Backward uniqueness results for some parabolic equations in an infinite rod

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Abstract

The goal of this article is to provide backward uniqueness results for several models of parabolic equations set on the half line, namely the heat equation, and the heat equation with quadratic potential and with purely imaginary quadratic potentials, with non-homogeneous boundary conditions. Such result can thus also be interpreted as a strong lack of controllability on the half line, as it shows that only the trivial initial datum can be steered to zero. Our results are based on the explicit knowledge of the kernel of each equation, and standard arguments from complex analysis, namely the Phragmén Lindelöf principle.

1 Introduction

1.1 Settings

This note aims at discussing the strong lack of controllability of the following parabolic equations:

- The classical heat equation:

$$\begin{cases} \partial_t y - \partial_{xx} y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ \partial_x y(t,0) = u(t), & \text{in } (0,T), \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}_+^*. \end{cases} \tag{1.1}$$

- A parabolic heat equation with real quadratic potential:

$$\begin{cases} \partial_t y - \partial_{xx} y + x^2 y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ \partial_x y(t,0) = u(t), & \text{in } (0,T), \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}_+^*. \end{cases} \tag{1.2}$$

- A parabolic heat equation with purely imaginary quadratic potential:

$$\begin{cases} \partial_t y - \partial_{xx} y + \imath x^2 y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ \partial_x y(t,0) = u(t), & \text{in } (0,T), \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}_+^*. \end{cases} \tag{1.3}$$

For each equation, y denotes the state, u is a control function acting on the boundary $x = 0$, and y^0 is the initial condition.

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Our goal is to discuss the strong lack of controllability of each of these models. Roughly speaking, we aim at showing that if the solution y of (1.1), respectively (1.2) or (1.3), satisfies for some time $T > 0$,

$$y(T, \cdot) = 0, \quad \text{ in } \mathbb{R}_+^*, \tag{1.4}$$

then necessarily $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.

Written as above, such property underlines the strong lack of controllability of the models (1.1), (1.2) and (1.3), asserting that there are no non-trivial datum which can be steered to zero. This property is also called backward uniqueness as it also states that any solution of (1.1), (1.2) or (1.3) satisfying the condition (1.4) should vanish identically.

1.2 Main results

From now on, we use the following normalization of the Fourier transform: for $g \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$,

$$\mathscr{F}(g)(\xi) = \int_{\mathbb{R}} g(x) e^{-\imath \xi x}\, dx.$$

We shall prove the following results.

Theorem 1.1 (The heat equation (1.1))**.** *Let $T > 0$. If y is a solution of* (1.1) *with an initial condition y^0 satisfying*

$$y_e^0 \in \mathscr{S}'(\mathbb{R}) \text{ and } \exists \rho \in [0, 2), \exists M > 0, \text{ s.t. } \forall \xi \in \mathbb{R}, \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant M \exp(M|\xi|^\rho), \tag{1.5}$$

where y_e^0 denotes the Fourier transform of the even extension of y^0, that is

$$y_e^0(x) = y^0(|x|), \forall x \in \mathbb{R},$$

and with a control function u satisfying

$$u \in L^2(0, T), \tag{1.6}$$

such that $y(T)$ satisfies condition (1.4)*, i.e. vanishes on $\mathbb{R}_+^*$, then necessarily $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.*

Theorem 1.2 (The parabolic equation with a real quadratic potential (1.2))**.** *Let $T > 0$. If y is a solution of* (1.2) *with an initial condition y^0 satisfying*

$$\exists \rho \in [0, 2), \exists M > 0, \text{ s.t. } \forall x \in \mathbb{R}_+^*, \quad |y^0(x)| \leqslant M \exp(M|x|^\rho), \tag{1.7}$$

and with a control function u satisfying (1.6)*, such that $y(T)$ satisfies condition* (1.4)*, i.e. vanishes on $\mathbb{R}_+^*$, then necessarily $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.*

Theorem 1.3 (The parabolic equation with a purely imaginary quadratic potential (1.3))**.** *Let $T > 0$. If y is a solution of* (1.3) *with an initial condition y^0 satisfying* (1.7)*, and with a control function u satisfying* (1.6)*, such that $y(T)$ satisfies condition* (1.4)*, i.e. vanishes on $\mathbb{R}_+^*$, then necessarily $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.*

Before commenting each of these results, let us briefly explain how we prove them (details of the proofs will be given in Section 3, and more details on the Cauchy theory corresponding to each equation will be given in Section 2.2). In fact, in each case, our approach is exactly the same. We first extend the solution y of (1.1), (1.2) or (1.3), in an even manner so that we can consider the equations in $\mathbb{R}$ with a source term $-2u(t)\delta_{x=0}$ involving the control function u. Then we use that in each of the above cases, we explicitly know the fundamental solutions G_0 of (1.1), G_1 of (1.2) and $G_\imath$ of (1.3), see Section 2.1 for their explicit form. We will thus be able to write down explicitly (1.4) as a relation on the initial datum y^0 and the control u taking the form

$$\forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} G(T, x, x_0) y_e^0(x_0)\, dx_0 = 2 \int_0^T u(t) G(T - t, x, 0)\, dt.,$$

where y_e^0 denotes the even extension of y^0. The next step then consists in taking the Fourier transform $x \to \xi$ of this identity. Introducing then some new function $\mathcal{F}(y^0)$ depending on $\xi \in \mathbb{R}$ (which coincides with the usual Fourier transform of y_e^0 when considering the case (1.1), but is slightly different in the other cases (1.2) and (1.3)), $\mathcal{F}(y^0)$ can necessarily be extended holomorphically on the whole complex plane, bounded by a constant in the two complex quadrants $\{\xi \in \mathbb{C},\text{ with } |\Re(\xi)| \leqslant |\Im(\xi)|\}$ (or some slightly smaller sets in the case of (1.3), see Lemma 3.2). The conditions (1.5) or (1.7) will then allow to show, using Phragmén Lindelöf principle, that $\mathcal{F}(y^0)$ is thus bounded on the whole complex plane, so that it is constant. We then get easily that necessarily $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.

The differences on the assumptions (1.5) and (1.7) may seem surprising at first. In fact, using the quasi-conformal transforms (sometimes called Appell transforms, see [41, 15]) for the heat equation will allow us to deduce the following corollary of Theorem 1.1, in which condition (1.5) is replaced by an assumption which is weaker than (1.7):

Corollary 1.4 (of Theorem 1.1). *Let $T > 0$ and y be a solution of* (1.1) *with a control function $u \in L^2(0, T)$, satisfying*

$$\exists M > 0,\ \forall (t, x) \in [0, T] \times \mathbb{R}_+^*, \quad |y(t, x)| \leqslant M \exp(M|x|^2), \tag{1.8}$$

and such that $y(T)$ vanishes on $\mathbb{R}_+^*$. Then $y^0 = 0$ in $\mathbb{R}_+^*$ and $u = 0$ in $(0, T)$.

We refer to Section 3.2.2 for the proof of Corollary 1.4.

1.3 Comments

1.3.1 Concerning Theorem 1.1 and Corollary 1.4

Backward uniqueness theorems for the heat equation (1.1) are well-known, and we shall refer in particular to the work [16] for such statements when the controlled trajectory y of (1.1) satisfies (1.8), which is a natural condition in view of the construction in [23]. Besides, the work [16] applies more generally to the case of heat equations with bounded potentials, which our method does not allow, and to higher dimensional cases.

Note however that the required conditions (1.8) and (1.5) do not seem to be comparable, in the sense that none implies the other. Somehow, condition (1.5) is a Fourier dual version of (1.8).

We also point out that several works have pursued the analysis performed in [16] and proved similar results in various geometric settings, in particular in cones with controls on the boundary of the cones, see [29], [42], [38]. In fact, when finishing our work, we find out that a complex analysis based argument was used in [39] to establish the backward uniqueness of the heat equation in half-space, in a very close spirit to the one we developed here.

Among the interests of backward uniqueness results, one should quote the work [21] which discusses application of these results to uniqueness results for Navier-Stokes equations. But it is also strongly related to controllability results for the heat equation. Indeed, Theorem 1.1 can be seen as a strong lack of controllability of the heat equation (1.1) as it states that only the zero initial condition can be driven to zero when acting on the boundary $x = 0$. This is in strong contrast to what happens in bounded domains (see [14, 17] in 1-d, [27, 19] in higher dimensions) in which any state in L^2 can be driven to zero with controls acting from any non-empty open subset of the boundary of the domain.

Regarding controllability aspects, Theorem 1.1 is strongly linked to the work [30], which proves that the solution of (1.1) cannot be controlled to zero when the initial datum y^0 belongs to the weighted Sobolev space $L^2(\mathbb{R}_+^*, \exp(x^2/4)dx)$ (and $y^0 \not\equiv 0$). Such problem has also been studied from the observability point of view, which is a dual property of the controllability one. For instance, it was proved in [32] that the heat equation (1.1) is not observable through $x = 0$, no matter what the time T is, meaning that there is no $T > 0$ and constant $C > 0$ such that for all solutions φ of

$$\begin{cases} \partial_t \varphi - \Delta \varphi = 0 & \text{in } (0, T) \times \mathbb{R}_+^* \\ \partial_x \varphi(t, 0) = 0, & \text{in } (0, T), \\ \varphi(0, \cdot) \in L^2(\mathbb{R}_+^*), \end{cases} \tag{1.9}$$

the observability inequality

$$\|\varphi(T)\|_{L^2(\mathbb{R}_+^*)} \leqslant C \, \|\varphi(t, 0)\|_{L^2(0,T)} \tag{1.10}$$

is satisfied.

But, from the controllability perspective, disproving the observability property (1.10) for (1.9) only shows that there exists initial datum y^0 which cannot be steered to 0 with a control acting in $x = 0$ as in (1.1). Thus, from the controllability point of view, Theorem 1.1 is stronger than the one in [32].

It is also of interest to note the work [1], which shows that, for all $T > 0$, the set of all functions $y(T, \cdot)$ which are obtained by solving

$$\begin{cases} \partial_t y - \partial_{xx} y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ y(t,0) = u(t), & \text{in } (0,T), \\ y(0,x) = 0 & \text{in } \mathbb{R}_+^*, \end{cases} \tag{1.11}$$

with $u \in L^2(0,T; t^{-1}dt)$ is exactly the set of traces on $\mathbb{R}_+^*$ of functions w which can be extended holomorphically in the complex sector $D = \{0 \leqslant |\Im(z)| < \Re(z)\}$ such that $we^{z^2/4T} \in L^2(D)$. Theorem 1.1 deals with the heat equation controlled from the Neumann boundary condition, but we claim that it can also be applied when the control acts in the Dirichlet boundary condition (see Section 4.2), and then, in view of the result of [1], it means that, for all non-trivial $y^0 \in L^2(\mathbb{R}_+^*)$ satisfying (1.5), the solution y of

$$\begin{cases} \partial_t y - \partial_{xx} y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ y(t,0) = 0, & \text{in } (0,T), \\ y(0,x) = y^0 & \text{in } \mathbb{R}_+^*, \end{cases} \tag{1.12}$$

cannot be extended as an holomorphic function w on D such that $we^{z^2/4T} \in L^2(D)$.

1.3.2 Concerning Theorem 1.2

The controllability properties of (1.2) were analyzed in [30] for initial data $y^0 \in L^2(\mathbb{R}_+^*)$, and it was shown that there is no non-trivial initial condition in $L^2(\mathbb{R}_+^*)$ which can be driven to 0 with control functions $u \in L^2(0,T)$ (in fact, [30] focuses of the equation $\partial_t z - \partial_{xx} z - x\partial_x z/2 = 0$ in $\mathbb{R}_+^*$ in the class $L^2(\mathbb{R}_+^*, \exp(x^2/4)\,dx)$, which corresponds to y solution of (1.2) by setting $\tilde{y}(t,x) = z(t,x)\exp(t/4 + x^2/8)$ and $y(t,x) = \tilde{y}(4t, 2x)$). Theorem 1.2 is more general, as we prove the same result under the only condition that y^0 satisfies the growth condition (1.7).

Equation (1.2) was studied from the observability point of view in [35], showing that equation (1.2) with $u = 0$ is not observable through $x = 0$. Similarly as for the heat equation, this result implies that there exists an initial condition $y^0 \in L^2(\mathbb{R}_+^*)$ which cannot be steered to 0 by a suitable choice of the control function $u \in L^2(0,T)$. In this sense, Theorem 1.2 generalizes the result of [35].

In fact, one of the specificity of equation (1.2) is that it somehow constitutes a limiting case of controllability. Indeed, the spectrum of the harmonic oscillator $\mathscr{H} = -\partial_{xx} + x^2$ in $L^2(\mathbb{R}_+^*)$ with domain $\mathscr{D}(\mathscr{H}) = \{\psi \in H^2(\mathbb{R}_+^*),\ x^2\psi \in L^2(\mathbb{R}_+^*), \partial_x\psi(0) = 0\}$ is given by $4\mathbb{N}+1$ (see e.g. [20, Section 2.1]). It is thus critical to apply Müntz-Szász theorem and spectral methods to prove observability properties, as in [17]. In fact, as pointed out in [35] (see also [13]), for each $k \in \mathbb{N} \setminus \{0,1\}$, the parabolic operators $\partial_t - \partial_{xx} + x^{2k}$ are controllable on $\mathbb{R}_+^*$ when controlled from $x = 0$. This critical behavior of the spectrum actually is something that is also present in the works on the controllability of parabolic equations $\partial_t + (-\partial_{xx})^{1/2}$ with scalar controls, where $(-\partial_{xx})^{1/2}$ is the square root of the Laplace operator $-\partial_{xx}$ in $L^2(0,1)$ with domain $H^2 \cap H_0^1(0,1)$. With this respect, we shall quote the work [31], which proves that no non-trivial initial data in $L^2(0,1)$ can be steered to 0 in this case, see also [34], and more recently [24], which shows that the operator $\partial_t + (-\partial_{xx})^{1/2}$ is not null-controllable when the control is a function in $L^2((0,T) \times (a,b))$.

In fact, one of the motivations of (1.2) is the strong links it has with the controllability of Grushin equations set in $\Omega = (0,L) \times (0,\pi)$:

$$\begin{cases} (\partial_t - \partial_1^2 - x_1^2\partial_2^2) f(t,x_1,x_2) = 0\,, & (t,x_1,x_2) \in (0,T) \times \Omega\,, \\ f(t,x_1,x_2) = u(t,x_2) 1_{x_1=0}\,, & (t,x_1,x_2) \in (0,T) \times \partial\Omega\,, \\ f(0,.,.) = f_0 \in L^2(\Omega)\,, \end{cases} \tag{1.13}$$

where $u \in L^2((0,T) \times (0,\pi))$ is the control function.

The Grushin equations (1.13) are degenerate in $\{x_1 = 0\}$, so the controllability properties of the usual heat equations may be modified: in fact, according to [3], there might exist a minimal time $T_* > 0$ such that system (1.13) is not null controllable when $T < T_*$ and system (1.13) is null controllable when $T > T_*$.

In the above case, similarly as for the usual heat equation, the critical time is $T_* = 0$ according to [6]. But when the control acts away from the singularity, the computation of the critical time is still not so clear: we refer to [6] when Ω is of the form $(-L, L) \times (0, \pi)$ and the controls act on both lateral boundaries, in which case the critical time is $T_* = L^2/2$, and to the recent work [4] when Ω is of the form $(-L, L) \times (0, \pi)$ and the control acts on one lateral boundary. Note also that when one horizontal strip does not meet the control region and Ω is of the form $(-L, L) \times (0, \pi)$, null-controllability never holds, whatever the time $T > 0$ is, see [24].

The link between equations (1.13) and (1.2) may not be completely obvious. In order to explain it, we use the fact that f in (1.13) can be expanded in Fourier in the x_2-variable, so that the controllability of (1.13) is equivalent to the uniform null controllability of the following family of 1-d heat equations indexed by the Fourier parameter n:

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) f_n(t, x) = 0\,, & (t, x) \in (0, T) \times (0, L)\,, \\ f_n(t, 0) = u_n,\ f_n(t, L) = 0\,, & t \in (0, T)\,, \\ f_n(0, .) = f_{0,n} \in L^2(0, L)\,, \end{cases} \tag{1.14}$$

Now, introducing the scaling

$$\begin{aligned} &g_n(\sqrt{n} t, n^{1/4} x) = f_n(t, x), \quad (t, x) \in (0, T) \times (-L, L), \quad i.e. \\ &g_n(\tilde{t}, \tilde{x}) = f_n(n^{-1/2} \tilde{t}, n^{-1/4} \tilde{x}), \quad (t, x) \in (0, \sqrt{n} T) \times (-n^{1/4} L, n^{1/4} L), \end{aligned}$$

the function g_n solves

$$\begin{cases} (\partial_t - \partial_x^2 + x^2) g_n(t, x) = 0\,, & (t, x) \in (0, T_n) \times (0, L_n)\,, \\ g_n(t, 0) = v_n(t),\ g_n(t, L_n) = 0\,, & t \in (0, T_n)\,, \\ g_n(0, .) = g_{0,n} \in L^2(0, L_n)\,, \end{cases}$$

with $T_n = \sqrt{n} T$, and $L_n = n^{1/4} L$, and $v_n(t) = u_n(n^{-1/2} t)$. When $n \to \infty$, it then means that we are looking at the controllability property of an asymptotic regime converging formally to (1.2) in infinite time. With this respect, Theorem 1.2 brings more light to the intricate controllability properties of the Grushin equations (1.2).

For instance, in view of the negative results in [24], which state that the Grushin equation (1.13) is not null-controllable when u is supported in $L^2((0, T) \times ((0, \pi) \setminus I))$ and $\Omega = (-L, L) \times (0, \pi)$, when I is any non-empty open set I of $(0, \pi)$, one can ask if there exists non-trivial initial data $f_0 \in L^2(\Omega)$ whose solution of (1.13) can be steered to zero with a control function $u \in L^2((0, T) \times ((0, \pi) \setminus I))$.

1.3.3 Concerning Theorem 1.3

The motivation to study the equation (1.3) is that it does not seem to be a case which can be handled easily using spectral techniques. Indeed, as the operator $-\partial_{xx} + \imath x^2$ is not self-adjoint, the spectral properties of this operator are not so obvious. We refer for instance to the works [10, 11, 8, 12] where the pseudo-spectrum of this operator is analyzed.

Similarly as above, we shall mention that one of the motivation for the study of (1.3) is the strong link between the equation (1.3) and the 'v^2' Kolmogorov equation set in $\Omega = (0, L) \times \mathbb{T}$:

$$\begin{cases} (\partial_t - \partial_1^2 + x_1^2 \partial_2) f(t, x_1, x_2) = 0\,, & (t, x_1, x_2) \in (0, T) \times \Omega\,, \\ f(t, x_1, x_2) = u(t, x_2) 1_{x_1 = 0}\,, & (t, x_1, x_2) \in (0, T) \times \partial\Omega\,, \\ f(0, ., .) = f_0 \in L^2(\Omega)\,, \end{cases} \tag{1.15}$$

which has been studied in the works [2, 5] and for which the null-controllability property depends on a critical time, similarly to what happens for (1.13). Namely, there might exist a critical time $T_* > 0$ such that for all $T < T_*$, the equation (1.15) is not null-controllable, while for $T > T_*$, equation (1.15) is null controllable. In fact, the value of the critical time in the above case is, to our knowledge, still unknown.

Again, after Fourier transform in the x_2-variable and a suitable scaling argument, this result amounts to analyze the controllability properties of the family of equations

$$\begin{cases} (\partial_t - \partial_x^2 + \imath x^2) g_n(t, x) = 0\,, & (t, x) \in (0, T_n) \times (0, L_n)\,, \\ g_n(t, 0) = v_n(t),\ g_n(t, L_n) = 0\,, & t \in (0, T_n)\,, \\ g_n(0, .) = g_{0,n} \in L^2(-L_n, L_n)\,, \end{cases}$$

with $T_n = \sqrt{n}T$ and $L_n = n^{1/4}L$, in the asymptotic $n \to \infty$. Theorem 1.3 thus highlights some properties of the formal limit system (1.3).

1.4 Outline

Section 2 aims at providing the reader some preliminaries and some more or less standard facts, including the fundamental solutions corresponding to (1.1), (1.2) and (1.3), some results on the well-posedness on each equation, and briefly recalls the Phragmén Lindelöf principle. Section 3 then presents the proofs of Theorem 1.1, Corollary 1.4, and Theorems 1.2 and 1.3. Section 4 provides further comments and open problems.

2 Preliminaries

2.1 Computations of fundamental solutions

In this paragraph, we do some computations which will be helpful in the sequel. For $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geqslant 0$ and $x_0 \in \mathbb{R}$, let us denote by $G_\alpha = G_\alpha(t, x, x_0)$ the solution of

$$\begin{cases} \partial_t G_\alpha - \partial_{xx} G_\alpha + \alpha x^2 G_\alpha = 0, & \text{in } (0,T) \times \mathbb{R}, \\ G_\alpha(0, \cdot, x_0) = \delta_{x_0}, & \text{in } \mathbb{R}. \end{cases} \tag{2.1}$$

When $\alpha = 0$, G_0 is the usual gaussian kernel:

$$G_0(t, x, x_0) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{(x-x_0)^2}{4t}\right), \quad (t, x, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^2. \tag{2.2}$$

When $\alpha = 1$, the kernel G_1 is known as the Mehler kernel, see e.g. [9, Section 4.3]: For $(t, x, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^2$,

$$G_1(t, x, x_0) = \frac{1}{\sqrt{2\pi \sinh(2t)}} \exp\left(-\coth(2t)\frac{x^2 + x_0^2}{2} + \frac{x x_0}{\sinh(2t)}\right) \tag{2.3}$$

$$= \frac{1}{\sqrt{2\pi \sinh(2t)}} \exp\left(-\frac{\coth(2t)}{2}\left(x - \frac{x_0}{\cosh(2t)}\right)^2 - \frac{\tanh(2t)}{2} x_0^2\right). \tag{2.4}$$

To compute the kernel G_α for general $\alpha \in \mathbb{C} \setminus \{0\}$ with $\Re(\alpha) \geqslant 0$, a nice trick is to remark that[1]

$$(t, x, x_0) \mapsto G_1(\sqrt{\alpha}t, \sqrt[4]{\alpha}x, \sqrt[4]{\alpha}x_0)$$

solves $(2.1)_{(1)}$, so that it is a good candidate for being G_α. Looking at the condition $(2.1)_{(2)}$ (see also [8, Section 4]), one easily gets

$$\begin{aligned} G_\alpha(t, x, x_0) &= \sqrt[4]{\alpha} G_1(\sqrt{\alpha}t, \sqrt[4]{\alpha}x, \sqrt[4]{\alpha}x_0), \\ &= \frac{\sqrt[4]{\alpha}}{\sqrt{2\pi \sinh(2\sqrt{\alpha}t)}} \exp\left(-\sqrt{\alpha}\coth(2\sqrt{\alpha}t)\frac{x^2 + x_0^2}{2} + \sqrt{\alpha}\frac{x x_0}{\sinh(2\sqrt{\alpha}t)}\right) \\ &= \frac{\sqrt[4]{\alpha}}{\sqrt{2\pi \sinh(2\sqrt{\alpha}t)}} \exp\left(-\sqrt{\alpha}\frac{\coth(2\sqrt{\alpha}t)}{2}\left(x - \frac{x_0}{\cosh(2\sqrt{\alpha}t)}\right)^2 - \sqrt{\alpha}\frac{\tanh(2\sqrt{\alpha}t)}{2} x_0^2\right). \end{aligned}$$

In this latter formula, the term $\sqrt{\sinh(2\sqrt{\alpha}t)}$ has to be understood in the usual sense when t is close to 0 (namely $t < \pi/(2|\Im(\sqrt{\alpha}|))$), but it has to be understood as $\exp(\sqrt{\alpha}t)\sqrt{1 - \exp(-4\sqrt{\alpha}t)}/\sqrt{2}$ when $t \geqslant \pi/(2|\Im(\sqrt{\alpha}|))$ (Indeed, when $\alpha \in \mathbb{C} \setminus \mathbb{R}_+$ with $\Re(\alpha) \geqslant 0$, $t \mapsto \sinh(2\sqrt{\alpha}t)$ cross the negative axis when t is larger than $\pi/(2|\Im(\sqrt{\alpha}|)))$.

In particular, in our case of interest $\alpha = \imath$, we obtain

$$G_\imath(t, x, x_0) = \frac{\sqrt[4]{\imath}}{\sqrt{2\pi \sinh(2\sqrt{\imath}t)}} \exp\left(-\sqrt{\imath}\frac{\coth(2\sqrt{\imath}t)}{2}\left(x - \frac{x_0}{\cosh(2\sqrt{\imath}t)}\right)^2 - \sqrt{\imath}\frac{\tanh(2\sqrt{\imath}t)}{2} x_0^2\right)$$

[1] In all what follows, we will take roots of complex numbers. Each time, we will use the complex root function which coincides with the function defined on $\mathbb{R}_+^*$ and which has $\mathbb{R}_-$ as a branch cut.

with

$$\sqrt{\imath} = e^{\imath\pi/4},\ \sqrt[4]{\imath} = e^{\imath\pi/8}.$$

For later use, let us compute the Fourier transform of the kernels $G_\alpha(t,\cdot,x_0)$, denoted by $\mathscr{F}(G_\alpha(t,\cdot,x_0)) = \widehat{G}_\alpha$: For $(t,x_0) \in (0,T)\times\mathbb{R}$ and $\xi \in \mathbb{R}$,

$$\begin{aligned}
&\widehat{G}_0(t,\xi,x_0) = \exp(-\xi^2 t - \imath\xi x_0),\\
&\widehat{G}_1(t,\xi,x_0) = \frac{1}{\sqrt{\cosh(2t)}}\exp\left(-\frac{\tanh(2t)}{2}x_0^2 - \frac{\tanh(2t)}{2}\xi^2 - \frac{\imath\xi x_0}{\cosh(2t)}\right),\\
&\widehat{G}_\imath(t,\xi,x_0) = \frac{1}{\sqrt{\cosh(2\sqrt{\imath}t)}}\exp\left(-\frac{\sqrt{\imath}\tanh(2\sqrt{\imath}t)}{2}x_0^2 - \frac{\tanh(2\sqrt{\imath}t)}{2\sqrt{\imath}}\xi^2 - \frac{\imath\xi x_0}{\cosh(2\sqrt{\imath}t)}\right),
\end{aligned}$$

or, more generally, for $\alpha \in \mathbb{C}\setminus\{0\}$ with $\Re(\alpha) \geqslant 0$,

$$\widehat{G}_\alpha(t,\xi,x_0) = \frac{1}{\sqrt{\cosh(2\sqrt{\alpha}t)}}\exp\left(-\frac{\sqrt{\alpha}\tanh(2\sqrt{\alpha}t)}{2}x_0^2 - \frac{\tanh(2\sqrt{\alpha}t)}{2\sqrt{\alpha}}\xi^2 - \frac{\imath\xi x_0}{\cosh(2\sqrt{\alpha}t)}\right). \tag{2.5}$$

Here again, due to the branch cut of the square root function on $\mathbb{C}$, when $\alpha \notin \mathbb{R}$, $\sqrt{\cosh(2\sqrt{\alpha}t)}$ should be replaced in the above formula by the expression $\exp(\sqrt{\alpha}t)\sqrt{1+\exp(-4\sqrt{\alpha}t)}/\sqrt{2}$ for large t, namely $t \geqslant \pi/(2|\Im(\sqrt{\alpha}|)$. In the following, we will keep the notation $\sqrt{\cosh(2\sqrt{\alpha}t)}$ for $\alpha \notin \mathbb{R}_+^*$, as it will underline the similarities between the various cases. (Besides, most of your computations in the case $\alpha = \sqrt{\imath}$ are in fact done for t small, see e.g. (3.28), thus allowing to take the formula $\sqrt{\cosh(2\sqrt{\alpha}t)}$ instead of the seemingly more intricate one $\exp(\sqrt{\alpha}t)\sqrt{1+\exp(-4\sqrt{\alpha}t)}/\sqrt{2}$.)

2.2 On the well-posedness of (1.1), (1.2) and (1.3)

2.2.1 Well-posedness of (1.1)

It is well-known that the classical heat equation

$$\begin{cases} \partial_t y - \partial_{xx} y = 0, & \text{in } (0,T)\times\mathbb{R},\\ y(0,x) = y^0(x), & \text{in } \mathbb{R}, \end{cases} \tag{2.6}$$

is well-posed for $y^0 \in L^2(\mathbb{R})$, in the sense that there exists a unique solution y of (2.6) in the class $C([0,T];L^2(\mathbb{R}))$ for instance. Besides, the solution y of (2.6) is given by the formula:

$$\forall (t,x) \in (0,T]\times\mathbb{R}, \quad y(t,x) = \int_{\mathbb{R}} G_0(t,x,x_0)y^0(x_0)\,dx_0. \tag{2.7}$$

One easily checks that the formula (2.7) still makes sense for y^0 satisfying, for some $\rho < 2$ and some constant $C > 0$,

$$\forall x \in \mathbb{R}, \quad |y^0(x)| \leqslant C\exp(C|x|^\rho), \tag{2.8}$$

and the formula (2.7) still provides a solution of (2.6) (One can even take $\rho = 2$ in (2.8), see [22, Chapter 7, Sections 1.(a) and 1.(b)], and get a local existence result via formula (2.7)). Besides, as underlined in [22, p.217], this solution is unique by the maximum principle among all solutions of (2.6) such that $y\exp(-C|x|^2) \in L^\infty((0,T)\times\mathbb{R})$. This actually is the functional setting corresponding to the one in Corollary 1.4.

In fact, formula (2.7) also shows that for $y^0 \in \mathscr{S}(\mathbb{R})$, the solution y of (2.6) belongs to $C^\infty([0,\infty),\mathscr{S}(\mathbb{R}))$ (see [37, Section 3.6 Theorem 1]. Consequently, formula (2.7) understood by duality also holds for $y^0 \in \mathscr{S}'(\mathbb{R})$ and gives a solution y of (2.6) in $C^\infty([0,\infty),\mathscr{S}'(\mathbb{R}))$.

Corresponding to the setting of Theorem 1.1 however, we shall rather assume that the initial datum y^0 belongs to $\mathscr{S}'(\mathbb{R})$ and satisfies, for some $\rho < 2$ and some constant $C > 0$,

$$\forall \xi \in \mathbb{R}, \quad |\mathscr{F}(y^0)(\xi)| \leqslant C\exp(C|\xi|^\rho). \tag{2.9}$$

Here, we impose $y^0 \in \mathscr{S}'(\mathbb{R})$ in order to be able to take the Fourier transform of y^0. When $y^0 \in \mathscr{S}'(\mathbb{R})$ and condition (2.9) is satisfied, we define the solution y of (2.6) as follows:

$$y(t,x) = (\mathscr{F})^{-1}(\exp(-\xi^2 t)\mathscr{F}(y^0)(\xi))(x). \tag{2.10}$$

It is clear that this formula makes sense for $y^0 \in \mathscr{S}'(\mathbb{R})$ satisfying (2.9). It is also clear that this formula yields the same solution as the one given by formula (2.7) for $y^0 \in \mathscr{S}(\mathbb{R})$.

Finally note that equation (1.1) is set on $\mathbb{R}_+^*$ with boundary conditions at $x = 0$. As we shall consider u in some $L^2(0,T)$ class, the well-posedness of (1.1) with $y^0 = 0$ is classical. Therefore, we shall only make precise the definition of the solution of (1.1) when $u = 0$: in this case, we will always consider the solution of (1.1) as being the solution of (2.6) obtained after even extension.

2.2.2 Well-posedness of (1.2)

When considering

$$\begin{cases} \partial_t y - \partial_{xx} y + x^2 y = 0, & \text{in } (0,T)\times\mathbb{R}, \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}, \end{cases} \tag{2.11}$$

with $y^0 \in L^2(\mathbb{R})$, there exists a unique solution y of (2.11) in the class $C^0([0,T];L^2(\mathbb{R}))$, which is given by

$$\forall (t,x) \in (0,T]\times\mathbb{R}, \quad y(t,x) = \int_{\mathbb{R}} G_1(t,x,x_0) y^0(x_0)\, dx_0. \tag{2.12}$$

Again, it is clear that this formula makes sense when y^0 only satisfies (2.8). In fact, arguing as in [22, Chapter 7 Section 1 paragraph (b)], one easily checks that the solution (2.12) is the unique solution of (2.11) in the class of functions y satisfying, for some $C > 0$, $y \exp(-C|x|^2) \in L^\infty((0,T)\times\mathbb{R})$.

2.2.3 Well-posedness of (1.3)

Let us finally mention the case of the equation

$$\begin{cases} \partial_t y - \partial_{xx} y + \imath x^2 y = 0, & \text{in } (0,T)\times\mathbb{R}, \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}, \end{cases} \tag{2.13}$$

with $y^0 \in L^2(\mathbb{R})$. Classical results give the existence of a unique solution y of (2.13) in $C^0([0,T];L^2(\mathbb{R}))$, (for instance, because the numerical range of the operator $-\partial_{xx} + \imath x^2$ is included in $\{\Re(z) \geqslant 0\}$ so that [36, Chapter 1, Theorem 3.9] applies), which is given by

$$\forall (t,x) \in (0,T]\times\mathbb{R}, \quad y(t,x) = \int_{\mathbb{R}} G_\imath(t,x,x_0) y^0(x_0)\, dx_0. \tag{2.14}$$

Again, it is clear that this formula makes sense for $T > 0$ small enough when y^0 only satisfies (2.8). However now, it is not clear that formula (2.14) gives the unique solution of (2.13) in such class as the uniqueness results we are aware of in such class all strongly rely on the maximum principle.

Still, in all that follows, we will consider the solution y of (2.13) given by formula (2.14).

For later use, we also mention the following lemma:

Lemma 2.1. *Let $T > 0$. Then there exists $\varepsilon > 0$ such that if y^0 satisfies, for some $C > 0$,*

$$|y^0(x)| \leqslant C \exp\left(\varepsilon \frac{|x|^2}{2}\right), \quad x \in \mathbb{R}, \tag{2.15}$$

then the solution y of (2.13) given by (2.14) satisfies $y(T) \in L^\infty(\mathbb{R})$.

Proof. Let $T > 0$ and y^0 satisfying (2.15) for some $\varepsilon > 0$ (to be determined). For $x \in \mathbb{R}$, we estimate the solution y of (2.13) as follows

$$\begin{aligned} |y(T,x)| &\leqslant \left| \int_{\mathbb{R}} G_\imath(T,x,x_0) y^0(x_0)\, dx_0 \right| \\ &\leqslant C \int_{\mathbb{R}} \exp\left(-\Re(\sqrt{\imath}\coth(2\sqrt{\imath}T)) \frac{x^2 + x_0^2}{2} + \Re\left(\frac{\sqrt{\imath}}{\sinh(2\sqrt{\imath}T)}\right) x x_0 + \varepsilon \frac{|x_0|^2}{2} \right) dx_0, \end{aligned}$$

where C does not depend on x. Thus, we aim at showing that

$$\sup_x \int_{\mathbb{R}} \exp\left(-A(T)\frac{x^2+x_0^2}{2} + B(T)xx_0 + \varepsilon\frac{|x_0|^2}{2}\right) dx_0 < \infty,$$

where we have set

$$A(T) = \Re(\sqrt{\imath}\coth(2\sqrt{\imath}T)), \quad B(T) = \Re\left(\frac{\sqrt{\imath}}{\sinh(2\sqrt{\imath}T)}\right).$$

Explicit computations yield:

$$A(T) = \frac{1}{2\sqrt{2}}\frac{\sinh(2\sqrt{2}T)+\sin(2\sqrt{2}T)}{\sinh^2(\sqrt{2}T)+\sin^2(\sqrt{2}T)}, \quad B(T) = \frac{1}{\sqrt{2}}\frac{\sinh(\sqrt{2}T)\cos(\sqrt{2}T)+\cosh(\sqrt{2}T)\sin(\sqrt{2}T)}{\sinh^2(\sqrt{2}T)+\sin^2(\sqrt{2}T)},$$

Under this form, we immediately check that $A(T) > 0$. We shall then take $\varepsilon \in (0, A(T))$, so that

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(-A(T)\frac{x^2+x_0^2}{2} + B(T)xx_0 + \varepsilon\frac{|x_0|^2}{2}\right) dx_0 \\
&\leqslant \exp\left(-A(T)\frac{x^2}{2}\right)\int_{\mathbb{R}} \exp\left(\frac{B(T)^2}{(A(T)-\varepsilon)}\frac{x^2}{2} - \frac{A(T)-\varepsilon}{2}\left(x_0 - \frac{B(T)x}{A(T)-\varepsilon}\right)^2\right) dx_0 \\
&\leqslant \frac{C}{\sqrt{A(T)-\varepsilon}}\exp\left(-\left(A(T) - \frac{B(T)^2}{A(T)-\varepsilon}\right)\frac{x^2}{2}\right).
\end{aligned}$$

We thus want to choose $\varepsilon \in (0, A(T))$ such that

$$A(T)(A(T)-\varepsilon) \geqslant B(T)^2.$$

A natural choice thus is to take $\varepsilon = (1-\lambda)A(T)$, where $\lambda \in (0,1)$ is such that

$$\lambda \geqslant \frac{B(T)^2}{A(T)^2}.$$

We shall then show that $B(T)/A(T)$ belongs to $(-1,1)$, so that we can choose $\varepsilon = (1-|B(T)|/|A(T)|)|A(T)| = A(T) - |B(T)|$. We thus introduce the function g defined on $(0,\infty)$ by

$$g(t) = \frac{B(t)}{A(t)} = 2\frac{\sinh(\sqrt{2}t)\cos(\sqrt{2}t)+\cosh(\sqrt{2}t)\sin(\sqrt{2}t)}{\sinh(2\sqrt{2}t)+\sin(2\sqrt{2}t)}.$$

It is easy to check that g can be extended by continuity as $t \to 0$ by $g(0) = 1$, and that

$$g'(t) = -4\sqrt{2}\sin(\sqrt{2}t)\sinh(\sqrt{2}t)\frac{\left(\sinh(2\sqrt{2}t) - \sin(2\sqrt{2}t)\right)}{(\sinh(2\sqrt{2}t)+\sin(2\sqrt{2}t))^2},$$

so that, to prove that $|g(T)| < 1$, it is enough to verify that $|g(t_k)| < 1$ for $t_k = k\pi/\sqrt{2}$ with $k \in \mathbb{N}^*$. We then check that for all $k \in \mathbb{N}^*$,

$$|g(t_k)| = \frac{2\sinh(k\pi)}{\sinh(2k\pi)},$$

which is obviously strictly smaller than 1 for all $k \in \mathbb{N}^*$ by strict convexity of sinh. □

2.3 Phragmén Lindelöf principle

We will repeatedly use the Phragmén Lindelöf principle in cones. We recall its precise statement here for the convenience of the reader.

Let us introduce the sector $S(\alpha,\beta)$, defined for $\alpha,\beta \in \mathbb{R}^2$ with $\alpha < \beta < \alpha + 2\pi$ by

$$S(\alpha,\beta) = \{z \in \mathbb{C}, \text{ such that } \operatorname{Arg}(z\exp(-\imath\alpha)) \in [0, \beta-\alpha]\}.$$

Phragmén Lindelöf principles in a sector of the form $S(\alpha,\beta)$ writes as follows:

Theorem 2.2 (Phragmén Lindelöf principle)**.** *Let $\alpha, \beta \in \mathbb{R}^2$ with $\alpha < \beta < \alpha + 2\pi$. Let us consider an holomorphic function f such that*

- *f is holomorphic on $S(\alpha, \beta)$,*
- *f satisfies*
$$\forall \xi \in \partial S(\alpha, \beta), \quad |f(z)| \leqslant 1,$$
- *f satisfies, for some γ, $0 < \gamma < \pi/(\beta - \alpha)$, and some constant $C > 0$,*
$$\forall \xi \in S(\alpha, \beta), \quad |f(z)| \leqslant C \exp(C|z|^\gamma).$$

Then
$$\forall \xi \in S(\alpha, \beta), \quad |f(z)| \leqslant 1.$$

We refer the interested reader to [28, Lecture 6] for the proof of this result.

3 Proofs of the main results

The proofs of Theorems 1.1, 1.2 and 1.3 all follow the same strategy and are thus all given in this section. They all start by writing condition (1.4) using the explicit expression of the kernels G_α given in Section 2.1, see Section 3.1.

We then focus on the proof of Theorem 1.1 and Corollary 1.4 in Section 3.2, of Theorem 1.2 in Section 3.3, and of Theorem 1.3 in Section 3.4. Each proof has some specificity coming from the specificity of their kernel, and we have chosen to present them in an increasing order of difficulty.

3.1 Writing Condition (1.4) using the kernels G_α

All the equations (1.1), (1.2) and (1.3) can be represented by the equation

$$\begin{cases} \partial_t y - \partial_{xx} y + \alpha x^2 y = 0, & \text{in } (0,T) \times \mathbb{R}_+^*, \\ \partial_x y(t,0) = u(t), & \text{in } (0,T), \\ y(0,x) = y^0(x), & \text{in } \mathbb{R}_+^*, \end{cases} \tag{3.1}$$

where α is a parameter satisfying $\Re(\alpha) \geqslant 0$. To be more precise, the equations (1.1), (1.2) and (1.3) respectively correspond to the cases $\alpha = 0$, $\alpha = 1$, and $\alpha = \imath$.

Let y be the solution of (3.1) and define y_e its even extension:

$$y_e(t,x) = y(t,|x|), \quad \text{for all } (t,x) \in (0,T) \times \mathbb{R}, \quad y_e^0(x) = y^0(|x|), \quad \text{for all } x \in \mathbb{R}.$$

Then one easily checks that y_e solves the equation:

$$\begin{cases} \partial_t y_e - \partial_{xx} y_e + \alpha x^2 y_e = -2\delta_{x=0} u(t), & \text{in } (0,T) \times \mathbb{R}, \\ y_e(0,x) = y_e^0(x), & \text{in } \mathbb{R}. \end{cases} \tag{3.2}$$

Note in particular that condition (1.4) rewrites in terms of y_e as follows:

$$\forall x \in \mathbb{R}, \quad y_e(T,x) = 0.$$

Now, using the fundamental solutions computed in Section 2.1, we obtain that the identity (1.4) is equivalent to the following one:

$$\forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} G_\alpha(T,x,x_0) y_e^0(x_0)\, dx_0 = 2 \int_0^T G_\alpha(T-t,x,0) u(t)\, dt. \tag{3.3}$$

In particular, when $\alpha \in \mathbb{C} \setminus \{0\}$ with $\Re(\alpha) \geqslant 0$, taking the Fourier transform in the variable x, we obtain, see (2.5),

$$\begin{aligned} \forall \xi \in \mathbb{R}, \quad \int_{\mathbb{R}} \frac{1}{\sqrt{\cosh(2\sqrt{\alpha}T)}} \exp\left(-\frac{\sqrt{\alpha}\tanh(2\sqrt{\alpha}T)}{2} x_0^2 - \frac{\tanh(2\sqrt{\alpha}T)}{2\sqrt{\alpha}} \xi^2 - \frac{\imath \xi x_0}{\cosh(2\sqrt{\alpha}T)} \right) y_e^0(x_0)\, dx_0 \\ = 2 \int_0^T \frac{1}{\sqrt{\cosh(2\sqrt{\alpha}(T-t))}} \exp\left(-\frac{\tanh(2\sqrt{\alpha}(T-t))}{2\sqrt{\alpha}} \xi^2 \right) u(t)\, dt, \end{aligned}$$

which can be rewritten as

$$\forall \xi \in \mathbb{R}, \quad \int_{\mathbb{R}} \exp\left(-\frac{\sqrt{\alpha}\tanh(2\sqrt{\alpha}T)}{2}x_0^2 - \frac{\imath\xi x_0}{\cosh(2\sqrt{\alpha}T)}\right) y_e^0(x_0)\, dx_0$$
$$= 2\int_0^T \sqrt{\frac{\cosh(2\sqrt{\alpha}T)}{\cosh(2\sqrt{\alpha}(T-t))}} \exp\left(\frac{\tanh(2\sqrt{\alpha}T) - \tanh(2\sqrt{\alpha}(T-t))}{2\sqrt{\alpha}}\xi^2\right) u(t)\, dt. \quad (3.4)$$

It is easy to check that, when $\alpha \in \mathbb{C}\setminus\{0\}$ with $\Re(\alpha) \geqslant 0$, condition (3.4) is equivalent to the condition (1.4) for solutions y of (3.1).

Similar computations also give the case $\alpha = 0$: When $\alpha = 0$, the solution y of (1.1) satisfies the condition (1.4) if and only if

$$\forall \xi \in \mathbb{R}, \quad \int_{\mathbb{R}} \exp\left(-\imath\xi x_0\right) y_e^0(x_0)\, dx_0 = 2\int_0^T \exp\left(t\xi^2\right) u(t)\, dt. \quad (3.5)$$

3.2 Proofs of Theorem 1.1 and Corollary 1.4

3.2.1 Proof of Theorem 1.1

Under the conditions of Theorem 1.1, using Section 3.1, we get formula (3.5). In particular, the Fourier transform $\mathscr{F}(y_e^0)$ satisfies, for all $\xi \in \mathbb{R}$,

$$\mathscr{F}(y_e^0)(\xi) = 2\int_0^T \exp\left(t\xi^2\right) u(t)\, dt.$$

The right hand side of this identity defines a holomorphic function on $\mathbb{C}$. Thus, $\mathscr{F}(y_e^0)$ can be extended on the whole complex plane $\mathbb{C}$ and we have

$$\forall \xi \in \mathbb{C}, \quad \mathscr{F}(y_e^0)(\xi) = 2\int_0^T \exp\left(t\xi^2\right) u(t)\, dt. \quad (3.6)$$

In particular, using $u \in L^2(0,T)$, for all $\xi \in \mathbb{C}$ with $\Re(\xi^2) \leqslant 0$, i.e. for all $\xi \in \{|\Re(\xi)| \leqslant |\Im(\xi)|\}$,

$$|\mathscr{F}(y_e^0)(\xi)| \leqslant 2\sqrt{T}\, \|u\|_{L^2(0,T)}, \quad (3.7)$$

while, for all $\xi \in \mathbb{C}$,

$$|\mathscr{F}(y_e^0)(\xi)| \leqslant 2\sqrt{T}\, \|u\|_{L^2(0,T)} \exp(|\xi|^2 T). \quad (3.8)$$

The next step is to use two successive Phragmén Lindelöf principles to get that the holomorphic function $\mathscr{F}(y_e^0)$ is bounded in $\mathbb{C}$. The first application of Phragmén Lindelöf principle is not completely classical and is thus detailed below.

We introduce $S(0,\pi/4) = \{\xi \in \mathbb{C} \text{ with } 0 \leqslant \Im(\xi) \leqslant \Re(\xi)\}$. With ρ and M as in (1.5), and $\varepsilon > 0$, we set

$$g_\varepsilon(\xi) = \mathscr{F}(y_e^0)(\xi) \exp\left(-M(\xi+\varepsilon)^\rho - \varepsilon(\xi e^{-\imath\pi/8})^3\right), \quad \xi \in S(0,\pi/4). \quad (3.9)$$

It is easy to check that:

- g_ε is holomorphic on $S(0,\pi/4)$.
- g_ε is bounded on $\{\Im(\xi) = \Re(\xi) > 0\}$ according to (3.7): for all $\xi \in \{\Im(\xi) = \Re(\xi) > 0\}$,

$$|g_\varepsilon(\xi)| \leqslant 2\sqrt{T}\, \|u\|_{L^2(0,T)}.$$

- g_ε is bounded on $\mathbb{R}_+$ according to assumption (1.5): for all $\xi \in \mathbb{R}_+$

$$|g_\varepsilon(\xi)| \leqslant M,$$

- g_ε goes to zero as $|\xi| \to \infty$ according to (3.8): there exists $C > 0$ such that

$$\forall \xi \in S(0,\pi/4), \quad |g_\varepsilon(\xi)| \leqslant C \exp\left(|\xi|^2 T - \varepsilon|\xi|^3 \cos\left(\frac{3\pi}{8}\right)\right).$$

As a consequence of the above properties, g_ε attains its maximum on the boundary of $S(0,\pi/4)$, i.e.

$$\forall \xi \in S(0,\pi/4), \quad |g_\varepsilon(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\}.$$

As the constant in the right hand-side does not depend on $\varepsilon > 0$, we pass to the limit in $\varepsilon \to 0$ and deduce that

$$\forall \xi \in S(0,\pi/4), \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\} e^{M|\xi|^\rho}.$$

The same arguments can be done in the sector $S(-\pi/4,0) = \{\xi \in \mathbb{C} \text{ with } -\Re(\xi) \leqslant \Im(\xi) \leqslant 0\}$, yielding that

$$\forall \xi \in S(-\pi/4,0), \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\} e^{M|\xi|^\rho}.$$

Consequently, in the sector $S(-\pi/4,\pi/4) = \{\xi \in \mathbb{C} \text{ with } |\Im(\xi)| \leqslant \Re(\xi)\}$, we have

$$\forall \xi \in S(-\pi/4,\pi/4), \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\} e^{M|\xi|^\rho}. \tag{3.10}$$

Now, the classical Phragmén Lindelöf theorem (Theorem 2.2) in the sector $S(-\pi/4,\pi/4)$ applies, as $\mathscr{F}(y_e^0)$ is bounded by a constant on the boundary $\partial S(-\pi/4,\pi/4)$ of the quadrant, see (3.7). Consequently,

$$\forall \xi \in S(-\pi/4,\pi/4), \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\}. \tag{3.11}$$

Of course, similar arguments can be performed in the sector $S(3\pi/4,5\pi/4) = \{\xi \in \mathbb{C} \text{ with } |\Im(\xi)| \leqslant -\Re(\xi)\}$, so that we also have

$$\forall \xi \in S(3\pi/4,5\pi/4), \quad |\mathscr{F}(y_e^0)(\xi)| \leqslant \max\{M, 2\sqrt{T}\,\|u\|_{L^2(0,T)}\}. \tag{3.12}$$

The bounds (3.7), (3.11) and (3.12) indicate that the holomorphic function $\mathscr{F}(y_e^0)$ is bounded everywhere in the complex plane $\mathbb{C}$. It is therefore constant on $\mathbb{C}$. We call this constant c_0:

$$\forall \xi \in \mathbb{C}, \quad \mathscr{F}(y_e^0)(\xi) = c_0. \tag{3.13}$$

Considering (3.6), it follows that the Fourier transform (in time) of the function $u 1_{[0,T]}$ is constant equals to $c_0/2$. As $u \in L^2(0,T)$, this is possible only if the constant c_0 is zero. This implies in particular that $u = 0$ in $(0,T)$ as its Fourier transform in time is zero. Besides, recalling the identity (3.13), this also implies that $y_e^0 = 0$ in $\mathbb{R}$.

3.2.2 Proof of Corollary 1.4

The basic idea to prove Corollary 1.4 is to use Appell transforms (see [41, 15]). Let us start by considering a trajectory y solving (1.1) satisfying condition (1.8) with constant M and vanishing at time $T > 0$.

If $T < 1/(16M)$, where M is the constant in (1.8), then we introduce the function z defined such that for all $(t,x) \in (0,T) \times \mathbb{R}_+^*$,

$$y(t,x) = \frac{1}{\sqrt{1-8Mt}}\, z\left(\frac{1}{8M}\frac{1}{1-8Mt}, \frac{x}{1-8Mt}\right) \exp\left(\frac{2M}{1-8Mt}x^2\right).$$

This identity defines z in the time interval (T_0,T_1) with $T_0 = 1/(8M)$ and $T_1 = 1/(8M(1-8MT))$, and in the space interval $\mathbb{R}_+^*$. Besides, computations show that z satisfies the heat equation

$$\begin{cases} \partial_t z - \partial_{xx} z = 0, & \text{in } (T_0,T_1) \times \mathbb{R}_+^*, \\ \partial_x z(t,0) = \tilde{u}(t), & \text{in } (T_0,T_1), \\ z(T_0,x) = y^0(x)\exp(-2Mx^2), & \text{in } \mathbb{R}_+^*, \end{cases}$$

where $\tilde{u}(t) \in L^2(T_0,T_1)$. According to (1.8), we can thus apply Theorem 1.1 to z and deduce that z and $\tilde{u}$ vanish identically on $[T_0,T_1]$, and consequently y and u vanish identically on the time interval $[0,T]$.

If $T > 1/(16M)$, we set $n = \lceil 32MT \rceil$ and $T_n = T/n$. We can thus apply the previous result on the time interval $[T-T_n,T]$ and deduce that y and u vanish on the time interval $[T-T_n,T]$. We then iterate this argument on each interval $[T-kT_n, T-(k-1)T_n]$ for $k \in \{1,\cdots,n\}$ to show that y and u vanish on each time interval of the form $[T-kT_n, T-(k-1)T_n]$ with $k \in \{1,\cdots,n\}$. Thus y and u vanish on the whole time interval $[0,T]$, concluding the proof of Corollary 1.4.

3.3 Proof of Theorem 1.2

Under the conditions of Theorem 1.2, using Section 3.1, we get formula (3.4). It is then natural to introduce, for $\xi \in \mathbb{R}$,

$$\mathcal{F}_1(y_e^0)(\xi) = \int_{\mathbb{R}} \exp\left(-\frac{\tanh(2T)}{2}x_0^2 - \frac{\imath \xi x_0}{\cosh(2T)}\right) y_e^0(x_0)\, dx_0. \tag{3.14}$$

Note that

$$\mathcal{F}_1(y_e^0)(\xi) = \mathscr{F}(e^{-\tanh(2T)x^2/2} y_e^0)\left(\frac{\xi}{\cosh(2T)}\right), \tag{3.15}$$

so that the function $\mathcal{F}_1(y_e^0)$ can be interpreted as a kind of Fourier transform of y_e^0. In particular, condition (1.7) easily implies $e^{-\tanh(2T)x^2/2} y_e^0 \in L^1(\mathbb{R})$, so that

$$\exists C > 0,\ \forall \xi \in \mathbb{R}, \quad |\mathcal{F}_1(y_e^0)(\xi)| \leqslant C. \tag{3.16}$$

As formula (3.4) reads

$$\forall \xi \in \mathbb{R}, \quad \mathcal{F}_1(y_e^0)(\xi) = 2\int_0^T \sqrt{\frac{\cosh(2T)}{\cosh(2(T-t))}} \exp\left(\frac{\tanh(2T) - \tanh(2(T-t))}{2}\xi^2\right) u(t)\, dt, \tag{3.17}$$

and the right hand-side of this identity is holomorphic in $\mathbb{C}$, the function $\mathcal{F}_1(y_e^0)$ can be extended holomorphically on $\mathbb{C}$ by the above identity. Besides, there exists $C > 0$ such that for all $\xi \in \mathbb{C}$ with $\Re(\xi^2) \leqslant 0$, i.e. for $\xi \in \{|\Re(\xi)| \leqslant |\Im(\xi)\}$,

$$|\mathcal{F}_1(y_e^0)(\xi)| \leqslant C, \tag{3.18}$$

while, for all $\xi \in \mathbb{C}$,

$$|\mathcal{F}_1(y_e^0)(\xi)| \leqslant C \exp\left(\tanh(2T)|\xi|^2/2\right). \tag{3.19}$$

Using Phragmén-Lindelöf principle (Theorem 2.2) in each sector $S(0, \pi/4)$, $S(-\pi/4, 0)$, $S(3\pi/4, \pi)$, and $S(-\pi, -3\pi/4)$ based on the bounds (3.16), (3.18) and (3.19), we easily derive that $\mathcal{F}_1(y_e^0)$ is bounded in the whole complex plane $\mathbb{C}$. As a consequence, $\mathcal{F}_1(y_e^0)$ is constant on $\mathbb{C}$. We call c_0 this constant:

$$\forall \xi \in \mathbb{C}, \quad \mathcal{F}_1(y_e^0)(\xi) = c_0. \tag{3.20}$$

As

$$\mathscr{F}\left(y_e^0 \exp\left(-\frac{\tanh(2T)}{2}x_0^2\right)\right)(\xi) = \mathcal{F}_1(y_e^0)(\xi\cosh(2T)), \tag{3.21}$$

we have that the Fourier transform of $y_e^0 \exp\left(-\tanh(2T)x^2/2\right)$ is constant. As this function belongs to $L^2(\mathbb{R})$ from (1.7), $y_e^0 \equiv 0$.

Using then (3.17), doing the change of variable $\tau = (\tanh(2T) - \tanh(2(T-t)))/2$, i.e. $t = T - \tanh^{-1}(\tanh(2T) - 2\tau)/2$ in the right hand-side of (3.17) and setting

$$\tilde{u}(\tau) = u\left(T - \frac{1}{2}\tanh^{-1}(\tanh(2T) - 2\tau)\right),$$

we have, for all $\xi \in \mathbb{C}$,

$$0 = \int_0^{\tanh(2T)/2} \sqrt{\frac{\cosh(2T)}{\cosh(\tanh^{-1}(\tanh(2T) - 2\tau))}} \exp(\tau\xi^2)\tilde{u}(\tau)\frac{d\tau}{1 + (\tanh(2T) - 2\tau)^2}.$$

In particular, the Fourier transform (in the time variable τ) of the function

$$2 \times 1_{\tau\in(0,\tanh(2T))} \sqrt{\frac{\cosh(2T)}{\cosh(\tanh^{-1}(\tanh(2T) - 2\tau))}} \frac{\tilde{u}(\tau)}{1 + (\tanh(2T) - 2\tau)^2}$$

equals to 0, hence u vanishes identically.

Remark 3.1. *One can in fact slightly weaken the growth assumption* (1.7) *and replace it by the following one instead: the trajectory y solving* (1.2) *satisfies, for some $\varepsilon = \varepsilon(T) > 0$,*

$$y \exp(-\varepsilon|x|^2) \in L^\infty((0, T) \times \mathbb{R}_+^*). \tag{3.22}$$

Indeed, taking $\varepsilon(T) = \tanh(2T)/4$, the proof above readily applies.

3.4 Proof of Theorem 1.3

Writing the null-controllability condition (1.4). Under the conditions of Theorem 1.3, using Section 3.1 formula (3.4), we get the identity:

$$\forall \xi \in \mathbb{R}, \quad \int_{\mathbb{R}} \exp\left(-\frac{\sqrt{\imath}\tanh(2\sqrt{\imath}T)}{2}x_0^2 - \frac{\imath \xi x_0}{\cosh(2\sqrt{\imath}T)}\right) y_e^0(x_0)\, dx_0$$
$$= 2\int_0^T \sqrt{\frac{\cosh(2\sqrt{\imath}T)}{\cosh(2\sqrt{\imath}(T-t))}} \exp\left(\frac{\tanh(2\sqrt{\imath}T) - \tanh(2\sqrt{\imath}(T-t))}{2\sqrt{\imath}}\xi^2\right) u(t)\, dt. \quad (3.23)$$

Similarly as in previous section, we thus introduce, for $\xi \in \mathbb{R}$,

$$\mathcal{F}_\imath(y_e^0)(\xi) = \int_{\mathbb{R}} \exp\left(-\frac{\sqrt{\imath}\tanh(2\sqrt{\imath}T)}{2}x_0^2 - \frac{\imath\xi x_0}{\cosh(2\sqrt{\imath}T)}\right) y_e^0(x_0)\, dx_0. \quad (3.24)$$

This definition makes sense for $\xi \in \mathbb{C}$ under condition (1.7) on y_e^0 as we have the identity

$$\tanh(2\sqrt{\imath}T) = \frac{\sinh(2\sqrt{2}T) + \imath\sin(2\sqrt{2}T)}{\cosh(2\sqrt{2}T) + \cos(2\sqrt{2}T)},$$

implying in particular,

$$\Re(\sqrt{\imath}\tanh(2\sqrt{\imath}T)) = \frac{1}{\sqrt{2}}\frac{\sinh(2\sqrt{2}T) - \sin(2\sqrt{2}T)}{\cosh(2\sqrt{2}T) + \cos(2\sqrt{2}T)} > 0. \quad (3.25)$$

Let us also note that condition (3.23), which *a priori* holds only for $\xi \in \mathbb{R}$, also holds in the whole complex plane as both sides of the identity (3.23) are entire functions of the variable ξ.

Bound on $\mathscr{F}_\imath(y_e^0)$. Setting

$$\Gamma_T = \{\xi \in \mathbb{C}, \text{ such that } \xi = \lambda\cosh(2\sqrt{\imath}T) \text{ for } \lambda \in \mathbb{R}\},$$

it is clear from (3.24), condition (1.7) and (3.25), that there exists a constant $C > 0$ such that

$$\forall \xi \in \Gamma_T, \quad |\mathcal{F}_\imath(y_e^0)(\xi)| \leqslant C. \quad (3.26)$$

Note that the line Γ_T can be rewritten as $e^{\imath\alpha_T}\mathbb{R}$, where

$$\alpha_T = \arctan\Big(\tanh(\sqrt{2}T)\tan(\sqrt{2}T)\Big), \quad (3.27)$$

which implies in particular that Γ_T is a line of $\mathbb{C}$ which is close to $\mathbb{R}$ when T is small.

Bound on the right hand-side of (3.23). We claim the following:

Lemma 3.2. *Let us define*

$$Q = \left\{\xi \in \mathbb{C} \text{ with } \operatorname{Arg}(\xi) \in \left[-\frac{2\pi}{3}, -\frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\right\}.$$

If T is such that

$$T \leqslant \frac{1}{2\sqrt{2}} \operatorname{Argsh}\left(\frac{1}{2}\right), \quad (3.28)$$

then for all $t \in [0, T]$ and $\xi \in Q$,

$$\Re\left(\frac{\tanh(2\sqrt{\imath}T) - \tanh(2\sqrt{\imath}(T-t))}{2\sqrt{\imath}}\xi^2\right) \leqslant 0. \quad (3.29)$$

Proof. Replacing t by $T-t$, we see that (3.29) is equivalent to show for all $t \in [0,T]$ and $\xi \in Q$,

$$\Re\left(\frac{\tanh(2\sqrt{\imath}T)-\tanh(2\sqrt{\imath}t)}{2\sqrt{\imath}}\xi^2\right) \leqslant 0. \tag{3.30}$$

Explicit computations show that

$$\begin{aligned}\frac{\tanh(2\sqrt{\imath}T)-\tanh(2\sqrt{\imath}t)}{2\sqrt{\imath}} =& \frac{1}{2\sqrt{2}}\left(\frac{\sinh(2\sqrt{2}T)+\sin(2\sqrt{2}T)}{\cosh(2\sqrt{2}T)+\cos(2\sqrt{2}T)} - \frac{\sinh(2\sqrt{2}t)+\sin(2\sqrt{2}t)}{\cosh(2\sqrt{2}t)+\cos(2\sqrt{2}t)}\right)\\ &+\frac{\imath}{2\sqrt{2}}\left(\frac{\sin(2\sqrt{2}T)-\sinh(2\sqrt{2}T)}{\cosh(2\sqrt{2}T)+\cos(2\sqrt{2}T)} - \frac{\sin(2\sqrt{2}t)-\sinh(2\sqrt{2}t)}{\cosh(2\sqrt{2}t)+\cos(2\sqrt{2}t)}\right).\end{aligned}$$

Writing $\xi \in Q$ as $\xi = |\xi|e^{i\theta}$ with $\cos(2\theta) \leqslant -1/2$, we see that condition (3.30) is equivalent to the following one: for all θ satisfying $\cos(2\theta) \leqslant -1/2$, the function h_θ defined on $(0,T)$ by

$$\begin{aligned}h_\theta(t) = \cos(2\theta)&\left(\frac{\sinh(2\sqrt{2}T)+\sin(2\sqrt{2}T)}{\cosh(2\sqrt{2}T)+\cos(2\sqrt{2}T)} - \frac{\sinh(2\sqrt{2}t)+\sin(2\sqrt{2}t)}{\cosh(2\sqrt{2}t)+\cos(2\sqrt{2}t)}\right)\\ &-\sin(2\theta)\left(\frac{\sin(2\sqrt{2}T)-\sinh(2\sqrt{2}T)}{\cosh(2\sqrt{2}T)+\cos(2\sqrt{2}T)} - \frac{\sin(2\sqrt{2}t)-\sinh(2\sqrt{2}t)}{\cosh(2\sqrt{2}t)+\cos(2\sqrt{2}t)}\right)\end{aligned} \tag{3.31}$$

satisfies

$$\sup_{[0,T]} h_\theta(t) \leqslant 0.$$

But the function h_θ satisfies $h_\theta(T) = 0$ and for all $t \in [0,T]$,

$$h'_\theta(t) = -4\sqrt{2}\left(\frac{\cos(2\theta)(1+\cos(2\sqrt{2}t)\cosh(2\sqrt{2}t)) + \sin(2\theta)\sinh(2\sqrt{2}t)\sin(2\sqrt{2}t)}{(\cosh(2\sqrt{2}t)+\cos(2\sqrt{2}t))^2}\right)$$

In particular, if $\cos(2\theta) \leqslant -1/2$ and T is such that for all $t \in [0,T]$, $\cos(2\sqrt{2}t) \geqslant 0$ and $\sinh(2\sqrt{2}t) \leqslant 1/2$, we get, for all $t \in [0,T]$,

$$\cos(2\theta)(1+\cos(2\sqrt{2}t)\cosh(2\sqrt{2}t)) + \sin(2\theta)\sinh(2\sqrt{2}t)\sin(2\sqrt{2}t) \leqslant -\frac{1}{2}+\frac{1}{2} \leqslant 0,$$

so that the function h_θ is increasing on $[0,T]$. We then simply note that if T satisfies (3.28), we have both conditions $\cos(2\sqrt{2}T) \geqslant 0$ and $\sinh(2\sqrt{2}T) \leqslant 1/2$, so that we can conclude that for all θ satisfying $\cos(2\theta) \leqslant -1/2$, for all $t \in [0,T]$, $h_\theta(t) \leqslant h_\theta(T) = 0$. This concludes the proof of Lemma 3.2. ☐

Of course, the main consequence of Lemma 3.2 is that, for all T satisfying (3.28), there exists a constant $C > 0$ such that for all ξ in Q,

$$\left|2\int_0^T \sqrt{\frac{\cosh(2\sqrt{\imath}T)}{\cosh(2\sqrt{\imath}(T-t))}}\exp\left(\frac{\tanh(2\sqrt{\imath}T)-\tanh(2\sqrt{\imath}(T-t))}{2\sqrt{\imath}}\xi^2\right)u(t)\,dt\right| \leqslant C\,\|u\|_{L^1(0,T)} \leqslant C. \tag{3.32}$$

Proof of Theorem 1.3 for T small enough. According to (3.27), there exists $T_0 > 0$ such that for all $T \in (0, T_0]$,

$$|\alpha_T| < \frac{\pi}{6},$$

and we have the following bound

$$\forall \xi \in e^{\imath\alpha_T}\mathbb{R}, \quad |\mathcal{F}_\imath(y_e^0)(\xi)| \leqslant C. \tag{3.33}$$

We now assume

$$T \in (0, T_1], \qquad \text{where} \qquad T_1 = \min\left\{T_0, \frac{\operatorname{Argsh}(1/2)}{2\sqrt{2}}\right\}, \tag{3.34}$$

so that from (3.23) and (3.32) we also get

$$\forall \xi \in Q, \quad |\mathcal{F}_\imath(y_e^0)(\xi)| \leqslant C. \tag{3.35}$$

Besides, we easily have from (3.23) that

$$\forall \xi \in \mathbb{C}, \quad |\mathcal{F}_\imath(y_e^0)(\xi)| \leqslant C \exp(C|\xi|^2). \tag{3.36}$$

We can then use Phragmén Lindelöf principle (Theorem 2.2) in each sector delimited by ∂Q and the line $e^{\imath\alpha_T}\mathbb{R}$, namely $S(\alpha_T, \pi/3)$, $S(-\pi/3, \alpha_T)$, $S(2\pi/3, \alpha_T + \pi)$, $S(\alpha_T - \pi, -2\pi/3)$, each of which is a cone of aperture at most $|\alpha_T| + \pi/3 < \pi/2$. As a consequence, $\mathcal{F}_\imath(y_e^0)$ is bounded on the whole complex plane, so it is constant: there exists c_0 such that

$$\forall \xi \in \mathbb{C}, \quad \mathcal{F}_\imath(y_e^0)(\xi) = c_0.$$

But for all $\xi \in \mathbb{R}$,

$$\mathscr{F}(y_e^0 e^{-\sqrt{\imath}\tanh(2\sqrt{\imath}T)x^2/2})(\xi) = \mathcal{F}_\imath(y_e^0)(\xi\cosh(2\sqrt{\imath}T)) = c_0.$$

As $y_e^0 e^{-\sqrt{\imath}\tanh(2\sqrt{\imath}T)x^2/2} \in L^2(\mathbb{R})$ from (1.7) and (3.25), we deduce that necessarily $c_0 = 0$ and consequently $y_e^0 = 0$ on $\mathbb{R}$.

Of course, for $t \in (0, T)$, the solution $y(t)$ of (1.3) is then given by

$$y(t, x) = \int_0^t G_\imath(t - \tau, x, 0)u(\tau)\, d\tau,$$

so that $y(t)$ belongs to $L^\infty(\mathbb{R})$. We can thus apply the same strategy as above on the time interval $[t, T]$, so that we get the same result: $y(t) \equiv 0$ on $\mathbb{R}$. As a consequence, the solution y of (1.3) vanishes for all $t \in [0, T]$ and $x \in \mathbb{R}$, so that we deduce from the equation (1.3) that u vanishes as well on $[0, T]$.

Proof of Theorem 1.3 for any T. If $T \leqslant T_1$, the previous paragraph allows to conclude.

If $T > T_1$, we set $n = \lceil T/T_1 \rceil$ and $T_n = T/n$. We then apply the previous paragraph on the time interval $[T - T_n, T]$, which is allowed as Lemma 2.1 easily implies that $y(T - T_n)$ belongs to $L^\infty(\mathbb{R})$ when y^0 satisfies (1.7). We can thus conclude that y and u vanish on the time interval $[T - T_n, T]$. Iterating this argument n times, we easily show that in fact y and u vanish identically on the whole time interval $[0, T]$.

Remark 3.3. *In view of* (3.25)*, the condition* (1.7) *in Theorem 1.3 on the initial condition can be slightly relaxed. Indeed, the above proof readily applies to prove that, replacing* (1.7) *by* (3.22) *with constant $\varepsilon = \varepsilon(T)$, the statement of Theorem 1.3 still holds, where $\varepsilon(T)$ is given by*

$$\varepsilon(T) = \frac{1}{2\sqrt{2}} \frac{\sinh(2\sqrt{2}T) - \sin(2\sqrt{2}T)}{\cosh(2\sqrt{2}T) + \cos(2\sqrt{2}T)} \quad \text{for } T \leqslant T_1,$$

and, for $T \geqslant T_1$,

$$\varepsilon(T) = \varepsilon(T/\lceil T/T_1 \rceil).$$

4 Further comments

4.1 Lack of controllability with distributed controls

Theorems 1.1, 1.2, and 1.3 focus on the case of boundary control, but our proof can be adapted to the case of distributed control. To fix the ideas, we focus on the case corresponding to (1.2).

Namely, we consider $L > 0$ and the controllability problem:

$$\begin{cases} \partial_t y - \partial_{xx} y + x^2 y = u 1_{(-L,L)}, & \text{in } (0, T) \times \mathbb{R}, \\ y(0, x) = y^0(x), & \text{in } \mathbb{R}, \end{cases} \tag{4.1}$$

where $y^0 \in L^2(\mathbb{R})$ and $u \in L^2((0, T) \times (-L, L))$.

We claim the following result:

Theorem 4.1. *Let $T > 0$. If y is a solution of* (4.1) *with an initial condition y^0 satisfying*

$$\exists \rho \in [0,2), \exists M > 0, \text{ s.t. } \forall x \in \mathbb{R}, \quad |y^0(x)| \leqslant M \exp(M|x|^\rho), \tag{4.2}$$

and with a control function $u \in L^2((0,T) \times (-L,L))$, such that $y(T)$ satisfies

$$y(T,\cdot) = 0 \quad \text{ in } \mathbb{R}, \tag{4.3}$$

then necessarily y^0 is supported in $[-L,L]$.

This result is a counterpart of Theorem 1.2 in the distributed case. Though, note that its conclusion is weaker than Theorem 1.2, as y^0 is not necessarily vanishing in the whole domain. This is in fact expected for the following reason:

Proposition 4.2. *If $y^0 \in L^2(\mathbb{R})$ is supported in $[-L_0, L_0]$ for some $L_0 < L$, then for any $T > 0$, one can construct a controlled trajectory of* (4.1) *satisfying* (4.3).

The proof of Proposition 4.2 is mainly straightforward for readers accustomed to control theory, and is therefore postponed to the end of this paragraph.

Proof of Theorem 4.1. Similarly as in the proof of Theorem 1.2, we consider y^0 and u such that the solution y of (4.1) satisfies (4.3). This implies that

$$\forall x \in \mathbb{R}, \quad \int_{\mathbb{R}} G_1(T,x,x_0) y^0(x_0)\, dx = -\int_0^T \int_{-L}^{L} G_1(T-t,x,x_0) u(t,x_0)\, dx_0 dt.$$

Taking the Fourier transform of this identity in the x variable, we get, instead of (3.17), for all $\xi \in \mathbb{R}$,

$$\mathcal{F}_1(y^0)(\xi) = -\int_0^T \int_{-L}^{L} \sqrt{\frac{\cosh(2T)}{\cosh(2(T-t))}} e^{\frac{\tanh(2T)-\tanh(2(T-t))}{2}\xi^2 - \frac{\tanh(2(T-t))}{2}x_0^2 - \frac{\imath \xi x_0}{\cosh(2(T-t))}} u(t,x_0)\, dx_0 dt, \tag{4.4}$$

where $\mathcal{F}_1$ is defined in (3.14).

Using the fact that $\mathcal{F}_1(y^0)$ is the Fourier transform of a $L^1(\mathbb{R})$ function due to the condition (4.2) (recall (3.15)), we have the estimate

$$\exists C > 0, \forall \xi \in \mathbb{R}, \quad |\mathcal{F}_1(y_e^0)(\xi)| \leqslant C, \tag{4.5}$$

which is similar to (3.16).

The right hand-side of identity (4.4) is an holomorphic function on $\mathbb{C}$, which allows to consider $\mathcal{F}_1(y^0)$ as an entire function defined by identity (4.4) for all $\xi \in \mathbb{C}$.

Besides, for all $\xi \in \mathbb{C}$ such that $\Re(\xi^2) \leqslant 0$, i.e. such that $|\Re(\xi)| \leqslant |\Im(\xi)|$, the identity (4.4) allows to immediately derive

$$\forall \xi \in \mathbb{C} \text{ with } |\Re(\xi)| \leqslant |\Im(\xi)|, \quad |\mathcal{F}_1(y^0)(\xi)| \leqslant C \exp\left(|\Im(\xi)| L\right). \tag{4.6}$$

Furthermore, identity (4.4) also gives

$$\forall \xi \in \mathbb{C}, \quad |\mathcal{F}_1(y^0)(\xi)| \leqslant C \exp\left(\tanh(2T)|\xi|^2 + |\Im(\xi)|L\right). \tag{4.7}$$

Using the bounds (4.5), (4.6), (4.7), the repeated use of Phragmén-Lindelöf principle (Theorem 2.2), to $\mathcal{F}_1(y^0)(\xi)\exp(\imath \xi L)$ in the sectors $S(0,\pi/4)$ and $S(3\pi/4,\pi)$, and to $\mathcal{F}_1(y^0)(\xi)\exp(\imath \xi L)$ in the sectors $S(-\pi/4,0)$ and $S(-\pi,-3\pi/4)$, shows that there exists a constant C such that

$$\forall \xi \in \mathbb{C}, \quad |\mathcal{F}_1(y^0)(\xi)| \leqslant C \exp\left(|\Im(\xi)| L\right). \tag{4.8}$$

Now, using (3.21), setting $z^0(x) = y^0(x) e^{-\tanh(2T)x^2/2}$, the Fourier transform of z^0 satisfies:

$$\forall \xi \in \mathbb{C}, \quad |\mathscr{F}(z^0)(\xi)| \leqslant C \exp\left(|\Im(\xi)| L \cosh(2T)\right).$$

As z^0 is in $L^2(\mathbb{R})$ by assumption (recall (4.2)), Paley-Wiener theorem (see [28, Lecture 10]) implies that z^0 is supported in $[-L\cosh(2T), L\cosh(2T)]$, which immediately implies the same for y^0. With the same arguments, it is not difficult to obtain that for all t in $[0,T)$, $y(t,.)$ is supported in $[-L\cosh(2T), L\cosh(2T)]$. But for all $(t,x) \in (0,T)\times\mathbb{R}$, we have

$$y(t,x) = \int_{\mathbb{R}} G_1(t,x,x_0) y^0(x_0)\,dx + \int_0^t \int_{-L}^{L} G_1(t-s,x,x_0) u(s,x_0)\,dx_0 dt,$$

which shows that, for all $t \in (0,T]$, $y(t,.)$ is analytic in $\mathbb{R}\setminus[-L,L]$. Hence, necessarily for all $t\in(0,T]$, $y(t,.)$ is supported in $[-L,L]$, which concludes the proof of Theorem 4.1. □

As a corollary of Theorem 4.1, we can also prove the following result:

Corollary 4.3. *Let $T>0$. If y is a solution of* (4.1) *with an initial condition y^0 satisfying* (4.2)*, and with a control function $u \in L^2((0,T)\times(-L,L))$, such that $y(T)$ satisfies*

$$y(T,\cdot) = 0 \quad \text{in } \mathbb{R}\setminus[-L,L], \tag{4.9}$$

then necessarily y^0 is supported in $[-L,L]$.

Proof. Let us consider a solution y of (4.1) with an initial condition y^0 satisfying (4.2), and with a control function $u\in L^2((0,T)\times(-L,L))$, such that $y(T)$ satisfies (4.9). Let then $L_*>L$, and $T_*>T$. According to Proposition 4.2, one can construct a control function $u_*\in L^2((T,T_*)\times(-L_*,L_*))$ such that the solution y of (4.1) satisfies $y(T_*)=0$ in $\mathbb{R}$. We can thus apply Theorem 4.1 to y, and we get $\mathrm{Supp}(y^0)\subset[-L_*,L_*]$. As L_* is any real number satisfying $L_*>L$, $\mathrm{Supp}(y^0)\subset[-L,L]$, which proves Corollary 4.3. □

We now prove Proposition 4.2.

Proof of Proposition 4.2. Let $y^0\in L^2(\mathbb{R})$ be supported in $[-L_0,L_0]$ for some $L_0<L$, and $T>0$. Setting

$$\theta\in C^\infty([0,T]), \quad \text{such that} \quad \theta(0)=1,\ \theta(T)=0,$$
$$\eta\in C^\infty(\mathbb{R}), \quad \text{such that } \mathrm{Supp}\,\eta\subset[-L,L], \text{ and } \forall x\in[-L_0,L_0],\ \eta(x)=1,$$

and z the solution of
$$\begin{cases} \partial_t z-\partial_{xx}z+x^2z=0, & \text{in } (0,T)\times\mathbb{R},\\ z(0,x)=y^0(x), & \text{in } \mathbb{R},\end{cases}$$
the trajectory y given by
$$y(t,x)=\theta(t)\eta(x)z(t,x), \quad (t,x)\in(0,T)\times\mathbb{R},$$
solves (4.1) with control function $u(t,x)=\theta'(t)\eta(x)z(t,x)-\theta(t)[\partial_{xx},\eta]z(t,x)$ satisfies $y(T)=0$ in $\mathbb{R}$. □

4.2 Lack of controllability with boundary controls on domains (a,∞)

It might seem from the previous results that our results apply only when the symmetry induced by the operator is preserved. This is not the case. Let us for instance consider, instead of equation (1.2) set on $(0,\infty)$ the following equation:

$$\begin{cases} \partial_t y-\partial_{xx}y+x^2y=0, & \text{in } (0,T)\times(a,\infty),\\ \partial_x y(t,a)=u(t), & \text{in } (0,T),\\ y(0,x)=y^0(x), & \text{in } (a,\infty),\end{cases} \tag{4.10}$$

which is set on (a,∞), where $a\in\mathbb{R}$.

We claim that we have the following result:

Theorem 4.4. *Let $T>0$ and $a\in\mathbb{R}$. If y is a solution of* (4.10) *with an initial condition y^0 satisfying*

$$\exists\rho\in[0,2), \exists M>0, \text{ s.t. } \forall x\in(a,\infty), \quad |y^0(x)|\leqslant M\exp(M|x|^\rho), \tag{4.11}$$

and with a control function $u\in L^2(0,T)$ such that $y(T)$ vanishes on (a,∞), then necessarily $y^0=0$ in (a,∞) and $u=0$ in $(0,T)$.

Proof. Let y be a solution of (4.10) with an initial condition y^0 satisfying (4.11) with a control function $u \in L^2(0,T)$ and such that $y(T,\cdot) = 0$ in (a,∞).

We shall distinguish the cases $a \geqslant 0$ and $a \leqslant 0$.

Case $a \geqslant 0$. Let us take $a_0 > a$, and introduce a cut-off function $\eta = \eta(x)$ such that $\eta(a) = \eta'(a) = 0$ and $\eta(x) = 1$ for all $x \geqslant a$. Then the function $\tilde{y}(t,x) = \eta(x) y(t,x) 1_{x>a}$ solves (4.1) with control function $\tilde{u}(t,x) = [\partial_{xx}, \eta] y(t,x) 1_{x>a}$ supported in space in $[-a_0, a_0]$, and initial condition $\eta(x) y^0(x)$. We can thus apply Theorem 4.1, and we get that $\eta(x) y^0(x)$ is supported in $[-a_0, a_0]$. In particular, $y^0(\cdot) = 0$ in (a_0, ∞). As a_0 can be chosen arbitrarily close to a, $y^0 = 0$ in (a,∞). Now, we can apply the same computations on each time interval (t,T), so that we can conclude that for all $t \in [0,T]$, $y(t,\cdot)$ vanishes on (a,∞). Consequently, u vanishes on $(0,T)$ as well.

Case $a \leqslant 0$. We can then take the trace at $x = 0$ and obtain a solution of (1.2) which satisfies all the assumptions of Theorem 1.2. Consequently, we get that $y(t,x) = 0$ for all $(t,x) \in (0,T) \times \mathbb{R}_+^*$. We then use the fact that solutions y of (4.10) are analytic in the space variable for all times $t \in (0,T]$. Indeed, the extension y_e of the solution y of (4.10) by 0 for $x < a$ satisfies

$$\begin{cases} \partial_t y_e - \partial_{xx} y_e + x^2 y_e = -u(t)\delta_a + y(t,x_a)\partial_x \delta_a, & \text{in } (0,T) \times \mathbb{R}, \\ y_e(0,x) = y^0(x_0) 1_{x>a}, & \text{in } \mathbb{R}, \end{cases}$$

so that for all $(t,x) \in (0,T) \times \mathbb{R}$,

$$y_e(t,x) = \int_a^\infty G_1(t,x,x_0) y^0(x_0)\, dx_0 - \int_0^t G_1(t-\tau,x,a) u(\tau)\, d\tau + \int_0^t \partial_x G_1(t-\tau,x,a) y(t,x,a) d\tau.$$

But the function y_e coincides with y in $(0,T) \times (a,\infty)$, and thus vanishes on $(0,T) \times (0,\infty)$. Now, the above formula and the analyticity properties of the fundamental solution G_1 in (2.3) imply that y necessarily vanish in $(0,T) \times (a,\infty)$. ☐

Let us finally end this section by pointing out that, although we focused on controls acting on the normal derivative of the solution, the same results also hold true when considering controls acting on the Dirichlet boundary conditions. We leave the details to the reader as it can be derived easily from Theorems 1.2 and 4.1 using the above proof of Theorem 4.4.

4.3 Higher dimensional settings

The approach presented in this article can be readily applied in tensorized situations allowing to reduce the problem to one of the 1-d problem (1.1), (1.2) or (1.3).

But it would be very interesting to further develop the strategy here to exhibit some non-trivial higher dimensional geometric settings in which only the trivial initial conditions (i.e. the initial condition $y^0 = 0$ or, in the case of distributed controls, the initial conditions supported in the control set) can be driven to zero with controls in L^2. In particular, can we characterize the sets for which there are only the trivial initial data which can be led to zero?

With that respect, let us emphasize that the situation is not completely clear, even for the heat equation in conical domains $\mathscr{C}_\theta = \{(x_1,x_2) \in \mathbb{R}^2, \text{ with } |x_2| \leqslant x_1 \tan(\theta/2)\}$ of aperture $\theta \in (0,2\pi)$. The best result so far is, to our knowledge, the work [38], which states that when $\theta \geqslant 95^\circ$, there is no non-trivial initial condition which can be steered to zero. On the other hand, an example of Escauriaza (see e.g. [29]) shows that when $\theta < \pi/2$, there exists non-trivial solutions of the heat equation which vanishes at time T.

Let us emphasize that there are several conditions in the literature ensuring that the multi-dimensional heat equation is observable from a given control set. We refer to the recent work [40] for a characterization of these sets (we also refer to [33, 26, 7]). Note that if the condition of γ-thickness (see [40] for a precise definition) is not satisfied, one can only conclude that there exists some initial condition which cannot be led to zero, which is of course a much weaker statement than the strong lack of controllability stated in Theorem 1.1. Thus, describing precisely the geometric settings in which there is a strong lack of controllability seems to be highly challenging.

4.4 On the cost of approximate controllability

Each of the equation (1.1), (1.2) or (1.3) is approximately controllable in any time $T > 0$ (for instance, because unique continuation for the adjoint equation is obvious by Holmgren's uniqueness theorem), but is strongly not null controllable according to Theorems 1.1, 1.2 and 1.3. It is thus natural to ask what is the cost of approximate controllability in each of the above cases. So far, this seems widely open.

Regarding approximate controllability, we should refer to the work [18] regarding the heat equation and to [25] for general hypoelliptic equations on compact manifolds. We shall also quote the more recent work [40], which considers weak observability inequalities in geometric settings in which observability does not hold, and which gives some leads to address this problem.

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