb{N}^*$,

$$\|\tilde{\alpha}\|^2_{L^2((n+1)T,(n+2)T;\mathbf{H}^2(\Omega))\cap H^1((n+1)T,(n+2)T;\mathbf{L}^2(\Omega))} \leqslant CC_k e^{-(2-2^{-k})\omega_1 nT} D_0,$$

so that there exists a constant C_{k+1} such that for all $n \in \mathbb{N}^*$,

$$\|\tilde{\alpha}\|_{L^2((n+1)T,(n+2)T;\mathbf{H}^2(\Omega))\cap H^1((n+1)T,(n+2)T;\mathbf{L}^2(\Omega))} \leqslant C_{k+1} e^{-(1-2^{-(k+1)})\omega_1(n+1)T} D_0^{1/2}.$$

We have thus proved (5.36) for all $k \in \mathbb{N}^*$.

We then take k large enough so that $(1-2^{-k})\omega_1 \geqslant \omega_0$, so that we have for all $n \in \mathbb{N}$,

$$\|\tilde{\alpha}\|_{L^2(nT,(n+1)T;\mathbf{H}^2(\Omega))\cap H^1(nT,(n+1)T;\mathbf{L}^2(\Omega))} \leqslant Ce^{-\omega_0 nT} D_0^{1/2}. \tag{5.37}$$

Combined with (5.34), this implies in particular the existence of a constant $C > 0$ such that for all $t \geqslant 0$,

$$\|\alpha(t)\|_{\mathbf{V}^1(\Omega)} \leqslant Ce^{-\omega_0 t} D_0^{1/2}. \tag{5.38}$$

5.5 End of the proof of Theorem 1.1

With $\gamma = \varepsilon^2$ in (1.9), where $\varepsilon > 0$ is given by (5.28), any initial datum u^0 satisfying (1.9) can be expanded as $u^0 = \varepsilon\alpha^0 + \varepsilon^2\beta^0$ with (α^0, β^0) as in (3.4) by taking $\alpha^0 = (I - \mathbb{P}_{uu})u^0$ and $\beta^0 = \mathbb{P}_{uu}u^0$.

The construction presented in Section 3 then yields controlled trajectories (α, β) and control functions (g_1, g_2) such that

- the function $u = \varepsilon\alpha + \varepsilon^2\beta$ solves the original problem (1.2) with control function $g = \varepsilon g_1 + \varepsilon g_2$;

- the couple (α, β) satisfies the decay estimates (5.38) and (5.33).

This entails estimate (1.10) on the solution u of (1.2) with control function g.

The last point which remains to be checked is that the control function $g = \varepsilon g_1 + \varepsilon^2 g_2$ belongs to $L^2(0,\infty; L^2_0(\mathbb{T}))$. In fact, this follows immediately from the above estimates (5.38) and (5.33) on α and β at times nT and the construction of g_1 and g_2 at each iteration, allowing to obtain from (3.6) and Theorem 3.2 for g_1 and from Proposition A.1 for g_2 that, for all $n \in \mathbb{N}$,

$$\|g_1\|_{H^1_0(nT,(n+1)T;L^2_0(\mathbb{T}))\cap L^2(nT,(n+1)T;H^2(\mathbb{T}))} + \|g_2\|_{H^1_0(nT,(n+1)T;L^2_0(\mathbb{T}))\cap L^2(nT,(n+1)T;H^2(\mathbb{T}))} \leqslant Ce^{-\omega_0 nT}.$$

As ω_0 is strictly positive, we immediately have that $g \in L^2(0,\infty; L^2_0(\mathbb{T}))$, and the additional estimate that $ge^{\omega_0 t}/(1+t^2) \in H^1(0,\infty; L^2_0(\mathbb{T})) \cap L^2(0,\infty; H^2(\mathbb{T}))$.

6 Further comments and open problems

6.1 Alternative constructions

Our construction gives the solution u of (1.2) under the form $u = \varepsilon\alpha + \varepsilon^2\beta$ with α and β of size one and ε small enough. Still, we draw the attention of the reader to the fact that, in this expansion, we can guarantee that the function α belongs to the stable space $\mathbf{Z}_s$ only at times nT, $n \in \mathbb{N}$. Indeed, the function $\tilde{\alpha}$ constructed from Theorem 3.2 has no reason to satisfy $\mathbb{P}_u\tilde{\alpha}(t) = 0$ for t not in $\{nT,\, n \in \mathbb{N}\}$. This could seem surprising at first, but actually, it does not impact our proof and result.

In fact, we could have made the choice to construct the solution $\tilde{\alpha}$ in Theorem 3.2 such that $\mathbb{P}_u\tilde{\alpha}(t) = 0$ by choosing $\tilde{\alpha}$ in the proof of Theorem 3.2 containing only Fourier modes k for large enough k (such that $\nu k^2 \geqslant \omega$) instead of considering $\tilde{\alpha}$ containing only Fourier mode 1 as we did.

6.2 Null controllability issues

It would be interesting to know if one can get the null-controllability of (1.2) by the choice of suitable controls $g \in L^2((0,T)\times\mathbb{T})$ satisfying (1.3).

As we said in Section 2, all the Fourier modes except the Fourier mode 0 satisfy the unique continuation property. The article [16], which basically extends Lemma 4.1 to any non-zero Fourier mode, shows that the linearized system (1.4) can be controlled to zero provided the initial datum contains no Fourier mode 0.

The question thus is reduced to control the 0-mode of the non-linear equation. Our approach can be seen as a preliminary step in this direction as Theorem 3.2 allows to control any finite number of eigenvectors of the 0-mode to zero for the second order approximation of (1.2).

As said in the introduction, this question is very much related to the works concerned with null-controllability of Navier-Stokes equations with controls having some vanishing components as in [19, 20] in which the case of distributed controls is considered.

A Controllability of the unstable space

The goal of this section is to prove the following classical result, slightly more general than Proposition 3.1:

Proposition A.1. *Given $\alpha^0 \in \mathbf{V}^1_0(\Omega)$ and $\alpha^f \in \mathbf{Z}_{ud}$, there exists a control function*

$$g_1 \in H^1_0(0,T; L^2_0(\mathbb{T})) \cap L^2(0,T; H^2(\mathbb{T})) \tag{A.1}$$

such that the solution α of (3.2) on $(0,T)$ satisfies the controllability requirement

$$\mathbb{P}_{ud}\alpha(T) = \alpha^f. \tag{A.2}$$

We can further impose the following estimates:

$$\|\mathbb{P}_u\alpha\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} + \|g_1\|_{H^1_0(0,T;L^2_0(\mathbb{T}))\cap L^2(0,T;H^2(\mathbb{T}))}$$
$$\leqslant C\left\|\mathbb{P}_u\alpha^0\right\|_{\mathbf{V}^1_0(\Omega)} + C\left\|\alpha^f\right\|_{\mathbf{V}^1_0(\Omega)}, \tag{A.3}$$

$$\|\alpha\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant C\left\|\alpha^0\right\|_{\mathbf{V}_0^1(\Omega)} + C\left\|\alpha^f\right\|_{\mathbf{V}_0^1(\Omega)}, \tag{A.4}$$

and

$$\|\mathbb{P}_s\alpha(T)\|_{\mathbf{V}_0^1(\Omega)} \leqslant e^{-\omega T}\left\|\mathbb{P}_s\alpha^0\right\|_{\mathbf{V}_0^1(\Omega)} + C\left\|\mathbb{P}_u\alpha^0\right\|_{\mathbf{V}_0^1(\Omega)} + C\left\|\alpha^f\right\|_{\mathbf{V}_0^1(\Omega)}. \tag{A.5}$$

The projection $\mathbb{P}_{uu}\alpha(T)$ cannot be controlled, but satisfies the following property:

$$\text{If } \mathbb{P}_{uu}\alpha^0 = 0, \text{ then } \mathbb{P}_{uu}\alpha(T) = 0. \tag{A.6}$$

Proof. Identity (A.6) follows immediately from the fact that if α solves (3.2), then for all $\ell \in \mathbb{N}$

$$\frac{d}{dt}\left(\langle\alpha(t), \Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)}\right) = -\frac{\nu\pi^2\ell^2}{L^2}\langle\alpha(t), \Psi_{0,\ell}\rangle_{\mathbf{L}^2(\Omega)},$$

where we have used the fact that for all $\ell \in \mathbb{N}$, $B^*\Psi_{0,\ell} = 0$.

In order to prove the existence of a control function g_1, we project the dynamical system (3.2) on $\mathbf{Z}_{ud}$ via the orthogonal projection operator $\mathbb{P}_{ud} : \mathbf{V}_n^0(\Omega) \to \mathbf{Z}_{ud}$ with kernel $\mathbf{Z}_s \oplus \mathbf{Z}_{uu}$:

$$\begin{cases} \mathbb{P}_{ud}\alpha' = (\mathbb{P}_{ud}A\mathbb{P}_{ud})\mathbb{P}_{ud}\alpha + \mathbb{P}_{ud}Bg_1, & \text{for } t \geqslant 0, \\ \mathbb{P}_{ud}\alpha(0) = \mathbb{P}_{ud}\alpha^0, \end{cases} \tag{A.7}$$

where we use the identities $\mathbb{P}_{ud}A = \mathbb{P}_{ud}A((Id - \mathbb{P}_{ud}) + \mathbb{P}_{ud}) = \mathbb{P}_{ud}A\mathbb{P}_{ud} = \mathbb{P}_{ud}A\mathbb{P}_{ud}^2$.

Kalman's criterion (see e.g. [36, Corollary 1.4.10]) then applies to $(\mathbb{P}_{ud}A\mathbb{P}_{ud}, \mathbb{P}_{ud}B)$ thanks to Proposition 2.1, so that given $\alpha^0 \in \mathbf{V}_0^1(\Omega)$ and $\alpha^f \in \mathbf{Z}_{ud}$, there exists a control g_1 such that the solution $\mathbb{P}_{ud}\alpha$ of (A.7) satisfies $\mathbb{P}_{ud}\alpha(0) = \mathbb{P}_{ud}\alpha^0$ and $\mathbb{P}_{ud}\alpha(T) = \alpha^f$.

We look for a control function $g_1 \in H_0^1(0,T;L_0^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$ so we need to make slightly more precise how we choose it. We introduce $\eta = \eta(t)$ a smooth non-negative cut-off function flat at $t = 0$ and at $t = T$ and $\eta(t) = 1$ for $t \in (T/3, 2T/3)$. Then we introduce the functional

$$J(\gamma^T) = \frac{1}{2}\int_0^T \eta(t)\left\|B^*\mathbb{P}_{ud}\gamma\right\|_{L_0^2(\mathbb{T})}^2 dt + \langle\gamma(0), \mathbb{P}_{ud}\alpha^0\rangle_{\mathbf{Z}_{ud}} - \langle\gamma^T, \alpha^f\rangle_{\mathbf{Z}_{ud}}, \tag{A.8}$$

defined for $\gamma^T \in \mathbf{Z}_{ud}$, where γ is the solution of the adjoint equation

$$\begin{cases} -\gamma' = \mathbb{P}_{ud}A\mathbb{P}_{ud}\gamma, & \text{for } t \in (0,T), \\ \gamma(T) = \gamma^T \in \mathbf{Z}_{ud}. \end{cases} \tag{A.9}$$

Thanks to Kalman's criterion, the functional J in (A.8) admits a unique minimizer $\widehat{\gamma}^T$ corresponding to a controlled trajectory $\widehat{\gamma}$ of (A.9), and

$$\left\|\widehat{\gamma}^T\right\|_{\mathbf{Z}_{ud}} \leqslant C\left(\left\|\mathbb{P}_{ud}\alpha^0\right\|_{\mathbf{V}^0(\Omega)} + \left\|\alpha^f\right\|_{\mathbb{Z}_{ud}}\right). \tag{A.10}$$

Setting

$$g_1(t) = \eta(t)B^*\mathbb{P}_{ud}^*\widehat{\gamma}(t), \quad t \in (0,T), \tag{A.11}$$

we easily check that if α solves (3.2) with initial data α^0 and control function g_1, α satisfies the controllability requirement (A.2). We therefore only need to check that g_1 and α satisfy the regularity results claimed by Proposition A.1. They can be easily deduced for g_1 from the formula (A.11) and the fact that $\widehat{\gamma}$ satisfies the finite dimensional equation (A.9) with an initial data $\widehat{\gamma}^T$ satisfying (A.10). The introduction of the cut-off function is used at that step to ensure $g_1(0) = g_1(T) = 0$. The fact that for all $t \in [0,T]$, $g_1(t) \in L_0^2(\mathbb{T})$ comes from (A.11) and the form of B^* in (2.12). This concludes the proof of estimate (A.3).

Once g_1 is estimated by (A.3), the estimate (A.4) follows easily from the results in [34] (recall (2.9)). The estimate (A.5) also comes easily from the equation satisfied by $\mathbb{P}_s\alpha$:

$$\begin{cases} \mathbb{P}_s\alpha' = (\mathbb{P}_sA\mathbb{P}_s)\mathbb{P}_s\alpha + \mathbb{P}_sBg_1, & \text{for } t \geqslant 0, \\ \mathbb{P}_s\alpha(0) = \mathbb{P}_s\alpha^0, \end{cases}$$

and the fact that the semi-group $\exp(t(\mathbb{P}_sA\mathbb{P}_s))$ satisfies, by definition of the set $\mathbf{Z}_s$ and self-adjointness of A, $\|\exp(t(\mathbb{P}_sA\mathbb{P}_s))\|_{\mathscr{L}(\mathbb{V}_0^1(\Omega))} \leqslant \exp(-t\omega)$ for all $t \geqslant 0$. □

B Estimates on Navier-Stokes equations

In this section, we present regularity results for the Navier-Stokes equations with homogeneous boundary conditions:

$$\begin{cases} \partial_t z^\varepsilon + \varepsilon^2 (z^\varepsilon \cdot \nabla) z^\varepsilon - \nu \Delta z^\varepsilon + \nabla p^\varepsilon = -\operatorname{div} F, & \text{in } (0,T)\times\Omega, \\ \operatorname{div} z^\varepsilon = 0, & \text{in } (0,T)\times\Omega, \\ z^\varepsilon(t,x_1,0) = z^\varepsilon(t,x_1,L) = (0,0), & \text{on } (0,T)\times\mathbb{T}, \\ z^\varepsilon(0,x_1,x_2) = z^0(x_1,x_2), & \text{in } \Omega. \end{cases} \tag{B.1}$$

We prove the following result based on the computations in [13, Chapter 5 Sections 1 and 2]:

Proposition B.1. *Let $z^0 \in \mathbf{V}^0_n(\Omega)$ and $F \in L^2(0,T;\mathbf{L}^2(\Omega))$. Then the solution z^ε of* (B.1) *belongs to $L^\infty(0,T;\mathbf{V}^0(\Omega)) \cap L^2(0,T;\mathbf{V}^1_0(\Omega))$ and the following estimate holds (uniformly with respect to $\varepsilon \in [0,1]$):*

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^0(\Omega))\cap L^2(0,T;\mathbf{V}^1_0(\Omega))} \leqslant C_0 \left(\|F\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \left\|z^0\right\|_{\mathbf{V}^0_n(\Omega)} \right). \tag{B.2}$$

If we further assume that $z^0 \in \mathbf{V}^1_0(\Omega)$ and $F \in L^2(0,T;\mathbf{H}^1(\Omega))$, the solution z^ε of (B.1) *belongs to $L^\infty(0,T;\mathbf{V}^1_0(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)^2) \cap H^1(0,T;\mathbf{V}^0(\Omega))$, and the following estimate holds: There exist $C_1 > 0$ and a decreasing function $e_1 : \mathbb{R}_+ \to (0,1]$, such that for all $d > 0$, if*

$$\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}^1_0(\Omega)} \leqslant d, \tag{B.3}$$

then, for all $\varepsilon \in [0, e_1(d)]$, the solution z^ε of (B.1) *satisfies*

$$\begin{aligned} &\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{V}^0(\Omega))} \\ &\qquad \leqslant C_1 e^{C_1 \varepsilon^6 \|z^\varepsilon\|^3_{L^\infty(0,T;\mathbf{V}^0(\Omega))\cap L^2(0,T;\mathbf{V}^1_0(\Omega))}} \left(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}^1_0(\Omega)} \right). \end{aligned} \tag{B.4}$$

Proof. Estimate (B.2) follows from a classical energy argument. Formally, multiplying equation (B.1) by z^ε, we get, for all $t \in (0,T)$,

$$\frac{1}{2}\frac{d}{dt}\left(\|z^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} \right) + \nu \|\nabla z^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} = \int_\Omega F \nabla z^\varepsilon,$$

hence

$$\frac{d}{dt}\left(\|z^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} \right) + \nu \|\nabla z^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} \leqslant \frac{1}{\nu} \|F(t)\|^2_{\mathbf{L}^2(\Omega)},$$

and we integrate this last estimate. To make this argument fully rigorous, we refer to the proof of Theorem V.1.4 in [13].

Estimate (B.4) is trickier. We first recall that setting $Az = \mathbb{P}(\Delta z) = \Delta z - \nabla p$, we have the following classical estimate (see e.g. [13, Th. IV.5.8]):

$$\|z\|_{\mathbf{H}^2(\Omega)} \leqslant C \|Az\|_{\mathbf{L}^2(\Omega)} \quad \text{for all } z \in \mathbf{H}^2(\Omega) \cap \mathbf{V}^1_0(\Omega).$$

As before, to justify completely the computations which are done afterwards, one should be cautious and perform these estimates for instance on Galerkin approximations of (B.1) following the proof of Theorem V.2.1 in [13]. We will skip these arguments for conciseness and refer the reader to [13, Section V.2] for precise justifications.

We thus formally multiply (B.1) by $Az^\varepsilon = \Delta z^\varepsilon - \nabla p$ and we obtain, for all $t \in (0,T)$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla z^\varepsilon\|^2_{\mathbf{L}^2(\Omega)} \right) + \nu \|Az^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} \\ &\leqslant \|F(t)\|_{\mathbf{H}^1(\Omega)} \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} + \varepsilon^2 \int_\Omega (z^\varepsilon \cdot \nabla z^\varepsilon) Az^\varepsilon \, dx \\ &\leqslant \frac{1}{\nu} \|F(t)\|^2_{\mathbf{H}^1(\Omega)} + \frac{\nu}{4} \|Az^\varepsilon(t)\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon^2 \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \|z^\varepsilon(t) \cdot \nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Using interpolation results, we have

$$\begin{aligned}
&\varepsilon^2 \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \|z^\varepsilon(t)\cdot\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \leqslant C\varepsilon^2 \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \|z^\varepsilon(t)\|_{\mathbf{L}^6(\Omega)} \|\nabla z^\varepsilon(t)\|_{\mathbf{L}^3(\Omega)} \\
&\leqslant C\varepsilon^2 \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \left(\|z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{1/3} \|\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{2/3}\right)\left(\|\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{2/3} \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{1/3}\right) \\
&\leqslant C\varepsilon^2 \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{4/3} \|z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{1/3} \|\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^{4/3} \\
&\leqslant \frac{\nu}{4} \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^2 + C\varepsilon^6 \|z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} \|\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^4 .
\end{aligned}$$

Therefore, we deduce, for all $t \in (0,T)$,

$$\frac{d}{dt}\left(\|\nabla z^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2\right) + \nu \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^2 \leqslant C\|F(t)\|_{\mathbf{H}^1(\Omega)}^2 + C\varepsilon^6 \|\nabla z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^4 \|z^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)} .$$

We introduce

$$G(t) = C\varepsilon^6 \int_0^t \|\nabla z^\varepsilon(\tau)\|_{\mathbf{L}^2(\Omega)}^2 \|z^\varepsilon(\tau)\|_{\mathbf{L}^2(\Omega)}\, d\tau, \qquad t \in (0,T),$$

and we write the above estimate in the form

$$\frac{d}{dt}\left(\|\nabla z^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2 e^{-G(t)}\right) + \nu \|Az^\varepsilon(t)\|_{\mathbf{L}^2(\Omega)}^2 e^{-G(t)} \leqslant C\|F(t)\|_{\mathbf{H}^1(\Omega)}^2 e^{-G(t)}.$$

Integrating this last expression in time and using

$$\|G\|_{L^\infty(0,T)} \leqslant C\varepsilon^6 \|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_n^0(\Omega))\cap L^2(0,T;\mathbf{V}_0^1(\Omega))}^3$$

we obtain

$$\begin{aligned}
&\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_0^1(\Omega))} + \|Az^\varepsilon\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \\
&\qquad\qquad \leqslant Ce^{C\varepsilon^6\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_n^0(\Omega))\cap L^2(0,T;\mathbf{V}_0^1(\Omega))}^3}\left(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}_0^1(\Omega)}\right).
\end{aligned}$$

To conclude the proof of Proposition B.1, we should add to the previous estimate that $\partial_t z^\varepsilon = \nu A z^\varepsilon - \mathbb{P}(\varepsilon^2 z^\varepsilon\cdot\nabla z^\varepsilon) - \mathbb{P}(\mathrm{div}\,(F))$, so an estimate on $\partial_t z^\varepsilon$ in $L^2(0,T;\mathbf{V}^0(\Omega))$ easily follows by choosing $e_1(d) = d^{-1/2}$, according to the following estimates:

$$\begin{aligned}
\left\|\mathbb{P}(\varepsilon^2 z^\varepsilon\cdot\nabla z^\varepsilon)\right\|_{L^2(0,T;\mathbf{V}^0(\Omega))} &\leqslant C\varepsilon^2 \|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_0^1(\Omega))} \|z^\varepsilon\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \\
&\leqslant C\varepsilon^2 e^{2C\varepsilon^6\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_n^0(\Omega))\cap L^2(0,T;\mathbf{V}_0^1(\Omega))}^3}\left(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}_0^1(\Omega)}\right)^2 \\
&\leqslant Ce^{2C\varepsilon^6\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_n^0(\Omega))\cap L^2(0,T;\mathbf{V}_0^1(\Omega))}^3}\left(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}_0^1(\Omega)}\right).
\end{aligned}$$

□

We will use Proposition B.1 in particular when F in (B.1) takes the form

$$F = F_0 + \varepsilon(F_1 \otimes z^\varepsilon + z^\varepsilon \otimes F_1). \tag{B.5}$$

In that case, as a Corollary of Proposition B.1, we get the following result:

Corollary B.2. *Let $z^0 \in \mathbf{V}_0^1(\Omega)$ and $F = F_0 + \varepsilon(F_1\otimes z^\varepsilon + z^\varepsilon\otimes F_1)$ as in* (B.5) *with $F_0 \in L^2(0,T;\mathbf{H}^1(\Omega))$ and $F_1 \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))$. There exist $C_2 > 0$ and a decreasing function $e_* : \mathbb{R}_+ \to (0,1]$, such that for all $d > 0$ if*

$$\left\|z^0\right\|_{\mathbf{V}_0^1(\Omega)} + \|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|F_1\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))} \leqslant d, \tag{B.6}$$

then, for all $\varepsilon \in [0, e_*(d)]$, the solution z^ε of (B.1) *satisfies*

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}_0^1(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{V}^0(\Omega))} \leqslant C_2\left(\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \left\|z^0\right\|_{\mathbf{V}_0^1(\Omega)}\right). \tag{B.7}$$

Proof. The existence of a solution z^ε of (B.1) with F as in (B.5) in the class of strong solutions, i.e. $L^\infty(0,T;\mathbf{V}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{V}^0(\Omega))$, can be done in a classical manner by using a fixed point argument for $\varepsilon>0$ small enough. Uniqueness can be done easily using for instance the energy estimates in Proposition B.2.

Using (B.2) with F as in (B.5), we get

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^0(\Omega))\cap L^2(0,T;\mathbf{V}^1_0(\Omega))} \leqslant C_0\Big(\|F_0\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \\ +\varepsilon\|F_1\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))}\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^0(\Omega))\cap L^2(0,T;\mathbf{V}^1_0(\Omega))}+\left\|z^0\right\|_{\mathbf{V}^0_n(\Omega)}\Big).$$

Thus, from (B.6), for $\varepsilon\in[0,e_2(d)]$ with $e_2(d)=\min\{1,1/(2C_0d)\}$, we have

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^0_n(\Omega))\cap L^2(0,T;\mathbf{V}^1_0(\Omega))}\leqslant 2C_0\left(\|F_0\|_{L^2(0,T;L^2(\Omega)^2)}+\left\|z^0\right\|_{\mathbf{V}^0_n(\Omega)}\right)\leqslant 2C_0d.$$

Plugging this estimate in (B.4), we get, for $\varepsilon\in[0,e_3(d)]$ with $e_3(d)=\min\{e_1(d),e_2(d)\}$,

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))}\leqslant C_1e^{8C_1C_0^3\varepsilon^6d^3}\Big(\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega))}+\left\|z^0\right\|_{\mathbf{V}^1_0(\Omega)} \\ +\varepsilon\|F_1\|_{L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))}\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))}\Big).$$

We then set $e_4(d)=\min\{1,e^{-8C_1C_0^3d^3}/(2C_1d)\}$, so that for all $\varepsilon\in[0,e_4(d)]$,

$$C_1de^{8C_1C_0^3\varepsilon^6d^3}\varepsilon\leqslant\frac{1}{2}.$$

Consequently, for all $\varepsilon\in[0,\min\{e_3(d),e_4(d)\}]$ we get

$$\|z^\varepsilon\|_{L^\infty(0,T;\mathbf{V}^1_0(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap H^1(0,T;\mathbf{L}^2(\Omega))}\leqslant 2C_1e^{8C_1C_0^3\varepsilon^6d^3}\left(\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega))}+\left\|z^0\right\|_{\mathbf{V}^1_0(\Omega)}\right).$$

Therefore, the estimate (B.7) holds for all $\varepsilon\in[0,e_*(d)]$ with $e_*(d)=\min\{e_3(d),e_4(d),1/\sqrt{d}\}$. As $d\mapsto e_3(d)$, $d\mapsto e_4(d)$ and $d\mapsto 1/\sqrt{d}$ are all decreasing functions of d, $d\mapsto e_*(d)$ also is a decreasing function of d. ☐

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