Dependence of high-frequency waves with respect to potentials*

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Abstract

In this article, we consider the wave equation in a bounded domain Ω of $\mathbb{R}^d$ with a potential q. Our goal then is to show that the high-frequency part of the corresponding solutions weakly depends on the potential. We will in particular focus on two instances of interest arising in data assimilation and control theory, respectively corresponding to the problem of recovering an initial data from a measurement and to the problem of computing a control. In these two cases, we derive an explicit bound on the error of the high-frequency part of the solution induced by a $W^{s,p}(\Omega)$-error on the potential for $s \in (0, 1]$ and $p \in (\max\{d, 2\}, \infty]$. In order to do that and to express it in a quantified form, we introduce spectral truncations. Our main tool is a commutator estimate.

Keywords: wave equation, controllability, high-frequency.

1 Introduction

The goal of this article is to study how the high-frequencies of waves depend on the potential.

More precisely, we focus on that particular system:

$$\begin{cases} \Box y + qy = f, & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y(0), \partial_t y(0)) = (y_0, y_1), \end{cases} \tag{1.1}$$

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Here, Ω will be assumed to be a smooth bounded domain of $\mathbb{R}^d$. The solution y of (1.1) is the displacement of the waves. The potential function $q = q(x)$ will always be assumed to be in $L^p(\Omega)$ with $p \in (\max\{d, 2\}, \infty]$ and we further assume that q takes real-value.

The source term f and the initial datum (y_0, y_1) will be taken in an appropriate functional setting that may change depending on the problem under consideration. We refer to [23, Chap.III, Sect. 8–9] for details on the well-posedness of (1.1) for real valued potentials in $L^p(\Omega)$ with $p \in (\max\{d, 2\}, \infty]$. In the following, we shall extensively use the two following settings: if $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^1((0,T); L^2(\Omega))$, the solution y of (1.1) belongs to the space of finite energy solutions $C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$; if $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $f \in L^1((0,T); H^{-1}(\Omega))$, there is a unique solution $y \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ of (1.1) in the sense of transposition.

Our goal is to derive new results on two related aspects:

- **Data assimilation:** When trying to recover the initial data of the waves from localized measurements, the knowledge of the potential in the wave equation is not really needed to get good approximations of the high-frequency part of the solutions.
- **Control theory:** When controlling the wave equation from an open subset $\omega \subset \Omega$, the control given by the Hilbert Uniqueness Method of Lions (see [22]) does not depend significantly on the potentials at high-frequency.

Of course, these statements are very natural since potentials appear as low order perturbations of the wave operator. The main novelty of our approach rather is to give a precise quantified description of these statements. This will be done using the Littlewood-Paley decomposition.

The setting. We now come back to the wave equation (1.1) and let ω be an open subdomain of Ω such that the Geometric Control Condition (GCC) of C. Bardos, G. Lebeau and J. Rauch [4] holds in some time $T > 0$. Let us here briefly recall this property.

Definition 1.1. *We say that the open set ω satisfies the geometric control condition (GCC in short) at time T if every geodesic ray of Ω traveling with speed one and starting at t=0 enters the open set ω in a time $t < T$.*

These geodesic rays have to be understood as the projection on the basis Ω of the generalized bicharacteristic rays of the wave operator, the so-called Melrose-Sjöstrand flow, for which we refer to [15, Chap. XXIV] and [24, 25]. We will always assume that there is no contact of infinite order between the boundary $\partial\Omega$ and the bicharacteristic rays of the wave operator in the free space, so that the Melrose-Sjöstrand flow is well defined.

When the GCC holds, it is by now well-known (see [8]) that, for any $q \in L^\infty(\Omega)$, there exists a constant $C(q) > 0$ such that any solution $\varphi[q]$ of

$$\begin{cases} \Box\varphi + q\varphi = 0, & \text{in } (0,T) \times \Omega, \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\varphi(0), \partial_t\varphi(0)) = (\varphi_0, \varphi_1), \end{cases} \tag{1.2}$$

satisfies the following observability inequality

$$\|(\varphi_0, \varphi_1)\|^2_{L^2 \times H^{-1}} \leq C(q) \int_0^T \int_\omega |\varphi[q]|^2 \, dxdt. \tag{1.3}$$

To be more precise, we will introduce a smooth (e.g. $W^{2,\infty}(\Omega)$) non-negative function $a_\omega = a_\omega(x)$ defined on Ω that approximates the characteristic function of ω and satisfies $\overline{\omega} \subset \{a_\omega > 0\}$ so that (1.3) can be weakened into

$$\|(\varphi_0, \varphi_1)\|^2_{L^2 \times H^{-1}} \leq C(q) \int_0^T \int_\Omega a_\omega^2 |\varphi[q]|^2 \, dxdt. \tag{1.4}$$

Our first result, proved in Section 3, provides a uniform observability estimate for potentials q lying only in $L^p(\Omega)$ for $p \in (\max\{d,2\},\infty]$, thus improving the results in [8] on the integrability of the potential.

Proposition 1.2. *Assume that (Ω,ω,T) satisfies the geometric control condition GCC and let $p \in (\max\{d,2\},\infty]$.*

For all $m > 0$, there exists a constant C_m such that for all $q \in L^p(\Omega)$ of $L^p(\Omega)$-norm smaller than m, any solution $\varphi[q]$ of (1.2) with initial data $(\varphi_0,\varphi_1) \in L^2(\Omega)\times H^{-1}(\Omega)$ satisfies

$$\|(\varphi_0,\varphi_1)\|^2_{L^2\times H^{-1}} \le C_m \int_0^T \int_\omega |\varphi[q]|^2\, dxdt. \tag{1.5}$$

At this stage, note that we strongly use the fact that the potential $q = q(x)$ does not depend on $t \in (0,T)$. Otherwise, the same question would be much harder since unique continuation properties may fail, see [1].

In the sequel, we always assume the geometric control condition (see Definition 1.1) and the integrability parameter $p \in (\max\{d,2\},\infty]$. Moreover, we shall denote solutions of a wave equation with a potential q using the notation $[q]$ to underline the dependence on the potential q.

Spectral decomposition. To state our results precisely, we shall first recall the Littlewood Paley decomposition. Let $-\Delta_D$ be the Laplace operator on Ω defined as an unbounded operator on $H^{-1}(\Omega)$ and with domain $H^1_0(\Omega)$. Since this operator is self-adjoint, positive definite and has compact resolvent, its spectrum is given by an increasing sequence of positive eigenvalues λ_j^2 corresponding to eigenfunctions e_j:

$$-\Delta e_j = \lambda_j^2 e_j \quad \text{in } \Omega, \qquad e_j = 0 \quad \text{on } \partial\Omega, \qquad j \ge 1. \tag{1.6}$$

In the following, the operator $-\Delta_D$ will simply be denoted by $-\Delta$.

If $\theta(\xi)$ denotes a smooth non-negative function taking value one between 0 and 1/2 and vanishing for $\xi > 1$, we can define

$$\psi(\xi) = \theta(\xi) - \theta(2\xi) \quad \text{ and, for } k \in \mathbb{N}^*, \quad \psi_k(\xi) = \psi(\xi 2^{-k}), \tag{1.7}$$

that satisfy

$$\forall \xi \in \mathbb{R}_+, \quad \theta(\xi) + \sum_{k=1}^{\infty} \psi_k(\xi) = 1. \tag{1.8}$$

We then define the operators $\psi_k(-\Delta)$ as follows: for $f = \sum_j a_j e_j$,

$$\psi_k(-\Delta) f = \sum_j \psi_k(\lambda_j^2) a_j e_j, \tag{1.9}$$

which correspond to localize in the set of frequencies in the dyadic ring $\lambda_j^2 \in (2^{k-2}, 2^k)$, and the operators $\eta_k(-\Delta)$ for $k \in \mathbb{N}^*$,

$$\eta_k(-\Delta) = \sum_{\ell=k}^{\infty} \psi_\ell(-\Delta), \tag{1.10}$$

which rather correspond to look at the frequencies λ_j such that $\lambda_j^2 \ge 2^{k-2}$.

In Section 2, we discuss some properties of these spectral operators with a particular emphasis on their commutation properties, as it will be the main tools to achieve our results.

A data assimilation problem. Let us consider the solution $\Phi[Q]$ of the wave equation

$$\begin{cases} \Box\Phi[Q] + Q\Phi[Q] = 0, & \text{in } (0,T)\times\Omega, \\ \Phi[Q] = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\Phi[Q](0), \partial_t\Phi[Q](0)) = (\Phi_0, \Phi_1). \end{cases} \tag{1.11}$$

and assume that we know the measurement of $a_\omega\Phi = a_\omega\Phi[Q]$ in $(0,T)\times\Omega$ (note that a_ω localizes the measurement close to the set $\bar{\omega}$), where ω is a subdomain of Ω such that (Ω,ω,T) satisfies the GCC.

It is by now well-known that, if the potential Q is known, then the initial datum $(\Phi_0,\Phi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ can be recovered directly from the knowledge of $a_\omega\Phi[Q]$, for instance by minimizing

$$J[Q](\varphi_0,\varphi_1) = \frac{1}{2}\int_0^T\int_\Omega a_\omega^2|\varphi[Q]-\Phi[Q]|^2\,dxdt, \tag{1.12}$$

over $(\varphi_0,\varphi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, where $\varphi[Q]$ is the solution of (1.2) with $q=Q$. Note that this minimization problem is well-posed due to (1.4).

Of course, the question is much more difficult when the potential Q is unknown. In that case, assume we have a guess q (for instance $q=0$ if we do not have any clever guess), and instead of minimizing $J[Q]$, minimize

$$J[q](\varphi_0,\varphi_1) = \frac{1}{2}\int_0^T\int_\Omega a_\omega^2|\varphi[q]-\Phi[Q]|^2\,dxdt, \tag{1.13}$$

over $(\varphi_0,\varphi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, where $\varphi[q]$ is the solution of (1.2). Note that this minimization problem is again well-posed due to (1.4) and therefore $J[q]$ admits a unique minimizer.

Before stating our next result, we also introduce, for $s\geq 0$, $p\in[1,\infty]$ and $m>0$ the class

$$W^{s,p}_{\leq m}(\Omega) = \{q\in W^{s,p}(\Omega) \text{ such that } \|q\|_{W^{s,p}}\leq m\}.$$

We then obtain the following result:

Theorem 1.3. *Assume the geometric control condition GCC. Let $m>0$, $p\in(\max\{d,2\},\infty]$ and $s\in[0,1]$, and assume that Q and q belong to $W^{s,p}_{\leq m}(\Omega)$.*

Let $(\Phi_0,\Phi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and let $\Phi[Q]$ be the corresponding solution of (1.11)*. Let $(\Phi_0[q],\Phi_1[q])$ be the minimizer of $J[q]$ in* (1.13)*.*

Then there exists a constant C independent of k, (Φ_0,Φ_1) and $q,Q\in W^{s,p}_{\leq m}(\Omega)$ such that for all $k\geq 1$,

$$\|\eta_k(-\Delta)((\Phi_0,\Phi_1)-(\Phi_0[q],\Phi_1[q]))\|_{L^2\times H^{-1}} \leq C2^{-ks/2}\|a_\omega\Phi[Q]\|_{L^2(L^2)}\|q-Q\|_{W^{s,p}}. \tag{1.14}$$

In other words, even without knowing precisely the potential Q, the measurement $a_\omega\Phi[Q]$ allows to recover a good approximation of the high-frequencies of the initial data.

The proof of Theorem 1.3 is given in Section 4.

A control problem. Let us consider the following control problem:

Find $u\in L^2((0,T)\times\omega)$ such that the solution y of

$$\begin{cases} \Box y + qy = a_\omega\, u, & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y(0),\partial_t y(0)) = (y_0,y_1), \end{cases} \tag{1.15}$$

satisfies

$$(y(T),\partial_t y(T)) = (0,0). \tag{1.16}$$

It is well-known that such problem can be solved for $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with control $u \in L^2((0,T)\times\Omega)$ when the observability estimate (1.4) holds for the adjoint system.

Besides, the control u of minimal $L^2((0,T)\times\Omega)$-norm can be computed through the minimization of the functional $K[q]$ defined by

$$K[q](\varphi_0, \varphi_1) = \frac{1}{2}\int_0^T \int_\Omega a_\omega^2 |\varphi[q]|^2 \, dxdt + \langle (\varphi_0, \varphi_1), (y_0, y_1) \rangle_{L^2\times H^{-1}, H_0^1 \times L^2} \tag{1.17}$$

over $(\varphi_0, \varphi_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where $\varphi[q]$ is the solution of (1.2) and the duality product is defined by

$$\langle (\varphi_0, \varphi_1), (y_0, y_1) \rangle_{L^2\times H^{-1}, H_0^1 \times L^2} = \int_\Omega \varphi_0 y_1 - \int_\Omega \nabla(-\Delta)^{-1}\varphi_1 \cdot \nabla y_0. \tag{1.18}$$

Then, if $(\Phi_0[q], \Phi_1[q])$ denotes the minimum of $K[q]$, which is uniquely defined thanks to (1.4), denoting by $\Phi[q]$ the corresponding solution of (1.2), we get that $u[q] = a_\omega \Phi[q]$ is the control of minimal $L^2((0,T)\times\Omega)$-norm for (1.15), see [22], which we refer to as the HUM control in the following (HUM stands for Hilbert Uniqueness Method, see [22]).

We then get the following result, proved in Section 5:

Theorem 1.4. *Assume the geometric control condition GCC. Let $m > 0$, $p \in (\max\{d, 2\}, \infty]$ and $s \in [0,1]$, and consider two potentials $q^a, q^b \in W^{s,p}_{\leq m}(\Omega)$.*

Let $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and denote by $(\Phi_0[q^a], \Phi_1[q^a])$ and $(\Phi_0[q^b], \Phi_1[q^b])$ the minimizers of, respectively, $K[q^a]$ and $K[q^b]$.

Then there exists a constant C independent of k, (y_0, y_1) and $q^a, q^b \in W^{s,p}_{\leq m}(\Omega)$ such that for all $k \geq 1$,

$$\left\| \eta_k(-\Delta) a_\omega (\Phi[q^a] - \Phi[q^b]) \right\|_{L^2(L^2)} \leq C 2^{-ks/2} \left\| (y_0, y_1) \right\|_{H_0^1 \times L^2} \left\| q^a - q^b \right\|_{W^{s,p}}. \tag{1.19}$$

Besides, we also have the following estimate:

$$\left\| \eta_k(-\Delta)(\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]) \right\|_{L^2 \times H^{-1}} \\ \leq C 2^{-ks/2} \left\| (y_0, y_1) \right\|_{H_0^1 \times L^2} \left\| q^a - q^b \right\|_{W^{s,p}}. \tag{1.20}$$

Again, this result means that the high-frequency components of the HUM controls do not see the potential in a significant manner. Indeed, setting

$$u[q^a] = a_\omega \Phi[q^a] \text{ and } u[q^b] = a_\omega \Phi[q^b],$$

which are the control functions corresponding to the initial data (y_0, y_1) for the potentials q^a and q^b respectively, (1.19) reads as:

$$\left\| \eta_k(-\Delta)(u[q^a] - u[q^b]) \right\|_{L^2(L^2)} \leq C 2^{-ks/2} \left\| (y_0, y_1) \right\|_{H_0^1 \times L^2} \left\| q^a - q^b \right\|_{W^{s,p}}.$$

Comments and related references. *On the observability of waves.* Numerous works have been devoted to study the observability properties of waves, for instance [22, 19] where multiplier methods were developed under the so-called Gamma condition (1.21): the set ω should be a neighborhood of a part Γ of the boundary for which there exists $x_0 \notin \Omega$ such that

$$\Gamma \supset \{x \in \partial\Omega, \text{ with } (x - x_0)\cdot n > 0\} \quad \text{and} \quad T > 2\sup_{x\in\Omega}\{|x - x_0|\}. \tag{1.21}$$

Later on, using microlocal analysis, it has been shown that the geometric control condition (see Definition 1.1) is a sufficient and necessary condition for observability to hold [3, 4, 8]. But the microlocal proof of observability relies on two fundamental facts: a propagation property (of wave front, compactness, etc) and a unique continuation result.

The first property is based on the hyperbolicity of the wave operator, see [15, 20]. For propagation of the H^1- wave front set, it applies to the wave equation with smooth coefficients and a potential $q \in L^\infty(\Omega)$. Our proof of Proposition 1.2 relies on an improvement of this propagation property for potentials in $L^p(\Omega)$ with $p > \max\{d, 2\}$ (see Corollary 3.2).

On the other hand, the unique continuation property is quite intricate when potentials are involved. Precisely, the question is the following: does any finite energy solution of the system

$$\begin{cases} \Box y + qy = 0, & \text{in } (0,T) \times \Omega, \\ y = 0 & \text{on } (0,T) \times \partial\Omega, \\ y = 0 & \text{on } (0,T) \times \omega, \end{cases} \tag{1.22}$$

vanish identically ?

If the potential does not depend on time, we have the original proof of [4]: one can show that the time derivative operator acts on the set of trajectories on which the observation vanishes (the invisible solutions) and thus that this set has to be reduced to the trivial one (see Section 3). Therefore, under the GCC, if the potential does not depend on time, the wave equation is observable. However, if the potential does depend on time, this argument is no longer available and one needs an "additional" unique continuation property. Indeed, in that case, one can actually construct counterexamples to unique continuation for (1.22) for general hypersurfaces, see [1]. On the other hand, by [27, 29], unique continuation holds true if q is analytic with respect to the time variable t. Moreover, it has been remarked that when $q = q(t, x)$ depends on both time and space variables, observability can be proved under the Gamma-conditions (1.21) for $q \in L^\infty(0, T; L^d(\Omega))$, by using Carleman estimates (see [16, 31, 13, 11]). For instance, these conditions are satisfied when ω surrounds the whole boundary.

To summarize, in the context of the microlocal condition GCC, it is natural to restrict our study to the case of a time independent potential $q = q(x)$. But we keep in mind that all our results can be stated for potentials $q = q(t, x)$ depending on both time and space variables under the Gamma-condition (1.21).

On the data assimilation problem. Though the approach presented here is well-known in the context of data assimilation, it seems that the dependences of this strategy with respect to possible errors on the potentials have not received a thorough study so far, but several related works are worth mentioning.

In the context of inverse problems, numerous results are concerned with the problem of recovering a potential in the equation from the flux measured on the boundary or an interior domain, but with known initial data. We refer to [7] for uniqueness without stability results, and then [26, 30, 17, 6] for stability issues. However, these works [26, 30, 17, 6] require the proof of Carleman estimates. In particular, all these works need a stronger geometric condition, namely the Gamma-conditions (1.21) of Lions [22]. Let us also mention the work [5] which precisely studies these issues and proposes an algorithm to recover the potential based on these Carleman estimates.

In a slightly different context but with a more geometric point of view that does not require the Gamma-conditions (1.21), we also refer to the recent work [28] for uniqueness and stability issues.

On the control problem. The works concerning the controllability properties of semilinear wave equations can be seen as studies of the dependence of the control with respect to the potential and we refer to [33, 32, 10, 9] for several results in this direction. Here again, the results in [5] (see also [11] for related results) study the dependence of controls with respect to potentials but for the controls constructed using the Carleman weights. In particular, it is proved that the relative errors for the controls can be reduced arbitrarily by taking large parameter in the Carleman weights. This Carleman parameter can be seen as a deformation of the $L^2(L^2)$ space that gives more weight to the high-frequency part of the solutions, thus making the results in [5] perfectly compatible with the ones above, though the controls computed using HUM given by the functional $K[q]$ in (1.17) are not the same as the ones in [5].

Outline. In Section 2, we discuss several commutator estimates involving the spectral projections $\psi_k(-\Delta)$, $\eta_k(-\Delta)$ that will be needed afterwards. In Section 3, we prove Proposition 1.2. In Section 4, we focus on the applications to data assimilation and prove Theorem 1.3. In Section 5, we give the proof of Theorem 1.4 along the same lines of the one of Theorem 1.3. Finally, in Section 6, we provide the reader with some further comments.

2 Commutator estimates

In this section we present several commutator estimates that will be usefull in the sequel. We work in the spirit of [9] where they have been used to study thoroughly the HUM control operator for the wave equation. One can also quote the work [21] for a numerical illustration of the results in [9] and [12] for a generalization to abstract conservative linear systems. Finally, let us also mention that related commutator estimates have been obtained independently in the recent work [2].

Our main goal is to prove the following result:

Proposition 2.1. *For all $p \in (\max\{d,2\},\infty]$ and $s \in [0,2]$, there exists a constant $C > 0$ such that for all potentials $q \in W^{s,p}(\Omega)$ and $k \in \mathbb{N}^*$,*

$$\|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))} \leq C 2^{-ks/2} \|q\|_{W^{s,p}(\Omega)}, \tag{2.1}$$

$$\|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega), H^{-1}(\Omega))} \leq C 2^{-ks/2} \|q\|_{W^{s,p}(\Omega)}. \tag{2.2}$$

This type of commutation properties has been proved in [9] for smooth functions. Their proof is based on the so-called Helffer-Sjöstrand formula which allows to compute functions of an operator through resolvent estimates and basic estimates on $[-\Delta, q]$. Here, dealing first with a $W^{2,p}$ potential, we follow the same strategy and get precise estimates with respect to the potential. Then, using interpolation arguments, we deduce Proposition 2.1 and variants of it with less regular potentials.

We end this section with an estimate guaranteeing the regularizing property of commutators of the form $[(-\Delta)^{\alpha/2}, \chi]$ for smooth function χ and $\alpha < 0$, which will be used in Section 3.

2.1 Main commutator estimates

In this section, we consider a smooth (C^∞) compactly supported function $\rho : \mathbb{R}_+^* \to \mathbb{R}$ and for $R > 0$, we introduce the operator $\rho_R(-\Delta)$ defined for $f = \sum_j a_j e_j$ by

$$\rho_R(-\Delta)\left(\sum_j a_j e_j\right) = \sum_j a_j e_j \rho(\lambda_j^2/R)$$

and work directly on $\rho_R(-\Delta)$.

We then prove the following result:

Lemma 2.2. *Let $\rho : \mathbb{R}_+^* \to \mathbb{R}$ be a smooth C^∞ compactly supported function.*

For all $p \in (\max\{2,d\}, \infty]$, there exists a constant C such that for all $R > 0$ and $\chi \in W^{2,p}(\Omega)$,

$$\begin{aligned}
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(L^2(\Omega))} &\leq CR^{-1/2}\|\chi\|_{W^{2,p}(\Omega)}, &(2.3)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(H_0^1(\Omega))} &\leq CR^{-1/2}\|\chi\|_{W^{2,p}(\Omega)}, &(2.4)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(H^{-1}(\Omega))} &\leq CR^{-1/2}\|\chi\|_{W^{2,p}(\Omega)}, &(2.5)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))} &\leq CR^{-1}\|\chi\|_{W^{2,p}(\Omega)}, &(2.6)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(L^2(\Omega), H^{-1}(\Omega))} &\leq CR^{-1}\|\chi\|_{W^{2,p}(\Omega)}, &(2.7)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(L^2(\Omega), H_0^1(\Omega))} &\leq C\|\chi\|_{W^{2,p}(\Omega)}, &(2.8)\\
\|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(H^{-1}(\Omega), L^2(\Omega))} &\leq C\|\chi\|_{W^{2,p}(\Omega)}. &(2.9)
\end{aligned}$$

Before going into the proof of this result, let us mention that it is available in the work [9]. Our proof follows the same lines, but will make precise the assumptions needed on the function χ and, more precisely, how the constants depend on χ.

Proof. The first remark is the following one:

$$(z+\Delta)^{-1}\left(\sum_j a_j e_j\right) = \sum_j \frac{a_j}{z-\lambda_j^2} e_j. \tag{2.10}$$

Therefore, if z lies in a bounded subset of $\mathbb{C} \setminus \mathbb{R}_+$, then there exists a constant C_1 such that

$$\begin{aligned}
\left\|(zR+\Delta)^{-1}f\right\|_{L^2(\Omega)} &\leq \frac{1}{R|\Im(z)|}\|f\|_{L^2(\Omega)}, &(2.11)\\
\left\|(zR+\Delta)^{-1}f\right\|_{H_0^1(\Omega)} &\leq \frac{1}{R|\Im(z)|}\|f\|_{H_0^1(\Omega)}, &(2.12)\\
\left\|(zR+\Delta)^{-1}f\right\|_{H_0^1(\Omega)} &\leq C_1\frac{1}{\sqrt{R}|\Im(z)|}\|f\|_{L^2(\Omega)}. &(2.13)
\end{aligned}$$

since for all z in a given bounded subset of $\mathbb{C} \setminus \mathbb{R}_+$, there exists $C_1 > 0$ such that

$$\begin{aligned}
\sup_{\omega\in\mathbb{R}}\left\{\frac{1}{|zR-\omega^2|}\right\} &\leq \frac{1}{R|\Im(z)|},\\
\sup_{\omega\in\mathbb{R}}\left\{\frac{|\omega|}{|zR-\omega^2|}\right\} &\leq \frac{1}{\sqrt{R}|\Im(z)|}\sup_{\omega\in\mathbb{R}}\left\{\frac{|\omega\Im(z)|}{|z-\omega^2|}\right\} \leq \frac{C_1}{\sqrt{R}|\Im(z)|}.
\end{aligned}$$

Hence we introduce an almost analytic extension $\tilde{\rho} \in C_0^\infty(\mathbb{C})$ of ρ, which satisfies $\tilde{\rho}(x) = \rho(x)$ for all $x \in \mathbb{R}$ and $\bar{\partial}\tilde{\rho}(z) = \mathcal{O}(|\Im(z)|^\infty)$ close to the real axis. Such extensions have been introduced in a lecture seminar by Hörmander in 1968. For instance, one can take

$$\tilde{\rho}(x+iy) = \sum_{k\geq 0}\frac{\rho^{(k)}(x)}{k!}(iy)^k\gamma(\alpha_k y),$$

where γ is a smooth compactly supported function and the coefficients α_k are chosen going to infinity fast enough to guarantee the convergence of the formula.

Then the Helffer-Sjöstrand formula yields

$$\rho_R(-\Delta) = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\bar{\partial}\tilde{\rho}(z)}{z+\frac{\Delta}{R}}\,dz = -\frac{R}{\pi}\int_{\mathbb{C}}\frac{\bar{\partial}\tilde{\rho}(z)}{zR+\Delta}\,dz.$$

In particular,

$$[\chi, \rho_R(-\Delta)] = -\frac{R}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{\rho}(z)[\chi, (zR+\Delta)^{-1}]\, dz. \tag{2.14}$$

But

$$\begin{aligned} [\chi, (zR+\Delta)^{-1}] &= \chi(zR+\Delta)^{-1} - (zR+\Delta)^{-1}\chi \\ &= (zR+\Delta)^{-1}\left((zR+\Delta)\chi - \chi(zR+\Delta)\right)(zR+\Delta)^{-1} \\ &= (zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}. \end{aligned}$$

Thus, formula (2.14) writes:

$$[\chi, \rho_R(-\Delta)] = -\frac{R}{\pi} \int_{\mathbb{C}} \bar{\partial}\tilde{\rho}(z)(zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}\, dz. \tag{2.15}$$

Of course, $[\Delta, \chi]$ can be computed explicitly:

$$[\Delta,\chi]f = \Delta(\chi f) - \chi\Delta f = 2\nabla\chi\cdot\nabla f + (\Delta\chi)f. \tag{2.16}$$

Hence, for $\chi \in W^{2,p}(\Omega)$ with $p > \max\{d, 2\}$,

$$\|[\Delta,\chi]\|_{\mathfrak{L}(H_0^1, L^2)} \le C\,\|\chi\|_{W^{2,p}}\,.$$

Therefore, using (2.11)–(2.13),

$$\begin{aligned} &\left\|(zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2)} \\ &\le \quad \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2)} \|[\Delta,\chi]\|_{\mathfrak{L}(H_0^1,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2,H_0^1)} \\ &\le \quad \frac{1}{R|\Im(z)|} C\,\|\chi\|_{W^{2,p}}\, \frac{1}{\sqrt{R}|\Im(z)|}. \end{aligned}$$

Using that $\bar{\partial}\tilde{\rho}(z) = \mathcal{O}(|\Im(z)|^\infty)$ and that the integral is on a compact set of $\mathbb{C}$, we thus obtain (2.3).

Similarly, estimate (2.4) is deduced from

$$\begin{aligned} &\left\|(zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}\right\|_{\mathfrak{L}(H_0^1)} \\ &\qquad \le \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2,H_0^1)} \|[\Delta,\chi]\|_{\mathfrak{L}(H_0^1,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(H_0^1)} \end{aligned}$$

and (2.11)–(2.13), whereas (2.5) follows from (2.4) by duality.

Estimate (2.6) is a consequence of the estimate

$$\begin{aligned} &\left\|(zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}\right\|_{\mathfrak{L}(H_0^1,L^2)} \\ &\qquad \le \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2)} \|[\Delta,\chi]\|_{\mathfrak{L}(H_0^1,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(H_0^1)} \end{aligned}$$

and (2.11)–(2.13). Again, estimate (2.7) follows from (2.6) by duality.

Finally, estimate (2.8) is a consequence of

$$\begin{aligned} &\left\|(zR+\Delta)^{-1}[\Delta,\chi](zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2,H_0^1)} \\ &\qquad \le \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2;H_0^1)} \|[\Delta,\chi]\|_{\mathfrak{L}(H_0^1,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(L^2,H_0^1)} \end{aligned}$$

and (2.11)–(2.13), and (2.9) follows by duality. □

Remark 2.3. *In this remark, we would like to point out that estimates* (2.3)*,* (2.8) *and* (2.9) *also hold true by replacing* $\|\chi\|_{W^{2,p}}$ *by* $\|\chi\|_{W^{1,\infty}}$*. Indeed, rewriting the formula* (2.16) *as*

$$[\Delta, \chi] f = \Delta(\chi f) - \chi \Delta f = div(f \nabla \chi) + \nabla \chi \cdot \nabla f := (div \circ A_1 + A_2) f, \\ \text{with } A_1 f = f \nabla \chi, \text{ and } A_2 f = \nabla \chi \cdot \nabla f,$$

for which we have

$$\|A_1\|_{\mathfrak{L}(L^2, L^2)} + \|A_2\|_{\mathfrak{L}(H_0^1, L^2)} \leq C \|\chi\|_{W^{1,\infty}}.$$

Then, if X and Y denote Hilbert spaces, we may write

$$\begin{aligned} &\|[\chi, \rho_R(-\Delta)]\|_{\mathfrak{L}(X,Y)} \\ \leq\ & CR \int_{\mathbb{C}} |\bar{\partial}\tilde{\rho}(z)| \left\|(zR + \Delta)^{-1} div\right\|_{\mathfrak{L}(L^2,Y)} \|A_1\|_{\mathfrak{L}(L^2,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(X,L^2)} dz \\ & + CR \int_{\mathbb{C}} |\bar{\partial}\tilde{\rho}(z)| \left\|(zR + \Delta)^{-1}\right\|_{\mathfrak{L}(L^2,Y)} \|A_2\|_{\mathfrak{L}(H_0^1,L^2)} \left\|(zR+\Delta)^{-1}\right\|_{\mathfrak{L}(X,H_0^1)} dz. \end{aligned}$$

Thus, using that

$$\left\|(zR + \Delta)^{-1} div\right\|_{\mathfrak{L}(L^2,Y)} = \left\|(zR + \Delta)^{-1}\right\|_{\mathfrak{L}(Y',H_0^1)},$$

the resolvent estimates (2.11)–(2.13) *and the additional estimate*

$$\left\|(zR + \Delta)^{-1} f\right\|_{H_0^1} \leq \frac{C}{|\Im(z)|} \|f\|_{H^{-1}(\Omega)}, \tag{2.17}$$

which can be proved similarly, one can improve estimates (2.3)*,* (2.8) *and* (2.9) *into*

$$\begin{aligned} \|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(L^2(\Omega))} &\leq CR^{-1/2} \|\chi\|_{W^{1,\infty}(\Omega)}, & (2.18) \\ \|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(L^2(\Omega), H_0^1(\Omega))} &\leq C \|\chi\|_{W^{1,\infty}(\Omega)}, & (2.19) \\ \|[\rho_R(-\Delta), \chi]\|_{\mathfrak{L}(H^{-1}(\Omega), L^2(\Omega))} &\leq C \|\chi\|_{W^{1,\infty}(\Omega)}. & (2.20) \end{aligned}$$

Note however that this trick does not improve the other estimates in Lemma 2.2, and in particular not (2.6)–(2.7)*, which are the critical ones for the proof of Proposition 2.1 and for the rest of the article.*

We would like to kindly acknowledge the anonymous referee for this remark.

2.2 Proof of Proposition 2.1 for $W^{2,p}$ potentials

In this paragraph, we prove the following result:

Lemma 2.4. *For all $p \in (\max\{d, 2\}, \infty]$, there exists a constant $C > 0$ such that for all potentials $q \in W^{2,p}(\Omega)$ and $k \in \mathbb{N}^*$,*

$$\begin{aligned} \|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega))} &\leq C 2^{-k/2} \|q\|_{W^{2,p}(\Omega)}, & (2.21) \\ \|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))} &\leq C 2^{-k} \|q\|_{W^{2,p}(\Omega)}, & (2.22) \\ \|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega), H^{-1}(\Omega))} &\leq C 2^{-k} \|q\|_{W^{2,p}(\Omega)}. & (2.23) \end{aligned}$$

Estimates (2.22)–(2.23) correspond to the estimates (2.1)–(2.2) in Proposition 2.1 for $s = 2$, and thus Lemma 2.4 should be seen as a first step toward Proposition 2.1.

Proof. Taking $R = 2^k$ and $\rho = \psi$, we get $\psi_k = \rho_R$, thus yielding the following estimates as an immediate consequence of Lemma 2.2: For all $p \in (\max\{2, d\}, \infty]$, there exists a constant C such that for all $k \in \mathbb{N}^*$ and $q \in W^{2,p}(\Omega)$,

$$\|[\psi_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega))} \leq C2^{-k/2} \|q\|_{W^{2,p}(\Omega)}, \tag{2.24}$$

$$\|[\psi_k(-\Delta), q]\|_{\mathfrak{L}(H_0^1(\Omega),L^2(\Omega))} \leq C2^{-k} \|q\|_{W^{2,p}(\Omega)}, \tag{2.25}$$

$$\|[\psi_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega),H^{-1}(\Omega))} \leq C2^{-k} \|q\|_{W^{2,p}(\Omega)}. \tag{2.26}$$

Consequently, for $q \in W^{2,p}(\Omega)$, estimate (2.21) can then be deduced simply by summing estimates (2.24) for ψ_ℓ for $\ell \geq k$. Similarly, the proof of estimates (2.22)–(2.23) can be deduced from the corresponding estimates in (2.25)–(2.26) by a summation argument. □

Remark 2.5. *Following Remark 2.3, estimate* (2.21) *can be improved into*

$$\|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega))} \leq C2^{-k/2} \|q\|_{W^{1,\infty}(\Omega)}.$$

2.3 The case of less regular potentials

We now focus on the proof of Proposition 2.1, which, as we will see, follows from Lemma 2.4 and suitable interpolation arguments.

Proof of Proposition 2.1. Note that we obviously have for some constant C independent of k, ($C = 2\|\psi\|_\infty$),

$$\|\eta_k(-\Delta)\|_{\mathfrak{L}(H_0^1(\Omega))} \leq C, \qquad \|\eta_k(-\Delta)\|_{\mathfrak{L}(L^2(\Omega))} \leq C.$$

Hence one easily has, for all $q \in L^p(\Omega)$ with $p > \max\{d, 2\}$ and $z \in H_0^1(\Omega)$,

$$\|\eta_k(-\Delta)(qz)\|_{L^2(\Omega)} \leq C\|qz\|_{L^2(\Omega)} \leq C\|q\|_{L^p(\Omega)} \|z\|_{H_0^1(\Omega)},$$

and

$$\|q\eta_k(-\Delta)(z)\|_{L^2(\Omega)} \leq C\|q\|_{L^p(\Omega)} \|\eta_k(-\Delta)z\|_{H_0^1(\Omega)} \leq C\|q\|_{L^p(\Omega)} \|z\|_{H_0^1(\Omega)}.$$

Thus, for $q \in L^p(\Omega)$ with $p > \max\{d, 2\}$,

$$\|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(H_0^1(\Omega),L^2(\Omega))} \leq C\|q\|_{L^p(\Omega)}. \tag{2.27}$$

Since the map

$$\mathscr{C}_k : q \mapsto [\eta_k(-\Delta), q]$$

is a linear map such that

- $\mathscr{C}_k$ is continuous on L^p for $p \in (\max\{d, 2\}, \infty]$ with values in $\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))$ and of norm $\leq C$, see (2.27);
- $\mathscr{C}_k$ is continuous on $W^{2,p}$ for $p \in (\max\{d, 2\}, \infty]$ with values in $\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))$ and of norm $\leq C2^{-k}$, see (2.22),

by interpolation, for all $s \in [0, 2]$, $\mathscr{C}_k$ is continuous on $W^{s,p}(\Omega)$ for $p > \max\{d, 2\}$ with values in $\mathfrak{L}(H_0^1(\Omega), L^2(\Omega))$ and of norm $\leq C2^{-ks/2}$.

We thus conclude (2.1) for all $p \in (\max\{d, 2\}, \infty]$, $s \in [0, 2]$, and potentials $q \in W^{s,p}(\Omega)$, and, by duality, (2.2). □

Remark 2.6. *In a similar way, using Remark 2.5 and spaces of potentials for which the multiplication operator acts on $L^2(\Omega)$, namely $L^\infty(\Omega)$, one can get: for all $s \in [0, 1]$ and $q \in W^{s,\infty}(\Omega)$,*

$$\|[\eta_k(-\Delta), q]\|_{\mathfrak{L}(L^2(\Omega))} \leq C2^{-ks/2} \|q\|_{W^{s,\infty}(\Omega)}. \tag{2.28}$$

Indeed, this result follows by interpolating $W^{1,\infty}(\Omega)$ with $L^\infty(\Omega)$.

2.4 A regularizing estimate

For later use, we also prove the following lemma:

Lemma 2.7. *Let $\alpha < 0$, and χ be a smooth function. Then there exists a constant $C > 0$ such that*

$$\forall f \in L^2(\Omega), \quad \left\| [(-\Delta)^{\alpha/2}, \chi] f \right\|_{H_0^1(\Omega)} \leq C \left\| f \right\|_{L^2(\Omega)} . \tag{2.29}$$

Proof. Writing $(-\Delta)^{\alpha/2}$ on its spectral basis and using (1.8), we have, for all sequence $(a_j)_{j \in \mathbb{N}^*}$ with a finite number of non-vanishing components,

$$\begin{aligned} (-\Delta)^{\alpha/2} \left(\sum a_j e_j \right) &= \sum_j \lambda_j^\alpha a_j e_j \\ &= \sum_j \lambda_j^\alpha \left(\theta(\lambda_j^2) + \sum_{k=1}^{\infty} \psi(\lambda_j^2 2^{-k}) \right) a_j e_j \\ &= \sum_j \lambda_j^\alpha \theta(\lambda_j^2) a_j e_j + \sum_{k=1}^{\infty} 2^{\alpha k/2} \sum_j \left(\lambda_j^2 2^{-k} \right)^{\alpha/2} \psi(\lambda_j^2 2^{-k}) a_j e_j . \end{aligned}$$

Hence, setting

$$\begin{aligned} A_{\alpha,0} \left(\sum_j a_j e_j \right) &= \sum_j \lambda_j^\alpha \theta(\lambda_j^2) a_j e_j , \\ A_{\alpha,k} \left(\sum_j a_j e_j \right) &= \sum_j \left(\lambda_j^2 2^{-k} \right)^{\alpha/2} \psi(\lambda_j^2 2^{-k}) a_j e_j , \qquad k \geq 1, \end{aligned}$$

we have, for all sequence $(a_j)_{j \in \mathbb{N}^*}$ with a finite number of non-vanishing components,

$$(-\Delta)^{\alpha/2} \left(\sum_j a_j e_j \right) = \sum_{k=0}^{\infty} 2^{\alpha k/2} A_{\alpha,k} \left(\sum_j a_j e_j \right) .$$

But, using that θ is compactly supported in $(0,1)$, for all $(a_j) \in \ell^2(\mathbb{N}^*)$,

$$\left\| [A_{\alpha,0}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H_0^1(\Omega)} \leq \left\| \left(\sum_j a_j e_j \right) \right\|_{L^2(\Omega)} .$$

For $k \geq 1$, one can use Lemma 2.2 with $\rho(\xi) = \psi(\xi)\xi^{\alpha/2}$, $R = 2^{-k}$ and $\chi = \chi$, and in particular estimate (2.8): there exists a constant $C > 0$ such that for all $(a_j) \in \ell^2(\mathbb{N}^*)$ and all $k \geq 1$,

$$\left\| [A_{\alpha,k}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H_0^1(\Omega)} \leq C \left\| \left(\sum_j a_j e_j \right) \right\|_{L^2(\Omega)} . \tag{2.30}$$

Combining the above estimates, for all sequence $(a_j)_{j \in \mathbb{N}^*}$ with a finite number of non-vanishing components,

$$\begin{aligned} \left\| [(-\Delta)^{\alpha/2}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H_0^1(\Omega)} &\leq \sum_{k=0}^{\infty} 2^{\alpha k/2} \left\| [A_{\alpha,k}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H_0^1(\Omega)} \\ &\leq C \left(\sum_{k=0} 2^{\alpha k/2} \right) \left\| \sum_j a_j e_j \right\|_{L^2(\Omega)} . \end{aligned} \tag{2.31}$$

We now conclude Lemma 2.7 by using the density in $\ell^2(\mathbb{N}^*)$ of sequences with a finite number of non-vanishing components. $\square$

Remark 2.8. *Note that, due to Remark 2.3, the above proof holds for $\chi \in W^{1,\infty}(\Omega)$.*

Remark 2.9. *Actually, we could have proved a slightly better estimate than the one in* (2.29). *Indeed, for all $\beta \in [0, 1-\alpha)$, there exists a constant $C > 0$ such that*

$$\forall f \in L^2(\Omega), \quad \left\| [(-\Delta)^{\alpha/2}, \chi] f \right\|_{H^\beta_{(0)}(\Omega)} \leq C \, \|f\|_{L^2(\Omega)} , \tag{2.32}$$

where $H^\beta_{(0)}(\Omega) = \mathcal{D}((-\Delta)^{\beta/2})$.

Indeed, similarly as in (2.17)*, one easily checks that*

$$\left\| (zR + \Delta)^{-1} f \right\|_{H^2_{(0)}} \leq \frac{C}{|\Im(z)|} \, \|f\|_{L^2(\Omega)} ,$$

which yields, within the setting of Lemma 2.2,

$$\| [\rho_R(-\Delta), \chi] \|_{\mathfrak{L}(H^2_{(0)}(\Omega), L^2(\Omega))} \leq C R^{1/2} \, \|\chi\|_{W^{2,p}(\Omega)} .$$

Hence one can interpolate between this estimate and (2.3) *to obtain*

$$\| [\rho_R(-\Delta), \chi] \|_{\mathfrak{L}(H^\beta_{(0)}(\Omega), L^2(\Omega))} \leq C R^{(\beta-1)/2} \, \|\chi\|_{W^{2,p}(\Omega)} .$$

Thus, we can replace estimate (2.30) *by*

$$\left\| [A_{\alpha,k}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H^\beta_{(0)}(\Omega)} \leq C 2^{(\beta-1)k/2} \left\| \left(\sum_j a_j e_j \right) \right\|_{L^2(\Omega)} .$$

Summing up these estimates for $k \geq 1$ and $k = 0$ (the case $k = 0$ being once again completely straightforward), we obtain, for all sequence $(a_j)_{j \in \mathbb{N}^*}$ with a finite number of non-vanishing components,

$$\left\| [(-\Delta)^{\alpha/2}, \chi] \left(\sum_j a_j e_j \right) \right\|_{H^\beta_{(0)}(\Omega)} \leq C \left(\sum_{k=0}^{\infty} 2^{(\alpha+\beta-1)k/2} \right) \left\| \sum_j a_j e_j \right\|_{L^2(\Omega)} ,$$

instead of (2.31)*, thus yielding* (2.32) *by density under the condition $\alpha + \beta - 1 < 0$.*

3 On the observability inequality with a potential

The goal of this section is to prove that the observability constant in (1.3) may be chosen uniformly for potentials in bounded sets of $L^p(\Omega)$ when $p \in (\max\{d, 2\}, \infty]$. Here, we only assume the control set ω to satisfy the geometric control condition (GCC, recall Definition 1.1) of C. Bardos, G. Lebeau and J. Rauch [3, 4]. The proof we present below is based on microlocal defect measures and their propagation and localization properties.

3.1 GCC and classical propagation properties

Let us recall that the geometric control condition implies the following propagation results:

- Propagation of microlocal defect measures, see [20]:

$$\left.\begin{array}{l}\varphi_n \underset{n\to\infty}{\rightharpoonup} 0 \text{ weakly in } L^2((0,T)\times\Omega)\\ \Box\varphi_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } H^{-1}((0,T)\times\Omega)\\ \varphi_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2((0,T)\times\omega)\end{array}\right\} \Rightarrow \varphi_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } L^2((0,T)\times\Omega). \tag{3.1}$$

- Propagation of regularity, see [15, Chap. XXIV] and [24, 25]:

$$\left.\begin{array}{l}\varphi \in L^2((0,T)\times\Omega)\\ \Box\varphi \in L^2((0,T)\times\Omega)\\ \varphi \in H^1_{loc}((0,T)\times\omega)\end{array}\right\} \Rightarrow \varphi \in L^2((0,T);H^1_0(\Omega))\cap H^1((0,T);L^2(\Omega)). \tag{3.2}$$

Let us be slightly more precise about the second condition in (3.1) and (3.2). In (3.1), we use the notation $\Box\varphi_n \underset{n\to\infty}{\longrightarrow} 0$ in $H^{-1}((0,T)\times\Omega)$ to say that each φ_n solves

$$\left\{\begin{array}{ll}\Box\varphi_n = f_n & \text{in } (0,T)\times\Omega,\\ \varphi_n = 0 & \text{on } (0,T)\times\partial\Omega,\end{array}\right.$$

where f_n is a sequence of functions of $H^{-1}((0,T)\times\Omega)$ which strongly converges to 0 in $H^{-1}((0,T)\times\Omega)$. Similarly, the second condition in (3.2) has to be understood as follows: φ is a weak solution of

$$\left\{\begin{array}{ll}\Box\varphi = f & \text{in } (0,T)\times\Omega,\\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega,\end{array}\right. \tag{3.3}$$

with $f \in L^2((0,T)\times\Omega)$.

Finally, let us notice that (3.2) states the propagation of the H^1 regularity from the observability set $(0,T)\times\omega$ to the whole space-time cylinder $(0,T)\times\Omega$. This result is a byproduct of the proof of the propagation of the wave front set in [24, 25], (or [15, Chap. XXIV]).

3.2 Propagation of regularity with potentials

We prove the following result, which slightly generalizes the propagation of regularity (3.2) to the case of more integrable source term.

Theorem 3.1. *Assume the geometric control condition GCC and $d \geq 3$.*

Let $r \in \left[\frac{2d}{d+2}, 2\right]$ and assume that $\varphi \in L^2((0,T)\times\Omega)$ is a weak solution of (3.3) with $f \in L^2((0,T);L^r(\Omega))$ such that $\varphi \in H^1((0,T)\times\omega)$. Then $\varphi \in L^2((0,T);L^s(\Omega))$ with

$$\frac{1}{s} = \frac{1}{r} - \frac{1}{d}.$$

Proof. By Sobolev's embedding, $L^r(\Omega)$ embeds into $H^{-k}(\Omega)$ with $k = d/r - d/2 \in [0,1]$. We thus have

$$\left\{\begin{array}{l}\varphi \in L^2((0,T)\times\Omega),\\ \Box\varphi \in L^2((0,T);H^{-k}(\Omega)),\\ \varphi \in H^1((0,T)\times\omega).\end{array}\right.$$

In order to apply (3.2), we set $v = (-\Delta)^{-k/2}\varphi$, for which we easily have

$$v \in L^2((0,T)\times\Omega), \quad \text{and} \quad \Box v \in L^2((0,T)\times\Omega).$$

We now prove that $v \in H^1_{loc}((0,T)\times\omega)$. Indeed, if $a = a(x)$ is a smooth function supported in ω, we can write

$$av = (-\Delta)^{-k/2}(a\varphi) + [a, (-\Delta)^{-k/2}]\varphi,$$

so that Lemma 2.7 implies $av \in L^2((0,T); H^1(\omega))$. Since this is true for all smooth function $a = a(x)$ supported in ω, $v \in L^2((0,T); H^1_{loc}(\omega))$.

Then, for any smooth function $\tilde{a} = \tilde{a}(x)$ supported in ω, we have

$$\Box(\tilde{a}v) = [-\Delta, \tilde{a}]v + \tilde{a}\Box v \in L^2((0,T)\times\Omega),$$

and thus multiplying by $\eta^2\tilde{a}v$ with $\eta = \eta(t)$ a smooth cut-off function of time supported in $(0,T)$, we immediately obtain $\eta\partial_t(\tilde{a}v) \in L^2((0,T)\times\Omega)$. Since this is true for all smooth function $\tilde{a} = \tilde{a}(x)$ supported in ω and $\eta = \eta(t)$ supported in $(0,T)$, we proved $v \in H^1_{loc}((0,T)\times\omega)$.

Hence, the propagation of regularity (3.2) implies that $v \in H^1((0,T)\times\Omega)$ and thus φ belongs to $L^2((0,T); H^{1-k}(\Omega))$. Using Sobolev's embedding, φ belongs to $L^2((0,T); L^s(\Omega))$ with

$$\frac{1}{s} = \frac{1}{2} - \frac{1-k}{d} = \frac{1}{2} - \frac{1}{d} + \frac{k}{d} = \frac{1}{r} - \frac{1}{d}$$

as announced. □

An important corollary of Theorem 3.1 is the following propagation result:

Corollary 3.2. *Assume the geometric control condition GCC and $d \geq 3$. Let $p \in (d,\infty]$ and $q \in L^p(\Omega)$. Then every function $\varphi \in L^2((0,T)\times\Omega)$ solution of system*

$$\begin{cases} \Box\varphi + q\varphi \in L^2((0,T)\times\Omega) \\ \varphi = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \varphi \in H^1_{loc}((0,T)\times\omega), \end{cases} \tag{3.4}$$

satisfies

$$\varphi \in L^2((0,T); H^1_0(\Omega)) \cap H^1((0,T); L^2(\Omega)). \tag{3.5}$$

Proof. We use the following bootstrap argument:

$$\left.\begin{array}{l} \Box\varphi + q\varphi \in L^2((0,T)\times\Omega) \\ \varphi \in L^2((0,T); L^{s_0}(\Omega)) \\ \text{ with } s_0 \geq 2 \\ \text{ and such that } \dfrac{1}{p} + \dfrac{1}{s_0} > \dfrac{1}{2} \end{array}\right\} \Rightarrow \varphi \in L^2((0,T); L^{s_1}(\Omega)) \text{ with } \frac{1}{s_1} = \frac{1}{s_0} + \frac{1}{p} - \frac{1}{d}. \tag{3.6}$$

Indeed, if $\varphi \in L^2((0,T); L^{s_0}(\Omega))$, $\Box\varphi = (\Box\varphi + q\varphi) - q\varphi$ belongs to $L^2((0,T); L^r(\Omega))$ with $1/r = 1/p + 1/s_0 \in [1/2, 1/2 + 1/d]$. Thus Theorem 3.1 applies and yields $\varphi \in L^2((0,T); L^{s_1}(\Omega))$ with $1/s_1 = 1/s_0 + 1/p - 1/d$.

Bootstrapping the property (3.6) implies that all φ satisfying (3.5) actually belong to $L^2((0,T); L^s(\Omega))$ for some $s \geq 2$ such that $1/p + 1/s \leq 1/2$, and thus $\Box\varphi = (\Box\varphi + q\varphi) - q\varphi$ belongs to $L^2((0,T)\times\Omega)$.

Using the propagation of regularity (3.2), all φ satisfying (3.4) belong to $L^2((0,T); H^1_0(\Omega)) \cap H^1((0,T); L^2(\Omega))$. □

Remark 3.3. *In the case $d = 2$, Theorem 3.1 holds for $r \in (1,2)$ and the proof follows the same lines: We only have to check that Sobolev's embeddings hold and thus we should avoid the cases $k = 0$ and $k = 1$ corresponding to $r = 1$ and $r = 2$. Consequently, Corollary 3.2 also applies in dimension $d = 2$ for $q \in L^p(\Omega)$ with $p > 2$.*

In the case $d = 1$, one can follow the proof of Theorem 3.1 to get, for $\varphi \in L^2((0,T)\times\Omega)$ satisfying $\varphi \in H^1((0,T)\times\omega)$:

- *If $\Box\varphi \in L^2((0,T); L^1(\Omega))$, $\varphi \in L^2((0,T); L^s(\Omega))$ for all $s < \infty$.*
- *If $\Box\varphi \in L^2((0,T); L^r(\Omega))$ for some $r \in (1,2)$, $\varphi \in L^2((0,T); L^\infty(\Omega))$.*

Following, Corollary 3.2 also applies for $d = 1$ and $q \in L^2(\Omega)$.

3.3 Proof of Proposition 1.2

Proof of Proposition 1.2. The proof is divided in several parts. To simplify the presentation, we only deal with the case $d \geq 3$. We omit the proof corresponding to the cases $d = 1$ or 2 which can be done similarly using Remark 3.3 and are left to the reader.

Step 1: A weak observability inequality. The first step of the proof consists of showing an "almost observability" inequality. More precisely, we are going to show that there exists a constant C_m depending only on $m > 0$ such that for all $q \in L^p(\Omega)$ with $L^p(\Omega)$-norm bounded by m, any solution $\varphi[q]$ of (1.2) satisfies:

$$\|(\varphi_0, \varphi_1)\|^2_{L^2 \times H^{-1}} \leq C_m \int_0^T \int_\omega |\varphi[q]|^2 \, dxdt + C_m \|\varphi[q]\|^2_{L^2(H^{-1})} . \tag{3.7}$$

We prove this result by contradiction. Assume (3.7) is false. Then there exist a sequence q_n of $L^p(\Omega)$-potentials bounded by m and a sequence of initial data $(\varphi_{0n}, \varphi_{1n}) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that:

$$\|(\varphi_{0n}, \varphi_{1n})\|_{L^2 \times H^{-1}} = 1, \quad \|q_n\|_{L^p} \leq m, \tag{3.8}$$

$$\lim_{n \to \infty} \int_0^T \int_\omega |\varphi_n[q_n]|^2 \, dxdt = 0, \tag{3.9}$$

$$\lim_{n \to \infty} \|\varphi_n[q_n]\|_{L^2(H^{-1})} = 0, \tag{3.10}$$

where $\varphi_n[q_n]$ is the solution of the wave equation (1.2) with initial data $(\varphi_{0n}, \varphi_{1n})$ and potential q_n.

Of course, the uniform bounds (3.8) imply that the solutions $\varphi_n[q_n]$ are uniformly bounded in $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$, therefore, up to a subsequence still denoted the same, they weakly converge to some Φ in $L^2((0,T); L^2(\Omega)) \cap H^1((0,T); H^{-1}(\Omega))$. Using (3.10), we necessarily have $\Phi \equiv 0$.

We then have to identify the limit of $q_n \varphi_n[q_n]$. First, the sequence $q_n\varphi_n[q_n]$ is bounded in $L^2((0,T); L^r(\Omega))$ with $1/r = 1/2 + 1/p$, hence it weakly converges in $L^2((0,T); L^r(\Omega))$ (again, up to a subsequence). Let us then show that it actually converges to 0 in $\mathcal{D}'((0,T)\times\Omega)$. In order to do that, we remark that $\varphi_n[q_n] \rightharpoonup 0$ weakly in $L^2((0,T) \times \Omega)$, hence $\Box\varphi_n[q_n] = -q_n\varphi_n[q_n]$ converges to 0 in $\mathcal{D}'((0,T) \times \Omega)$. But, by Aubin-Lions' theorem, $H_0^1((0,T) \times \Omega)$ compactly embeds into $L^2((0,T); L^{r'}(\Omega))$ since

$$\frac{1}{r'} = 1 - \frac{1}{r} = \frac{1}{2} - \frac{1}{p} > \frac{1}{2} - \frac{1}{d},$$

and thus, by duality, $L^2((0,T); L^r(\Omega))$ compactly embeds into $H^{-1}((0,T) \times \Omega)$. We therefore obtain that $q_n\varphi_n[q_n]$ strongly converges to 0 in $H^{-1}((0,T) \times \Omega)$.

We can then use (3.1) and (3.9), which guarantee that $\varphi_n[q_n]$ strongly converges to 0 in $L^2((0,T) \times \Omega)$, from which we immediately deduce the convergence of $\partial_t\varphi_n[q_n]$ to 0 in $L^2((0,T); H^{-1}(\Omega))$ (for instance multiplying the equation of $\varphi_n[q_n]$ by $(-\Delta)^{-1}\varphi_n[q_n]\eta$, where $\eta = \eta(t)$ is a smooth cut-off function in time). Using the energy estimates in $L^2(\Omega) \times H^{-1}(\Omega)$, we readily obtain the strong convergence of $(\varphi_{0n}, \varphi_{1n})$ to $(0,0)$ in $L^2(\Omega) \times H^{-1}(\Omega)$. This is in contradiction with (3.8).

Step 2: A unique continuation result. Based on (3.7), we will prove the following fact: for all q in $L^p(\Omega)$, the set $X[q]$ defined by

$$X[q] = \{\varphi[q] \in L^2((0,T)\times\Omega) \text{ solution of (1.2) such that } \varphi[q] = 0 \text{ on } (0,T)\times\omega\} \tag{3.11}$$

is reduced to the singleton $\{0\}$.

Step 2.a. ∂_t acts on $X[q]$. As a consequence of Corollary 3.2, if $\varphi[q] \in X[q]$, it automatically belongs to $H^1((0,T)\times\Omega)$ and thus $\partial_t\varphi[q]$ also belongs to $X[q]$.

Step 2.b: $X[q]$ is trivial. If we endow $X[q]$ with the $L^2((0,T)\times\Omega)$-topology, the balls of $X[q]$ are compact. Indeed, take $\varphi_n[q] \in X[q]$ of bounded $L^2((0,T)\times\Omega)$-norms. Then applying (3.7) to $\partial_t\varphi_n[q]$ -which belongs to $L^2((0,T)\times\Omega)$ from Step 2.a- one obtains that $\varphi_n[q]$ is bounded in the space $H^1((0,T);L^2(\Omega)) \cap H^2((0,T);H^{-1}(\Omega))$, hence in $L^2((0,T);H_0^1(\Omega))$ from the equation (1.2). But the space $L^2((0,T);H_0^1(\Omega)) \cap H^1((0,T);L^2(\Omega))$ compactly embeds into $L^2((0,T)\times\Omega)$. Hence the balls of $X[q]$ are compact and thus $X[q]$ is a finite-dimensional space.

It follows that, if $X[q]$ were nontrivial, there would exist an eigenvector of ∂_t on $X[q]$, i.e. a non-trivial solution $\varphi = \varphi[q] \in L^2((0,T)\times\Omega) \cap H^1((0,T);H^{-1}(\Omega))$ of (1.2) such that $\partial_t\varphi = \lambda\varphi$ and $\varphi = 0$ on $(0,T)\times\omega$. The unique continuation property of the Laplace operator (see [18]) implies that such non-trivial φ cannot exist.

Hence, necessarily,

$$X[q] = \{0\}. \tag{3.12}$$

Step 3: A compactness argument. We now argue by contradiction to show (1.5). Let q_n be a sequence of potentials in $L^p(\Omega)$ and $(\varphi_{0n}, \varphi_{1n}) \in L^2(\Omega)\times H^{-1}(\Omega)$ such that

$$\|q_n\|_{L^p} \le m, \qquad \|(\varphi_{0n}, \varphi_{1n})\|_{L^2\times H^{-1}} = 1, \qquad \text{and } \lim_{n\to\infty} \|\varphi_n[q_n]\|_{L^2((0,T)\times\omega)} = 0. \tag{3.13}$$

where $\varphi_n[q_n]$ is the solution of (1.2) with initial data $(\varphi_{0n}, \varphi_{1n})$ and potential q_n.

From the above estimates, we have the existence of $(\Phi_0, \Phi_1) \in L^2(\Omega)\times H^{-1}(\Omega)$, of $Q \in L^p(\Omega)$ and of $f \in L^2((0,T);L^{2p/(p+2)}(\Omega))$ such that, extracting subsequences if necessary,

$$(\varphi_{0n}, \varphi_{1n}) \underset{n\to\infty}{\rightharpoonup} (\Phi_0, \Phi_1) \text{ weakly in } L^2(\Omega)\times H^{-1}(\Omega), \tag{3.14}$$

$$q_n \underset{n\to\infty}{\rightharpoonup} Q \text{ weakly in } L^2(\Omega), \qquad Q \in L^p(\Omega) \text{ and } \|Q\|_{L^p} \le m, \tag{3.15}$$

$$q_n\varphi_n[q_n] \underset{n\to\infty}{\rightharpoonup} f \text{ weakly in } L^2((0,T);L^{2p/(p+2)}(\Omega)). \tag{3.16}$$

$$q_n\varphi_n[q_n] \underset{n\to\infty}{\rightarrow} f \text{ strongly in } H^{-1}((0,T)\times\Omega), \tag{3.17}$$

where the strong convergence in (3.17) is deduced by duality from the compactness of the embedding of $H_0^1((0,T)\times\Omega)$ into $L^2((0,T);L^{2p/(p-2)}(\Omega))$ (Aubin-Lions' compactness theorem).

It is then an easy matter to show that, introducing Φ the solution of

$$\begin{cases} \square\Phi = -f & \text{in } (0,T)\times\Omega, \\ \Phi = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\Phi(0,\cdot), \partial_t\Phi(0,\cdot)) = (\Phi_0, \Phi_1), & \end{cases} \tag{3.18}$$

we have

$$\varphi_n[q_n] \underset{n\to\infty}{\rightharpoonup} \Phi \text{ weakly in } L^2((0,T)\times\Omega) \cap H^1((0,T)\times H^{-1}(\Omega)) \tag{3.19}$$

$$\Phi = 0 \text{ on } (0,T)\times\omega. \tag{3.20}$$

The main difficulty now is to check that Φ actually vanishes everywhere. In order to prove it, we prove $f = Q\Phi$, which, by (3.19)–(3.20), would imply $\Phi \in X[Q]$, hence $\Phi \equiv 0$ from (3.12).

We thus want to identify f. This cannot be done directly since the convergences of q_n and $\varphi_n[q_n]$ are only weak convergences in $L^2((0,T)\times\Omega)$. But we can argue as in [14, Lemma 3.1] and use the fact that the micro-local defect measures corresponding to $\varphi_n[q_n]-\Phi$ and $q_n - Q$ are supported on disjoint subsets. We prove it below for completeness.

We are going to identify f in the sense of $\mathcal{D}'((0,T)\times\Omega)$. Let ψ and $\tilde\psi$ be two smooth compactly supported functions in $(0,T)\times\Omega$.

Using (3.17) and (3.18), $\Box(\varphi_n[q_n]-\Phi)$ strongly converges to 0 in $H^{-1}((0,T)\times\Omega)$. Combined with the weak convergence (3.19), the $L^2((0,T)\times\Omega)$-micro-local defect measure corresponding to $\psi(\varphi_n[q_n]-\Phi)$ is supported on the bicharacteristic set $\mathcal{C}_\Box = \{(t,x,\tau,\xi)\in(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^d$ such that $|\tau| = |\xi| = 1/\sqrt{2}\}$.

Let us also remark that $\partial_t(\tilde\psi q_n) = (\partial_t\tilde\psi)q_n$ strongly converges to $(\partial_t\tilde\psi)Q$ in $H^{-1}((0,T)\times\Omega)$. According to (3.15), this implies that the $L^2((0,T)\times\Omega)$ micro-local defect measure corresponding to $\tilde\psi(q_n-Q)$ is supported on the set $\mathcal{C}_{\partial_t} = \{(t,x,\tau,\xi)\in(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^d$ such that $\tau = 0$ and $|\xi| = 1\}$.

Regarding these two facts, it is natural to introduce a self-adjoint operator P of order 0 which localizes around the characteristic set $\mathcal{C}_\Box$ and vanishes identically on a neighborhood of $\mathcal{C}_{\partial_t}$. In order to do that, we define a smooth function $\chi = \chi(\tau,\xi)$ on $\mathbb{R}\times\mathbb{R}^d$ as follows:

$$\begin{cases} \chi : \mathbb{R}\times\mathbb{R}^d \to [0,1], & \\ \chi(\tau,\xi) = 0 & \text{if } \tau^2+|\xi|^2 \geq 1 \text{ and } |\tau| \leq |\xi|/4, \\ \chi(\tau,\xi) = 1 & \text{if } \tau^2+|\xi|^2 \geq 1 \text{ and } |\tau| \geq 3|\xi|/4, \\ \chi(\tau,\xi) = 0 & \text{if } \tau^2+|\xi|^2 \leq 1/2, \\ \chi(\alpha\tau,\alpha\xi) = \chi(\tau,\xi) & \text{for all } (\tau,\xi) \text{ with } \tau^2+|\xi|^2\geq 1 \text{ and } \alpha \geq 1. \end{cases} \tag{3.21}$$

We then define the pseudo-differential operator P on smooth functions $z \in \mathcal{S}'(\mathbb{R}\times\mathbb{R}^d)$ by

$$Pz = \mathcal{F}^{-1}\left(\chi(\mathcal{F}z)\right), \quad \text{where } \mathcal{F} \text{ denotes the Fourier transform in space-time.} \tag{3.22}$$

Due to the localization properties of the micro-local defect measures of $\psi(\varphi_n[q_n]-\Phi)$ and $\tilde\psi(q_n-Q)$, the localization of the principal symbol of P implies the following convergence results:

$$\begin{cases} (I-P)(\psi\varphi_n[q_n]) \underset{n\to\infty}{\to} (I-P)(\psi\Phi) & \text{strongly in } L^2((0,T)\times\Omega), \\ P(\tilde\psi q_n) \underset{n\to\infty}{\to} P(\tilde\psi Q) & \text{strongly in } L^2((0,T)\times\Omega). \end{cases} \tag{3.23}$$

In particular,

$$\begin{aligned} \iint \psi\varphi_n[q_n]\tilde\psi q_n &= \iint P(\psi\varphi_n[q_n])\tilde\psi q_n + \iint (I-P)(\psi\varphi_n[q_n])\tilde\psi q_n \\ &= \iint \psi\varphi_n[q_n]P(\tilde\psi q_n) + \iint (I-P)(\psi\varphi_n[q_n])\tilde\psi q_n, \end{aligned}$$

and the convergences (3.23) and (3.15) and (3.19) imply

$$\lim_{n\to\infty}\iint_{(0,T)\times\Omega} \psi\varphi_n[q_n]\tilde\psi q_n = \iint_{(0,T)\times\Omega} \psi\Phi\tilde\psi Q.$$

Of course, since ψ and $\tilde\psi$ were arbitrary, we have proved that the weak limit f of $q_n\varphi_n[q_n]$ coincides with $Q\Phi$.

If follows that Φ actually belongs to $X[Q]$, hence it vanishes identically from (3.12). But, by Aubin-Lions' lemma, the convergence in (3.19) implies the strong convergence of $\varphi_n[q_n]$ to 0 in $L^2((0,T);H^{-1}(\Omega))$. Combined with the strong convergence of $\varphi_n[q_n]$ to 0 in $L^2((0,T)\times\omega)$ and $\|(\varphi_{0n},\varphi_{1n})\|_{L^2\times H^{-1}} = 1$ in (3.13), we get a contradiction with (3.7). This finishes the proof of Proposition 1.2. $\square$

4 Data assimilation with unknown potential

In this section, we focus on the proof of Theorem 1.3 presented in the introduction.

Proof of Theorem 1.3. We fix Q and q in $W^{s,p}_{\leq m}(\Omega)$ for some $s\in[0,1]$, $p\in(\max\{d,2\},\infty]$ and $m>0$. The proof will be done in several steps.

Preliminaries. Let $\Phi[Q]$ be the trajectory (1.11) on which $a_\omega\Phi[Q]$ is assumed to be known. Note that, according to Proposition 1.2, one immediately has the following a priori bounds:

$$\|(\Phi_0,\Phi_1)\|_{L^2\times H^{-1}} + \|\Phi[Q]\|_{C(L^2)\cap C^1(H^{-1})} \leq C\,\|a_\omega\Phi[Q]\|_{L^2(L^2)}\,. \tag{4.1}$$

To avoid possible confusion, in the following we denote (Φ_0,Φ_1) by $(\Phi_0[Q],\Phi_1[Q])$.

Let $\Phi[q]$ be the solution of (1.2) corresponding to the initial data $(\Phi_0[q],\Phi_1[q])$ which minimizes the functional $J[q]$. The first remark that can be done is that $J[q](\Phi_0[q],\Phi_1[q])\leq J(0,0)$, which yields that

$$\|a_\omega\Phi[q]\|_{L^2(L^2)} \leq C\,\|a_\omega\Phi[Q]\|_{L^2(L^2)}\,.$$

Of course, using Proposition 1.2, similarly to (4.1), this yields the following bounds on $\Phi[q]$:

$$\|(\Phi_0[q],\Phi_1[q])\|_{L^2\times H^{-1}} + \|\Phi[q]\|_{C(L^2)\cap C^1(H^{-1})} \leq C\,\|a_\omega\Phi[Q]\|_{L^2(L^2)}\,. \tag{4.2}$$

Moreover, the Euler-Lagrange equation satisfied by $(\Phi_0[q],\Phi_1[q])$ implies that for all initial data $(\varphi_0,\varphi_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, the solution $\varphi[q]$ of (1.2) satisfies

$$\int_0^T\int_\Omega a_\omega^2\varphi[q](\Phi[q]-\Phi[Q])\,dx\,dt = 0. \tag{4.3}$$

This implies in particular that the solution y of

$$\begin{cases} \Box y + qy = a_\omega^2(\Phi[q]-\Phi[Q]), & \text{in } (0,T)\times\Omega,\\ y=0 & \text{on } (0,T)\times\partial\Omega,\\ (y(0),\partial_t y(0)) = (0,0), \end{cases} \tag{4.4}$$

satisfies

$$(y(T),\partial_t y(T)) = (0,0). \tag{4.5}$$

Indeed, when multiplying the solution y of (4.4) by $\varphi[q]$ solution of (1.2), by (4.3) we obtain, for any $\varphi[q]$ solution of (1.2),

$$\langle(\varphi(T),\partial_t\varphi(T)),(y(T),\partial_t y(T)\rangle_{L^2\times H^{-1},H_0^1\times L^2} = \int_0^T\int_\Omega a_\omega^2(\Phi[q]-\Phi[Q])\varphi[q] = 0,$$

which is equivalent to (4.5).

A priori estimates. The function $\Psi = \Phi[q]-\Phi[Q]$ solves

$$\begin{cases} \Box\Psi + q\Psi = (Q-q)\Phi[Q], & \text{in } (0,T)\times\Omega,\\ \Psi = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{4.6}$$

In particular, it can be decoupled as $\Psi = z + w$, where z solves

$$\begin{cases} \Box z + qz = 0, & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ (z(0), \partial_t z(0)) = (\Psi(0), \partial_t \Psi(0)), \end{cases} \tag{4.7}$$

and w solves

$$\begin{cases} \Box w + qw = (Q-q)\Phi[Q], & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times\partial\Omega, \\ (w(0), \partial_t w(0)) = (0,0). \end{cases} \tag{4.8}$$

Moreover y solves the control problem

$$\begin{cases} \Box y + qy = a_\omega^2 w + a_\omega^2 z, & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y(0), \partial_t y(0)) = (0,0), \\ (y(T), \partial_t y(T)) = (0,0), \end{cases} \tag{4.9}$$

Multiplying by z, we easily obtain

$$\int_0^T \int_\Omega a_\omega^2 |z|^2 = - \int_0^T \int_\Omega a_\omega^2 z w,$$

so that

$$\|a_\omega z\|_{L^2(L^2)} \leq \|a_\omega w\|_{L^2(L^2)} . \tag{4.10}$$

Since z solves the wave equation (4.7), thanks to the observability estimate (1.5), we have

$$\|z\|_{C(L^2)\cap C^1(H^{-1})} \leq C \|a_\omega w\|_{L^2(L^2)} . \tag{4.11}$$

Now, we estimate the solution w of (4.8): energy estimates yield

$$\begin{aligned} \|w\|_{C(L^2)\cap C^1(H^{-1})} &\leq C \|(q-Q)\Phi[Q]\|_{L^1(H^{-1})} \\ &\leq C \|q-Q\|_{L^p} \|\Phi[Q]\|_{L^1(L^2)} \leq C \|q-Q\|_{L^p} \|a_\omega \Phi[Q]\|_{L^2(L^2)} , \end{aligned} \tag{4.12}$$

where the last estimate is a consequence of (4.1). We thus obtain

$$\|z\|_{C(L^2)\cap C^1(H^{-1})} + \|\Psi\|_{C(L^2)\cap C^1(H^{-1})} \leq C \|q-Q\|_{L^p} \|a_\omega \Phi[Q]\|_{L^2(L^2)} , \tag{4.13}$$

and, using energy estimates for y solution of (4.9),

$$\|y\|_{C(H_0^1)\cap C^1(L^2)} \leq C \|q-Q\|_{L^p} \|a_\omega \Phi[Q]\|_{L^2(L^2)} . \tag{4.14}$$

Strategy. We consider $y_k = \eta_k(-\Delta) y$. Since y solves the control problem (4.9), y_k solves the control problem

$$\begin{cases} \Box y_k + q y_k = \tilde{f}_k + a_\omega^2 \eta_k(-\Delta) z, & \text{in } (0,T)\times\Omega, \\ y_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y_k(0), \partial_t y_k(0)) = (0,0), \\ (y_k(T), \partial_t y_k(T)) = (0,0), \end{cases} \tag{4.15}$$

where $\tilde{f}_k$ is given by

$$\tilde{f}_k = [q, \eta_k(-\Delta)] y + \eta_k(-\Delta)(a_\omega^2 w) + [\eta_k(-\Delta), a_\omega^2] z. \tag{4.16}$$

But, as z solves (4.7), the function $\eta_k(-\Delta)z$ writes

$$\eta_k(-\Delta)z = z_k + w_k,$$

where z_k solves

$$\begin{cases} \Box z_k + q z_k = 0 & \text{in } (0,T)\times\Omega, \\ z_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ (z_k(0), \partial_t z_k(0)) = (\eta_k(-\Delta)z(0), \partial_t(\eta_k(-\Delta)z)(0)), \end{cases} \tag{4.17}$$

and w_k solves

$$\begin{cases} \Box w_k + q w_k = [q, \eta_k(-\Delta)]z & \text{in } (0,T)\times\Omega, \\ w_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ (w_k(0), \partial_t w_k(0)) = (0,0). \end{cases} \tag{4.18}$$

Hence we can rewrite (4.15) as

$$\begin{cases} \Box y_k + q y_k = f_k + a_\omega^2 z_k, & \text{in } (0,T)\times\Omega, \\ y_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y_k(0), \partial_t y_k(0)) = (0,0), \\ (y_k(T), \partial_t y_k(T)) = (0,0), \end{cases} \tag{4.19}$$

where f_k is defined by

$$f_k = [q, \eta_k(-\Delta)]y + \eta_k(-\Delta)(a_\omega^2 w) + [\eta_k(-\Delta), a_\omega^2]z + a_\omega^2 w_k. \tag{4.20}$$

Multiplying then the equation of y_k in (4.19) by z_k solution of (4.17), we easily get

$$0 = \int_0^T \int_\Omega f_k z_k + \int_0^T \int_\Omega a_\omega^2 |z_k|^2.$$

Hence, using the observability inequality (1.5) for z_k, which is a solution of (4.17), we get

$$\|z_k\|_{C(L^2)\cap C^1(H^{-1})} \le C \|f_k\|_{L^2(L^2)}. \tag{4.21}$$

The rest of the proof then focuses on estimating f_k defined by (4.20).

Estimating f_k. We do it term by term.
Estimating $[q, \eta_k(-\Delta)]y$: From (2.1) and (4.14),

$$\|[q, \eta_k(-\Delta)]y\|_{L^2(L^2)} \le C 2^{-ks/2} \|q - Q\|_{W^{s,p}} \|a_\omega \Phi[Q]\|_{L^2(L^2)}.$$

Estimating $\eta_k(-\Delta)(a_\omega^2 w)$: We first improve the estimates (4.12) on w solution of (4.8) by remarking that $(q-Q)\Phi[q]$ belongs to $L^1((0,T); H^{s-1}(\Omega))$. Indeed, the multiplication operator

$$\mathscr{M} : (\tilde{q}, \tilde{\varphi}) \mapsto \tilde{q}\tilde{\varphi}$$

maps $W^{1,p}(\Omega)\times L^2(\Omega)$ to $L^2(\Omega)$ and $L^p(\Omega)\times L^2(\Omega)$ to $H^{-1}(\Omega)$. Therefore, by interpolation, for all $s\in[0,1]$, $\mathscr{M}$ maps $W^{s,p}(\Omega)\times L^2(\Omega)$ to $H^{s-1}(\Omega)$.

Thus, energy estimates for the solution w of (4.8)give

$$\begin{aligned} \|w\|_{C(H_0^s)\cap C^1(H^{s-1})} &\le C \|(q-Q)\Phi[Q]\|_{L^1(H^{s-1})} \le C \|q-Q\|_{W^{s,p}} \|\Phi[Q]\|_{L^2(L^2)} \\ &\le C \|q-Q\|_{W^{s,p}} \|a_\omega \Phi[Q]\|_{L^2(L^2)}, \end{aligned} \tag{4.22}$$

where the last estimate is a consequence of (4.1). Using then that $\|\eta_k(-\Delta)\|_{\mathfrak{L}(H_0^s, L^2)} \leq C2^{-ks/2}$, and the fact that the multiplication by a_ω^2 continuously acts on $H_0^s(\Omega)$, we get

$$\begin{aligned}\left\|\eta_k(-\Delta)(a_\omega^2 w)\right\|_{L^2(L^2)} \leq C2^{-ks/2}\left\|a_\omega^2 w\right\|_{L^2(H_0^s)} &\leq C2^{-ks/2}\|w\|_{L^2(H_0^s)}\\ &\leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.23)\end{aligned}$$

Note that we have also proved the following estimate:

$$\|\eta_k(-\Delta)w\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.24)$$

Estimating $[\eta_k(-\Delta), a_\omega^2]z$*:* Using the bound (4.13) and the bound (2.21), we obtain

$$\left\|[\eta_k(-\Delta), a_\omega^2]z\right\|_{L^2(L^2)} \leq C2^{-k/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.25)$$

Estimating $a_\omega^2 w_k$*:* As w_k solves (4.18), by energy estimates we obtain

$$\begin{aligned}\|w_k\|_{L^2(L^2)} \leq C\|w_k\|_{C(L^2)\cap C^1(H^{-1})} &\leq C\|[q,\eta_k(-\Delta)]z\|_{L^2(H^{-1})}\\ &\leq C2^{-ks/2}\|z\|_{L^2(L^2)} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}, \quad (4.26)\end{aligned}$$

where we used (2.2) and (4.13). Hence we have

$$\left\|a_\omega^2 w_k\right\|_{L^2(L^2)} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}.$$

Combining all these estimates, we derived

$$\|f_k\|_{L^2(L^2)} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.27)$$

Conclusion. According to (4.21), estimate (4.27) yields

$$\|z_k\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.28)$$

Combined with (4.26), we derive

$$\|\eta_k(-\Delta)z\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}. \quad (4.29)$$

As $\Phi[q]-\Phi[Q] = z + w$, estimate (4.24) yields

$$\|\eta_k(-\Delta)(\Phi[q]-\Phi[Q])\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2}\|q-Q\|_{W^{s,p}}\|a_\omega\Phi[Q]\|_{L^2(L^2)}.$$

This concludes the proof of Theorem 1.3. □

Remark 4.1. *It might be surprising that, though the commutator estimates in* (2.1) *are valid for* $s \in [0,2]$*, we do not have any better estimate when* $s > 1$*. This is due to the the estimate in* (4.25)*, since the regularity of* $\Phi[Q]$ *is limited to* $C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$*.*

Remark 4.2. *The precise smoothness assumptions required on* a_ω *to get estimate* (1.14) *is* $a_\omega \in W^{s,\infty}(\Omega)$*. Indeed, this regularity is required to estimate* $\|[a_\omega, \eta_k(-\Delta)]\|_{\mathfrak{L}(L^2)}$ *in* (4.25) *by* $C2^{-ks/2}$*, see* (2.28)*. (Note that this regularity is also needed to derive* (4.23)*.)*

5 Controllability with unknown potential

In this section, we prove Theorem 1.4. As we will see, its proof is very similar to the one of Theorem 1.3.

Proof of Theorem 1.4. As in the proof of Theorem 1.3, we divide the proof in several paragraphs.

Below, we fix the initial data $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and the constants $m > 0$, $s \in [0,1]$ and $p \in (\max\{d,2\}, \infty]$. We shall also denote by $q = q(x)$ a generic potential $q(x) \in W^{s,p}_{\leq m}(\Omega)$.

Preliminaries. Similarly to the proof of Theorem 1.3, we begin with some classic bounds on the optimizers $(\Phi_0[q], \Phi_1[q])$ of $K[q]$ in (1.17). Indeed, using the coercivity of the functional $K[q]$ on $L^2(\Omega) \times H^{-1}(\Omega)$, which is uniform with respect to potentials $q \in W^{s,p}_{\leq m}(\Omega)$ according to Proposition 1.2, and the fact that $K[q](\Phi_0[q], \Phi_1[q]) \leq K[q](0,0) = 0$, we immediately obtain that, uniformly with respect to $q \in W^{s,p}_{\leq m}(\Omega)$,

$$\|(\Phi_0[q], \Phi_1[q])\|_{L^2 \times H^{-1}} + \|\Phi[q]\|_{C(L^2) \cap C^1(H^{-1})} \leq C \|(y_0, y_1)\|_{H_0^1 \times L^2}, \tag{5.1}$$

where $\Phi[q]$ is the solution of (1.2) with initial data $(\Phi_0[q], \Phi_1[q])$.

Let us then denote by $Y[q^a]$, respectively $Y[q^b]$ the controlled trajectories solutions of (1.15) with controls $u[q^a] = a_\omega \Phi[q^a]$, respectively $u[q^b] = a_\omega \Phi[q^b]$. Note that, due to (5.1), the controlled trajectories $Y[q^a]$ and $Y[q^b]$ satisfy:

$$\|Y[q^a]\|_{C^0(H_0^1) \cap C^1(L^2)} + \left\|Y[q^b]\right\|_{C^0(H_0^1) \cap C^1(L^2)} \leq C \|(y_0, y_1)\|_{H_0^1 \times L^2}. \tag{5.2}$$

A priori estimates. Let us set

$$Y^{ab} = Y^a - Y^b, \quad \Phi^{ab} = \Phi[q^a] - \Phi[q^b].$$

By construction, Φ^{ab} solves

$$\begin{cases} \Box \Phi^{ab} + q^a \Phi^{ab} = (q^b - q^a)\Phi[q^b], & \text{in } (0,T) \times \Omega, \\ \Phi^{ab} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (\Phi^{ab}(0), \partial_t \Phi^{ab}(0)) = (\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]). \end{cases} \tag{5.3}$$

It is then natural to write $\Phi^{ab} = z + w$, where z solves

$$\begin{cases} \Box z + q^a z = 0, & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ (z(0), \partial_t z(0)) = (\Phi_0[q^a] - \Phi_0[q^b], \Phi_1[q^a] - \Phi_1[q^b]). \end{cases} \tag{5.4}$$

and w solves

$$\begin{cases} \Box w + q^a w = (q^b - q^a)\Phi[q^b], & \text{in } (0,T) \times \Omega, \\ w = 0 & \text{on } (0,T) \times \partial\Omega, \\ (w(0), \partial_t w(0)) = (0,0). \end{cases} \tag{5.5}$$

One can then check that Y^{ab} solves the control problem

$$\begin{cases} \Box Y^{ab} + q^a Y^{ab} = f + a_\omega^2 z, & \text{in } (0,T) \times \Omega, \\ Y^{ab} = 0 & \text{on } (0,T) \times \partial\Omega, \\ (Y^{ab}(0), \partial_t Y^{ab}(0)) = (0,0), \\ (Y^{ab}(T), \partial_t Y^{ab}(T)) = (0,0). \end{cases} \tag{5.6}$$

where f is given by

$$f = (q^b - q^a)Y[q^b] + a_\omega^2 w. \tag{5.7}$$

Multiplying the equation (5.6) by z solution of (5.4), we obtain

$$0 = \int_0^T \int_\Omega fz + \int_0^T \int_\Omega a_\omega^2 |z|^2. \tag{5.8}$$

Thus, using the observability inequality (1.5) and the fact that for solutions φ of (1.2), the quantities $\|(\varphi_0, \varphi_1)\|_{L^2\times H^{-1}}$ and $\|\varphi\|_{C(L^2)\cap C^1(H^{-1})}$ are equivalent, we obtain

$$\|z\|_{C(L^2)\cap C^1(H^{-1})} \leq C \|f\|_{L^2(L^2)}. \tag{5.9}$$

We are thus reduced to estimate the source term f in (5.7): But on one hand,

$$\left\|(q^b - q^a)Y[q^b]\right\|_{L^2(L^2)} \leq \left\|q^b - q^a\right\|_{L^p} \left\|Y[q^b]\right\|_{L^2(H_0^1)} \leq C \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1 \times L^2}, \tag{5.10}$$

and on the other hand, using classical energy estimates,

$$\begin{aligned} \|w\|_{C(L^2)\cap C^1(H^{-1})} \leq C \left\|(q^b - q^a)\Phi[q^b]\right\|_{L^1(H^{-1})} &\leq C \left\|q^b - q^a\right\|_{L^p} \left\|\Phi[q^b]\right\|_{L^1(L^2)} \\ &\leq C \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1 \times L^2}. \end{aligned} \tag{5.11}$$

All together, these estimates yield

$$\|z\|_{C(L^2)\cap C^1(H^{-1})} \leq C \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1 \times L^2}. \tag{5.12}$$

With (5.11), we thus have

$$\left\|\Phi^{ab}\right\|_{C(L^2)\cap C^1(H^{-1})} = \left\|\Phi[q^a] - \Phi[q^b]\right\|_{C(L^2)\cap C^1(H^{-1})} \leq C \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1 \times L^2}, \tag{5.13}$$

and, thanks to a straightforward energy estimate on (5.6),

$$\left\|Y^{ab}\right\|_{C(H_0^1)\cap C^1(L^2)} = \left\|Y[q^a] - Y[q^b]\right\|_{C(H_0^1)\cap C^1(L^2)} \leq C \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1 \times L^2}. \tag{5.14}$$

Strategy. We first remark that $y_k = \eta_k(-\Delta)Y^{ab}$ solves the control problem

$$\begin{cases} \Box y_k + q^a y_k = \tilde{f}_k + a_\omega^2 \eta_k(-\Delta) z, & \text{in } (0,T)\times\Omega, \\ y_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ (y_k(0), \partial_t y_k(0)) = (0,0), \\ (y_k(T), \partial_t y_k(T)) = (0,0), \end{cases} \tag{5.15}$$

where $\tilde{f}_k$ is defined by

$$\tilde{f}_k = [q^a, \eta_k(-\Delta)]Y^{ab} + \eta_k(-\Delta)((q^b - q^a)Y[q^b]) + \eta_k(-\Delta)(a_\omega^2 w) + [\eta_k(-\Delta), a_\omega^2]z. \tag{5.16}$$

It is then natural to decompose $\eta_k(-\Delta)z$ as $\eta_k(-\Delta)z = z_k + w_k$, where z_k solves

$$\begin{cases} \Box z_k + q^a z_k = 0, & \text{in } (0,T)\times\Omega, \\ z_k = 0, & \text{on } (0,T)\times\partial\Omega, \\ (z_k(0), \partial_t z_k(0)) = (\eta_k(-\Delta)(\Phi_0[q^a] - \Phi_0[q^b]), \eta_k(-\Delta)(\Phi_1[q^a] - \Phi_1[q^b])), \end{cases} \tag{5.17}$$

and w_k solves

$$\begin{cases} \Box w_k + q^a w_k = [q^a, \eta_k(-\Delta)]z, & \text{in } (0,T)\times\Omega,\\ w_k = 0 & \text{on } (0,T)\times\partial\Omega,\\ (w_k(0), \partial_t w_k(0)) = (0,0). \end{cases} \tag{5.18}$$

This allows in particular to rewrite the first line of (5.15) as follows:

$$\Box y_k + q^a y_k = f_k + a_\omega^2 z_k, \quad \text{in } (0,T)\times\Omega, \tag{5.19}$$

where

$$f_k = [q^a, \eta_k(-\Delta)]Y^{ab} + \eta_k(-\Delta)((q^b - q^a)Y[q^b]) + \eta_k(-\Delta)(a_\omega^2 w) + [\eta_k(-\Delta), a_\omega^2]z + a_\omega^2 w_k. \tag{5.20}$$

We can then multiply equation (5.19) by z_k: using that y_k solves the control problem (5.15), we obtain, similarly as in (5.8), that

$$0 = \int_0^T \int_\Omega f_k z_k + \int_0^T \int_\Omega a_\omega^2 |z_k|^2.$$

Since z_k solves the wave equation (5.17) without source term, we can then use the observability inequality (1.5) to derive from this identity that

$$\|z_k\|_{C(L^2)\cap C^1(H^{-1})} \leq C \|f_k\|_{L^2(L^2)}. \tag{5.21}$$

The rest of the proof then consists in estimating f_k in (5.20).

Estimating f_k.
Estimating $[q^a, \eta_k(-\Delta)]Y^{ab}$: We have

$$\begin{aligned} \left\|[q^a, \eta_k(-\Delta)]Y^{ab}\right\|_{L^2(L^2)} &\leq \|[q^a, \eta_k(-\Delta)]\|_{\mathfrak{L}(H_0^1, L^2)} \left\|Y^{ab}\right\|_{L^2(H_0^1)} \\ &\leq C 2^{-ks/2} \left\|q^b - q^a\right\|_{L^p} \|(y_0, y_1)\|_{H_0^1\times L^2}, \end{aligned}$$

where we used (2.1) and (5.14).
Estimating $\eta_k(-\Delta)((q^b - q^a)Y[q^b])$: The bilinear multiplication operator maps $L^p(\Omega)\times H_0^1(\Omega)$ to $L^2(\Omega)$ and $W^{1,p}(\Omega)\times H_0^1(\Omega)$ to $H_0^1(\Omega)$. Thus by interpolation, for $s\in(0,1)$, it maps $W^{s,p}(\Omega)\times H_0^1(\Omega)$ to $H_0^s(\Omega)$. Hence we get

$$\begin{aligned} \left\|\eta_k(-\Delta)(q^b - q^a)Y[q^b]\right\|_{L^2(L^2)} &\leq C 2^{-ks/2} \left\|(q^b - q^a)Y[q^b]\right\|_{L^2(H_0^s)} \\ &\leq C 2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \left\|Y[q^b]\right\|_{L^2(H_0^1)} \\ &\leq C 2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0, y_1)\|_{H_0^1\times L^2}, \end{aligned}$$

where we used $\|\eta_k(-\Delta)\|_{\mathfrak{L}(H_0^s, L^2)} \leq C 2^{-ks/2}$ and the bound (5.2).

Estimating $\eta_k(-\Delta)(a_\omega^2 w)$: Improving estimate (5.11), we obtain

$$\begin{aligned} \|w\|_{C(H_0^s)\cap C^1(H^{s-1})} \leq C \left\|(q^b - q^a)\Phi[q^b]\right\|_{L^1(H^{s-1})} &\leq C \left\|q^b - q^a\right\|_{W^{s,p}} \left\|\Phi[q^b]\right\|_{L^1(L^2)} \\ &\leq C \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0, y_1)\|_{H_0^1\times L^2}. \end{aligned}$$

Hence, using that $\|\eta_k(-\Delta)\|_{\mathfrak{L}(H_0^s,L^2)} \leq C2^{-ks/2}$ and that the multiplication by a_ω^2 continuously acts on $H_0^s(\Omega)$, we obtain

$$\left\|\eta_k(-\Delta)(a_\omega^2 w)\right\|_{L^2(L^2)} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}.$$

Note that we also proved

$$\|\eta_k(-\Delta)w\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}. \tag{5.22}$$

Estimating $[\eta_k(-\Delta), a_\omega^2]z$*:* Here, we use estimates (5.12):

$$\left\|[\eta_k(-\Delta), a_\omega^2]z\right\|_{L^2(L^2)} \leq C2^{-k/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}.$$

Estimating $a_\omega^2 w_k$*:* Using energy estimates for the solution w_k of (5.18) and (2.2) we obtain

$$\begin{aligned}\|w_k\|_{C(L^2)\cap C^1(H^{-1})} &\leq C\,\|[q^a,\eta_k(-\Delta)]z\|_{L^1(H^{-1})} \\ &\leq C2^{-ks/2}\,\|z\|_{L^1(L^2)} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2},\end{aligned} \tag{5.23}$$

where the last estimate follows from (5.12). Hence we get

$$\left\|a_\omega^2 w_k\right\|_{L^2(L^2)} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}.$$

Collecting all the above estimates, we obtain

$$\|f_k\|_{L^2(L^2)} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}. \tag{5.24}$$

Conclusion. From (5.21), we get

$$\|z_k\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}.$$

As $\eta_k(-\Delta)(\Phi[q^a] - \Phi[q^b]) = \eta_k(-\Delta)(z+w) = z_k + w_k + \eta_k(-\Delta)w$, from estimates (5.22) and (5.23), we derive

$$\left\|\eta_k(-\Delta)(\Phi[q^a] - \Phi[q^b])\right\|_{C(L^2)\cap C^1(H^{-1})} \leq C2^{-ks/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2}. \tag{5.25}$$

Of course, this yields (1.20).

To get (1.19), we write

$$\eta_k(-\Delta)(a_\omega\Phi[q^a] - a_\omega\Phi[q^b]) = a_\omega\eta_k(-\Delta)(\Phi[q^a] - \Phi[q^b]) + [\eta_k(-\Delta), a_\omega](\Phi[q^a] - \Phi[q^b]).$$

We then obtain (1.19) from (5.25) and

$$\begin{aligned}\left\|[\eta_k(-\Delta), a_\omega](\Phi[q^a] - \Phi[q^b])\right\|_{L^2(L^2)} &\leq C2^{-k/2} \left\|\Phi[q^a] - \Phi[q^b]\right\|_{L^2(L^2)} \\ &\leq C2^{-k/2} \left\|q^b - q^a\right\|_{W^{s,p}} \|(y_0,y_1)\|_{H_0^1\times L^2},\end{aligned}$$

where the last estimate comes from (5.13). This concludes the proof of Theorem 1.4. □

Remark 5.1. *Since the proof of Theorem 1.4 closely follows the one of Theorem 1.3, Remarks 4.1 and 4.2 also apply in that case:*

- *Our proof does not allow to improve the rate of decay as* $k \to \infty$ *in the estimates of Theorem 1.4 by taking potentials lying in bounded sets of* $W^{s,p}(\Omega)$ *for* $s > 1$.
- *Theorem 1.4 also holds for* $a_\omega \in W^{s,\infty}(\Omega)$.

6 Further comments

Several remarks are in order.

Integrability conditions on the potentials. In Theorems 1.3 and 1.4, one can replace, for $d \geq 3$ and $s \in (0,1]$, the spaces $W^{s,p}(\Omega)$ with $p \in (d, \infty]$ by the interpolated spaces

$$\widetilde{W^{s,d}}(\Omega) = [L^d(\Omega), W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega)]_{s/2}, \tag{6.1}$$

since Section 2 can easily be adapted to that case. Note that these spaces are *a priori* slightly smaller than the classical Sobolev $W^{s,d}(\Omega)$. Also note that when $s = 0$, i.e. when the potentials simply belong to $L^d(\Omega)$, the observability inequality is not known under the geometric control condition, see Proposition 1.2.

Optimality of the rate of decay as $k \to \infty$ in the estimates of Theorems 1.3 and 1.4. Theorems 1.3 and 1.4 state that the high-frequency components weakly depend on the errors on the potential function. But the optimality of the rate of decay as $k \to \infty$ in the estimates (1.14) and (1.19)–(1.20) is completely open.

Potentials depending on time. As said in the introduction, when the potentials depend on time, one basically needs the strength of Carleman estimates [11] to prove observability properties, uniformly with respect to the $L^\infty((0,T); L^d(\Omega))$-norm of the potentials, thus requiring stronger geometric assumptions, namely the Gamma-conditions (1.21) recalled in the introduction.

Except for that part, one easily checks that if we are working on potentials in balls of $L^\infty((0,T); W^{s,p}(\Omega))$ for $s \in [0,1]$ and $p \in (\max\{d,2\}, \infty]$ (or in balls of $L^\infty((0,T); \widetilde{W^{s,d}}(\Omega))$, see (6.1)), one can prove similar results as in Theorems 1.3 and 1.4 by replacing in their statements the norms $\|q - Q\|_{W^{s,p}}$, respectively $\|q^a - q^b\|_{W^{s,p}}$, by $\|q - Q\|_{L^\infty(W^{s,p})}$, respectively $\|q^a - q^b\|_{L^\infty(W^{s,p})}$.

The case of boundary observation. Our method does not apply in the context of boundary observation/control. Indeed, in the above proofs, we have to estimate the commutator of $\eta_k(-\Delta)$ with the control operator, which in our case simply is the multiplication by a smooth function a_ω. Whether this simply is a technical issue or not is an open problem.

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