Resolvent estimates in controllability theory and applications to the discrete wave equation

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Abstract

We briefly present the difficulties arising when dealing with the controllability of the discrete wave equation, which are, roughly speaking, created by high-frequency spurious waves which do not travel. It is by now well-understood that such spurious waves can be dealt with by applying some convenient filtering technique. However, the scale of frequency in which we can guarantee that none of these non-traveling waves appears is still unknown in general. Though, using Hautus tests, which read the controllability of a given system in terms of resolvent estimates, we are able to prove that these spurious waves do not appear before some frequency scale. This document is based on the articles [12, 13, 14].

1 Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator with compact resolvent. Let us consider the following abstract system:

$$\dot{z}(t) = Az(t), \quad z(0) = z_0. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, and $z = z(t)$ is the *state* of the system. Such systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation).

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A), Y)$ be an observation operator and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

In order to give a sense to (1.2), we make the assumption that B is an admissible observation operator in the following sense (see, e.g., [47]):

Definition 1.1. The operator B is an admissible observation operator for system (1.1)–(1.2) if for every $T > 0$ there exists a constant $K_T > 0$ such that

$$\int_0^T \|y(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2, \qquad \forall \, z_0 \in \mathcal{D}(A). \qquad (1.3)$$

This implies in particular that for any $z_0 \in X$, the observation $y(t) = Bz(t)$ lies in $L^2(0, T; Y)$.

Note that if B is *bounded* in X, i.e. if it can be extended such that $B \in \mathfrak{L}(X, Y)$, then B is obviously an admissible observation operator. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property of the solutions of the evolution equation (1.1), see e.g. [31].

In the following, we will always assume that the continuous model (1.1)–(1.2) is admissible, so that the observation $y(t)$ in (1.2) always belongs to $L^2(0, T; Y)$.

Our main interest here is to study the exact observability property for system (1.1)–(1.2), which can be formulated as follows:

Definition 1.2. System (1.1)–(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|y(t)\|_Y^2 \, dt, \qquad \forall \, z_0 \in X. \qquad (1.4)$$

Moreover, system (1.1)–(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that these observability issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [31]). Indeed, controllability and observability are dual notions, and therefore each statement concerning observability has its counterpart in controllability. In the sequel, we focus on the observability properties of (1.1)–(1.2).

There is an extensive literature providing observability results for wave, plate and Schrödinger equations, among other models, and by various methods including microlocal analysis [3, 4], multipliers techniques [29, 40], Carleman estimates [25, 48], Ingham type inequalities [27, 20], etc. Our goal in this paper is to develop a theory allowing to get observability results for space semidiscrete systems as a direct consequence of those corresponding to the continuous ones, thus avoiding technical developments in the discrete setting.

One of the interesting features of the approach we shall present here (developed in the articles [14, 13, 12]) is that it works in any dimension and in a very general setting. To our knowledge, these were the first works which prove in a systematic way observability properties for time and space semidiscrete systems from the ones of the continuous setting.

Let us briefly comment some related works. Similar problems have been extensively studied in the last decade for various space semi-discretizations of the 1d wave equation, see for instance the review article [51] and the references therein. The numerical schemes on uniform meshes given by the finite difference and finite element methods do not have uniform observability properties, whatever the time T is ([26]), thus yielding blow-up behaviors of the discrete controls. This is due to high frequency waves which do not propagate, see [46, 33]. In other words, these numerical schemes create some spurious high-frequency wave solutions which are localized and cannot be controlled.

Actually, as observed in [31], the norm of the control map (which maps initial data to be controlled to the corresponding control of minimal L^2-norm) coincides with the constant of observability $1/\sqrt{k_T}$ in (1.4). This is why we shall consider only *uniform* observability properties for the discrete schemes, where *uniform* has to be understood as *uniformly with respect to the discretization parameters* in both the time and spaces variables.

In this context, filtering techniques have been extensively developed. It has been proved in [26, 50] that filtering the initial data removes these spurious waves, and makes possible uniform observability properties to hold. Other ways to filter these spurious waves exist, for instance using a wavelet filtering approach [38] or bi-grids techniques [18, 39]. However, to the best of our knowledge, these methods have been analyzed only for uniform grids in small dimensions (namely in 1d or 2d). Also note that these results prove uniform observability properties for larger classes of initial data than the ones we shall state here, but in more particular cases. In particular, our results below depend on neither the dimension nor the uniformity of the meshes.

Let us also mention that observability properties are equivalent to stabilization properties (see [21]), when the observation operator is bounded. Therefore, observability properties can be deduced from the literature in stabilization theory. In particular, we refer to the works [45, 44, 37, 15], which prove uniform stabilization results for damped space semidiscrete wave equations in 1d and 2d, discretized on uniform meshes using finite difference approximation schemes, in which a numerical viscosity term has been added. Again, these results are better than the ones derived here, but apply in the more restrictive context of 1d or 2d wave equations on uniform meshes. Similar results have also been proved in [42], but using a non trivial spectral condition on A, which reduces the scope of applications mainly to 1d equations.

To the best of our knowledge, there are very few paper dealing with nonuniform meshes. A first step in this direction can be found in the context of the stabilization of the 1d wave equation in [42]: Indeed, stabilization properties are equivalent (see [21]) to observability properties for the corresponding undamped systems. The results in [42] can therefore be applied to 1d wave equations on nonuniform meshes to derive uniform observability results at a scale $1/h$ (in the sense given below). However, they strongly use a spectral gap condition on the eigenvalues of the operator A, which does not hold for the wave operator in dimension higher than one. In the following, we will explain how to get rid of that additional assumption and consider more general observation operator B.

Another result in this direction is presented in [10], in the context of the 1d wave equation discretized using a mixed finite element method as in [2, 6]. In [10], it is proved that observability properties for schemes derived from a mixed finite element method hold uniformly within a large class of nonuniform meshes.

For time semidiscrete equations, very few results are available. In [36], the 1d wave equation is considered in a fully discrete setting on uniform meshes, using Ingham type inequalities and the explicit knowledge of the discrete eigenvalues, which is therefore hard to generalize in a more general setting. In [49], a time-discrete multiplier technique is developed and proved that time semidiscrete N-dimensional equations are not uniformly observable if some filtering condition is not added. However, the proof strongly uses the well-known space structure of the continuous wave equation and then cannot be generalized easily to the fully discrete case.

We shall also mention recent works on spectral characterizations of exact observability for abstract conservative linear systems, which will be the basis of our approach. We refer to [5, 34] for a very general approach of observability properties for such conservative linear systems, which yields a necessary and sufficient resolvent condition for exact observability to hold. Let us also mention the articles [32, 41], which derived several spectral conditions for the exact observability of wave type equations. In [41], a spectral characterization of the exact observability property based on wave packets is also given. Our approach is inspired in all these works.

The outline of this article is the following.

We shall first present how our approach applies to time discretization schemes of (1.1)–(1.2). We will then explain how these spectral approaches apply in the context of space semidiscrete schemes. Finally, we give some further comments.

2 Time semidiscrete schemes

2.1 The midpoint scheme

Let us first present a natural discretization of the continuous system (1.1). For any $\Delta t > 0$, we denote by z^k and y^k respectively the approximations of the solution z and the output function y of system (1.1)–(1.2) at time $t_k = k\Delta t$ for $k \in \mathbb{Z}$. Consider the following *implicit midpoint* time discretization of system (1.1):

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\Delta t} = A\left(\dfrac{z^{k+1} + z^k}{2}\right), & \text{in } X, \quad k \in \mathbb{Z}, \\ z^0 \text{ given.} \end{cases} \tag{2.1}$$

The output function of (2.1) is now given by

$$y^k = Bz^k, \qquad k \in \mathbb{Z}. \tag{2.2}$$

Note that (2.1)–(2.2) is a discrete version of (1.1)–(1.2).

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Taking into account that the spectrum of A is skew-adjoint, it is easy to show that the energy $\left\|z^k\right\|_X$ is conserved in the discrete time variable $k \in \mathbb{Z}$, i.e. $\left\|z^k\right\|_X = \left\|z^0\right\|_X$, similarly as for solutions of (1.1). Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting.

The uniform exact observability problem for system (2.1)–(2.2) is formulated as follows:

To find a positive constant $\tilde{k}_T$, independent of $\Delta t > 0$, such that the solutions z^k of system (2.1) satisfy:

$$\tilde{k}_T \left\|z^0\right\|_X^2 \leq \Delta t \sum_{k\Delta t\in(0,T)} \left\|Bz^k\right\|_Y^2, \tag{2.3}$$

for all initial data z^0 in an appropriate class.

Clearly, (2.3) is a discrete version of (1.4).

In the sequel, we are interested in understanding under which assumptions inequality (2.3) holds uniformly on $\Delta t > 0$. One expects to do it so that, when letting $\Delta t \to 0$, one recovers the observability property of the continuous model.

It can be done by means of a spectral filtering mechanism. More precisely, since A is skew-adjoint with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : j \in \mathbb{Z}\}$, where $(\mu_j)_{j\in\mathbb{Z}}$ is an increasing sequence of real numbers. Set $(\Psi_j)_{j\in\mathbb{Z}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{Z}}$:

$$A\Psi_j = i\mu_j\Psi_j. \tag{2.4}$$

Moreover, we define the filtered class

$$\mathcal{C}[A](s) = \text{Span}\{\Psi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}. \tag{2.5}$$

We will prove that inequality (2.3) holds uniformly (with respect to $\Delta t > 0$) in the class $\mathcal{C}[A](\delta/\Delta t)$ for any $\delta > 0$ and for T_δ large enough, depending on the filtering parameter δ.

Theorem 2.1 ([14]). *Assume that the continuous system (1.1)–(1.2) is admissible and exactly observable.*

Then, for any $\delta > 0$, there exist T_δ such that for any $T > T_\delta$, there exists two positive constants $k_{T,\delta}, K_{T,\delta} > 0$, independent of Δt, such that for $\Delta t > 0$ small enough, the solutions z^k of (2.1) with initial data $z^0 \in \mathcal{C}[A](\delta/\Delta t)$ satisfy

$$k_{T,\delta} \left\|z^0\right\|_X^2 \leq \Delta t \sum_{k\Delta t\in(0,T)} \left\|Bz^k\right\|_Y^2 \leq K_{T,\delta} \left\|z^0\right\|_X^2. \tag{2.6}$$

Theorem 2.1 states uniform observability and admissibility properties within the class $\mathcal{C}[A](\delta/\Delta t)$. As one can check by using the counterexample in [49], one cannot go in general beyond the scale $1/\Delta t$: the observability estimate (2.6) is false in any class $\mathcal{C}[A](1/f(\Delta t))$ with $f(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.

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Sketch of the proof. We only focus on the proof of the observability estimate in (2.6). The admissibility property can be proved similarly by using Theorem 2.2 in [13].

The proof of Theorem 2.1 is based on the following result derived in [5, 34]:

Theorem 2.2 ([5, 34]). *Assume that system* (1.1)–(1.2) *is admissible.*

Then system (1.1)–(1.2) *is exactly observable if and only if there exist positive constants* $m, M > 0$ *such that*

$$M^2 \left\| (i\omega I - A)z \right\|_X^2 + m^2 \left\| Bz \right\|_Y^2 \geq \left\| z \right\|_X^2, \qquad \forall \omega \in \mathbb{R}, \, \forall z \in \mathcal{D}(A). \qquad (2.7)$$

Besides, the proof is entirely constructive. In particular, the constants in the observability estimate (1.4) *can be made explicit: The observability estimate* (1.4) *holds for any time* $T > T^*$, *with*

$$T^* = \pi M, \qquad k_T = \frac{2m^2 T}{T^2 - \pi^2 M^2}. \qquad (2.8)$$

Proof of Theorem 2.2. The complete proof can be found in [34]. Here, we are only interested in proving that if the resolvent estimate (2.7) is satisfied, then the observability property (1.4) holds. The proof presented below is the one in [34].

Set $z_0 \in \mathcal{D}(A)$, and let $z(t)$ be the corresponding solution of (1.1). Set $w(t) = \chi(t)z(t)$, where $\chi(t)$ is a cut-off function of time lying in $H^1(\mathbb{R})$ and compactly supported. Then w solves $w' = Aw + \chi' z$, and its time Fourier transform satisfies
$$(i\omega - A)\hat{w}(\omega) = \widehat{\chi' z}(\omega).$$

In particular, plugging $\hat{w}(\omega)$ in (2.7) and integrating in the frequency variable $\omega \in \mathbb{R}$, we obtain

$$M^2 \int_{\mathbb{R}} \left\| \widehat{\chi' z}(\omega) \right\|_X^2 d\omega + m^2 \int_{\mathbb{R}} \| B\hat{w}(\omega) \|_Y^2 \, d\omega \geq \int_{\mathbb{R}} \| \hat{w}(\omega) \|_X^2 \, d\omega.$$

Using Parseval's identity and the fact that $\| z(t) \|_X^2 = \| z_0 \|_X^2$ for solutions of (1.1), this yields

$$\| z_0 \|_X^2 \int_{\mathbb{R}} \left(|\chi(t)|^2 - M^2 |\chi'(t)|^2 \right) dt \leq m^2 \int_R |\chi(t)|^2 \, \| Bz(t) \|_Y^2 \, dt.$$

We then choose

$$\chi(t) = \begin{cases} \sin\left(\dfrac{t\pi}{T} \right), & \text{for } t \in (0, T), \\ 0, & \text{for } t \notin (0, T). \end{cases}$$

Explicit computations then yields the observability estimate (1.2) with explicit constants as in (2.8). □

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Let us now explain how the proof of Theorem 2.1 works.

Assume that the continuous system (1.1)–(1.2) is admissible and exactly observable. Then, according to Theorem 2.2, the operators (A, B) should satisfy the resolvent estimate (2.7).

We then mimic the above proof (of Theorem 2.2) with time-discrete Fourier transform instead of time-continuous ones. This makes appear some extra term that can be handled thanks to the filtering condition. Indeed, in the filtered space $\mathcal{C}[A](\delta/\Delta t)$, we have the following property: for $\kappa \in [0, 1]$,

$$\|A^\kappa z\|_X \leq \left(\frac{\delta}{\Delta t}\right)^\kappa \|z\|_X, \quad \forall z \in \mathcal{C}[A]\left(\frac{\delta}{\Delta t}\right).$$

In particular, our proof is explicit and we can even prove an estimate on the time of observability: If $B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y)$ for some $\kappa \in [0, 1)$, then the discrete observability estimates (2.6) hold uniformly with respect to $\Delta t > 0$ in any time $T > T_\delta$, with

$$T_\delta = \pi M \left(1 + \frac{\delta^2}{4}\right). \tag{2.9}$$

Note that this estimate is not the one obtained in [14], which is given for a general observation operator $B \in \mathfrak{L}(\mathcal{D}(A), Y)$. Estimate (2.9) has been derived afterwards in [11]. □

2.2 General time-discretization schemes

In this section, we deal with more general time-discretization schemes of the form

$$z^{k+1} = \mathbb{T}_{\Delta t} z^k, \qquad y^k = B z^k \tag{2.10}$$

We will show that, under some appropriate assumptions on the operator $\mathbb{T}_{\Delta t}$, inequality (2.6) holds uniformly on Δt for solutions of (2.10) when the initial data are taken in the class $\mathcal{C}[A](\delta/\Delta t)$.

More precisely, we assume that the discrete system (2.10) is conservative in the sense that there exist real numbers $\mu_{j,\Delta t}$ such that

$$\mathbb{T}_{\Delta t} \Psi_j = \exp(i\mu_{j,\Delta t}\Delta t)\Psi_j. \tag{2.11}$$

Moreover, we assume that there is an explicit relation between $\mu_{j,\Delta t}$ and μ_j of the following form:

$$\mu_{j,\Delta t} = \frac{1}{\Delta t} \, h(\mu_j \Delta t), \tag{2.12}$$

where $h : (-R, R) \to [-\pi, \pi]$ is a smooth strictly increasing function, with $R \in (0, \infty]$, i.e.

$$\forall \eta \in (-R, R), \quad |h(\eta)| \leq \pi, \quad \text{and} \quad \forall \delta < R, \quad \inf\{h'(\eta), \, |\eta| \leq \delta\} > 0. \tag{2.13}$$

The parameter R corresponds to a frequency limit $R/\Delta t$ imposed by the discretization scheme. Roughly speaking, the first part of (2.13) reflects the fact

that one cannot measure frequencies higher than $\pi/\Delta t$ in a mesh of size Δt. The second part is a non-degeneracy condition on the group velocity (see [46]) of solutions of (2.10) which is necessary to guarantee the propagation of solutions that is required for observability to hold.

We also assume

$$\lim_{\eta \to 0} \frac{h(\eta)}{\eta} = 1. \tag{2.14}$$

This guarantees the consistency of the time-discrete scheme with the continuous model (1.1).

Before going further, let us remark that the midpoint scheme (2.1) is a particular instance of such time-discretization and the corresponding function h simply is

$$h(\eta) = 2 \arctan\left(\frac{\eta}{2}\right). \tag{2.15}$$

We also point out that several time-discretization schemes fit this abstract setting, as for instance the fourth-order Gauss method (a Runge-Kutta discretization which preserves the energy), see e.g. [19].

We have the following theorem:

Theorem 2.3 ([14]). *Assume that the continuous system (1.1)–(1.2) is admissible and exactly observable.*

Under assumptions (2.11), (2.12), (2.13) and (2.14), for any $\delta \in (0, R)$, there exists a time T_δ such that for all $T > T_\delta$, there exist two positive constants $k_{T,\delta}, K_{T,\delta} > 0$ such that for all $\Delta t > 0$ small enough, any solution of (2.10) with initial value $z^0 \in \mathcal{C}[A](\delta/\Delta t)$ satisfies

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \Delta t \sum_{k\Delta t \in (0,T)} \left\| B\left(\frac{z^k + z^{k+1}}{2}\right) \right\|_Y^2 \leq K_{T,\delta} \left\| z^0 \right\|_X^2. \tag{2.16}$$

Besides, if $B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y)$ for some $\kappa \in [0, 1)$, we have the following estimate on T_δ:

$$T_\delta \leq \pi M \left(1 + \tan^2\left(\frac{h(\delta)}{2}\right)\right) \sup_{|\eta| \leq \delta} \left\{\frac{\cos^2(h(\eta)/2)}{h'(\eta)}\right\}. \tag{2.17}$$

The proof of Theorem 2.3 follows the same lines of the one of Theorem 2.1, except that the estimates are slightly more technical. This illustrates the robustness of these resolvent estimates based techniques.

Also note that

$$\frac{h'(\eta)}{\cos^2(h(\eta)/2)} = \frac{d}{d\eta}\left(2\tan\left(\frac{h(\eta)}{2}\right)\right),$$

thus explaining why this term does not appear in the case of the midpoint scheme, see (2.15).

2.3 Application to family of operators

Since all the above proofs are constructive, one easily checks that family of systems of the form (1.1) that are uniformly observable can be discretized in time so that the corresponding time-discrete systems are uniformly observable.

More precisely, for $h > 0$, let X_h be a Hilbert space endowed with the norm $\|\cdot\|_h$ and let $A_h : \mathcal{D}(A_h) \to X_h$ be a skew-adjoint operator with compact resolvent. Consider then the systems

$$\dot{z}_h = A_h z_h, \quad z_h(0) = z_{0h}, \tag{2.18}$$

observed by

$$y_h(t) = B_h z_h(t), \tag{2.19}$$

for some operator $B_h \in \mathfrak{L}(\mathcal{D}(A_h), Y_h)$.

Assume then that the systems (2.18)–(2.19) are admissible and observable uniformly with respect to $h > 0$, and such that

$$\sup_h \left\{ \|B_h\|_{\mathfrak{L}(\mathcal{D}(A_h), Y_h)} \right\} < \infty.$$

Then they satisfy uniformly the resolvent estimate (2.7), that is: There exists $m, M > 0$ such that for all $h > 0$,

$$M^2 \|(i\omega I - A_h)z_h\|_h^2 + m^2 \|B_h z_h\|_{Y_h}^2 \geq \|z_h\|_h^2, \quad \forall \omega \in \mathbb{R}, \forall z_h \in \mathcal{D}(A_h). \tag{2.20}$$

One can then follow the constructive proof of Theorem 2.1 to prove that, for any $\delta > 0$, the systems

$$\begin{cases} \dfrac{z_h^{k+1} - z_h^k}{\Delta t} = A_h \left(\dfrac{z_h^{k+1} + z_h^k}{2} \right), & \text{in } X_h, \ k \in \mathbb{Z}, \\ z^0 \in X_h \text{ given,} \end{cases} \qquad y_h^k = B_h z_h^k, \tag{2.21}$$

are admissible and exactly observable uniformly with respect to $h > 0$ and $\Delta t > 0$ within the class of filtered data $\mathcal{C}[A_h](\delta/\Delta t)$.

Of course, in our mind, systems (2.18)–(2.19) refer to space semidiscrete approximation schemes of conservative systems, but it can be any family of systems depending on a parameter, as for instance in homogenization theory.

Also note that, when X_h is a finite dimensional vector space, A_h is bounded, and under the CFL type condition $\sup_{(h,\Delta t) \to (0,0)} \{\|A_h\|_{\mathfrak{L}(X_h)} \Delta t\} < \infty$, the filtered space $\mathcal{C}[A_h](\delta/\Delta t)$ coincide with the whole space X_h by choosing $\delta > 0$ large enough.

This argument allows then to reduce the study of the fully discrete approximations of (1.1)–(1.2) to the study of its space semi-discretizations. We will present in the next section some partial answers to that question.

3 Space semidiscrete finite element approximations

In the following, we focus on two particular instances of (1.1)–(1.2), namely Schrödinger type equations, which write

$$\begin{cases} i\dot{z} = A_0 z, & t \geq 0, \\ z(0) = z_0, \end{cases} \qquad y(t) = B_0 z(t), \ t \geq 0, \qquad (3.1)$$

and wave type equations

$$\begin{cases} \ddot{u} + A_0 u = 0, & t \geq 0, \\ u(0) = u_0, & \dot{u}(0) = u_1. \end{cases} \qquad y(t) = B_0 \dot{u}(t), \quad t \geq 0, \qquad (3.2)$$

where, in both cases, A_0 stands for a self adjoint positive definite operator on an Hilbert space H. The operator B_0 in (3.1) or in (3.2) is assumed to be in $\mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$.

These two cases indeed fit the abstract setting of (1.1)–(1.2), and the observability inequalities corresponding to (1.4) are, respectively,

$$k_T \|z_0\|_X^2 \leq \int_0^T \|B_0 z(t)\|_Y^2 \ dt, \qquad (3.3)$$

$$k_T \left(\left\|A_0^{1/2} u_0\right\|_X^2 + \|u_1\|_X^2 \right) \leq \int_0^T \|B_0 \dot{u}(t)\|_Y^2 \ dt. \qquad (3.4)$$

Let us now describe the finite element method we use to discretize (3.1)–(3.2).

Consider $(V_h)_{h>0}$ a sequence of vector spaces of finite dimension n_h that embed into H using a linear morphism $\rho_h : V_h \to H$. For each $h > 0$, the inner product $< \cdot, \cdot >_H$ in H induces a structure of Hilbert space for V_h endowed by the scalar product $< \cdot, \cdot >_h = < \rho_h \cdot, \rho_h \cdot >_H$.

We assume that for each $h > 0$, the vector space $\rho_h(V_h)$ is a subspace of $\mathcal{D}(A_0^{1/2})$. We thus define the linear operator $A_{0h} : V_h \to V_h$ by

$$< A_{0h}\phi_h, \psi_h >_h = < A_0^{1/2}\pi_h\phi_h, A_0^{1/2}\pi_h\psi_h >_H, \quad \forall \phi_h, \psi_h \in V_h. \qquad (3.5)$$

The operator A_{0h} defined in (3.5) obviously is self-adjoint and positive definite. If we introduce the adjoint ρ_h^* of ρ_h, definition (3.5) implies that

$$A_{0h} = (A_0^{1/2}\rho_h)^* A_0^{1/2}\rho_h = \rho_h^* A_0 \rho_h. \qquad (3.6)$$

This operator A_{0h} corresponds to the finite element discretization of A_0. Systems (3.1) and (3.2) are then discretized into

$$i\dot{z}_h = A_{0h}z_h, \quad z_h(0) = z_{0h} \in V_h, \qquad (3.7)$$

and

$$\ddot{u}_h + A_{0h}u_h = 0, \quad u_h(0) = u_{0h} \in V_h, \quad \dot{u}_h(0) = u_{1h} \in V_h, \qquad (3.8)$$

respectively.

In this context, for all $h > 0$, the observation operator naturally becomes $B_{0h} = B_0 \rho_h$, which obviously belongs to $\mathfrak{L}(\mathcal{D}(A_{0h}^{1/2}), Y)$.

We now make precise the assumptions we have, usually, on ρ_h, and which will be needed in our analysis. One easily checks that $\rho_h^* \rho_h = Id_{V_h}$. Besides, the embedding ρ_h describes the finite element approximation we have chosen. In particular, the vector space $\rho_h(V_h)$ approximates, in the sense given hereafter, the space $\mathcal{D}(A_0^{1/2})$: There exist $\theta > 0$ and $C_0 > 0$, such that for all $h > 0$,

$$
\begin{cases}
\left\| A_0^{1/2}(\rho_h \rho_h^* - I)\phi \right\|_H \leq C_0 \left\| A_0^{1/2}\phi \right\|_H, & \forall \phi \in \mathcal{D}(A_0^{1/2}), \\
\left\| A_0^{1/2}(\rho_h \rho_h^* - I)\phi \right\|_H \leq C_0 h^\theta \left\| A_0 \phi \right\|_H, & \forall \phi \in \mathcal{D}(A_0).
\end{cases}
\tag{3.9}
$$

Note that in many applications, and in particular for A_0 the Laplace operator on a bounded domain with Dirichlet boundary conditions, estimates (3.9) are satisfied for $\theta = 1$ when discretizing on regular meshes (see [43]).

We will not discuss convergence results for the numerical approximation schemes presented here, which are classical under assumption (3.9), and which can be found for instance in the textbook [43].

Let us mention that this question has already been investigated in [26] for the 1d wave equation observed from the boundary on a 1d mesh. In [26], it has been proved that, using a space semidiscrete approximation scheme for the 1d wave equation on uniform meshes, discrete versions of (3.4) do not hold uniformly with respect to the discretization parameter $h > 0$, because of the presence of spurious high frequency solutions that do not travel. However, if the initial data are filtered in a suitable way, then observability inequalities hold uniformly with respect to the space discretization parameter.

Therefore, it is natural to restrict ourselves to filtered initial data. For all $h > 0$, since A_{0h} is a self adjoint positive definite matrix, the spectrum of A_{0h} is given by a sequence of positive eigenvalues

$$
0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{n_h}^h
\tag{3.10}
$$

and normalized (in V_h) eigenvectors $(\Phi_j^h)_{1 \leq j \leq n_h}$. For any s, we can now define, for each $h > 0$, the filtered space

$$
\mathfrak{C}_h(s) = \mathrm{Span}\Big\{ \Phi_j^h \text{ such that the corresponding eigenvalue satisfies } |\lambda_j^h| \leq s \Big\}.
$$

We are now in position to state the following results:

Theorem 3.1 ([13, 12]). *Let A_0 be a self-adjoint positive definite operator with compact resolvent, and $B_0 \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. Assume that the maps $(\rho_h)_{h>0}$ satisfy property* (3.9).

Schrödinger type equations: *Set*

$$
\sigma = \theta \min \Big\{ 2(1 - 2\kappa), \frac{2}{3} \Big\}.
\tag{3.11}
$$

11

Assume that system (3.1) is admissible and exactly observable.

Then there exist $\varepsilon > 0$, a time T^* and two positive constants $k_*, K_* > 0$ such that, for any $h \in (0,1)$, any solution of (3.7) with initial data

$$z_{0h} \in \mathfrak{C}_h(\varepsilon/h^\sigma) \tag{3.12}$$

satisfies

$$k_* \left\| z_{0h} \right\|_h^2 \leq \int_0^{T^*} \left\| B_{0h} z_h(t) \right\|_Y^2 \ dt \leq K_* \left\| z_{0h} \right\|_h^2. \tag{3.13}$$

Wave type equations: *Set*

$$\varsigma = \theta \min\{2(1 - 2\kappa), 1\}. \tag{3.14}$$

Assume that system (3.2) is admissible and exactly observable.

Then there exist $\epsilon > 0$, a time T^* and two positive constants $k_*, K_* > 0$ such that, for any $h \in (0,1)$, any solution of (3.8) with initial data

$$(u_{0h}, u_{1h}) \in \mathfrak{C}_h(\epsilon/h^\varsigma)^2 \tag{3.15}$$

satisfies

$$k_* \left(\left\| A_{0h}^{1/2} u_{0h} \right\|_h^2 + \left\| u_{1h} \right\|_h^2 \right) \leq \int_0^{T^*} \left\| B_{0h} \dot{u}_h(t) \right\|_Y^2 \ dt$$
$$\leq K_* \left(\left\| A_{0h}^{1/2} u_{0h} \right\|_h^2 + \left\| u_{1h} \right\|_h^2 \right). \tag{3.16}$$

Note in particular that this yields the same results as the one obtained in [42] in a 1d framework and generalizes it to any dimension.

One of the interesting features of these results is that they hold in any dimension and in a very general setting. To our knowledge, [13, 12] are the first works which proved in such a systematic way admissibility and observability properties for space semidiscrete approximation schemes as a consequence of the ones of the continuous setting.

The next sections will explain how the proofs of these statements can be derived using resolvent estimates. As before, we will only explain how the uniform observability properties can be proved, since the admissibility ones can be derived similarly (and are straightforward anyway when the observation operator B_0 is bounded on H).

3.1 The Schrödinger equation

Sketch of the proof (Schrödinger). Here, we focus on the case of Schrödinger equations.

First, one easily checks that the Schrödinger equation (3.1) can be written similarly as (1.1)–(1.2) by setting $X = H$, $A = -iA_0$, $B = B_0$. Therefore the

admissibility and exact observability property of the continuous system can be read as a resolvent estimate: there exist positive constants m and M such that

$$\|z\|_H^2 \le M^2 \|(A_0 - \omega I)z\|_H^2 + m^2 \|B_0 z\|_Y^2, \quad \forall z \in \mathcal{D}(A_0),\ \forall \omega \in \mathbb{R}. \quad (3.17)$$

One wants to prove a uniform observability result for the semidiscrete schemes (3.7) within the class of filtered data $\mathfrak{C}_h(\varepsilon/h^\sigma)$. According to Theorem 2.2, this is equivalent to prove the existence of $m_*, M_* > 0$ such that for all $h > 0$,

$$\|z_h\|_h^2 \le M_*^2 \|(A_{0h} - \omega I)z_h\|_h^2 + m_*^2 \|B_{0h} z_h\|_Y^2, \quad \forall \omega \in \mathbb{R},\ \forall z_h \in \mathfrak{C}_h(\varepsilon/h^\sigma). \tag{3.18}$$

Studying rapidly the dependence of this estimate of ω_h, one easily checks that the minimum of the right hand-side is achieved for $\omega \in [0, \varepsilon h^{-\sigma}]$, and thus we only need to check (3.18) for $\omega \in [0, \varepsilon h^{-\sigma}]$.

We shall then explain how to prove (3.18) when (3.17) holds. This can be done in the following way.

Fix $z_h \in \mathfrak{C}_h(\epsilon/h^\sigma)$ and $\omega \in [0, \varepsilon h^{-\sigma}]$.

Then take $Z_h \in \mathcal{D}(A_0)$ such that

$$A_0 Z_h = \rho_h A_{0h} z_h. \tag{3.19}$$

Such a Z_h exists and is unique since A_0 is self-adjoint positive definite. We shall now plug Z_h into (3.17) and measure the errors terms $Z_h - \rho_h z_h$ in terms of z_h. Note that this is precisely the idea of the *a posteriori* error estimates developed, among others, by Babuska, see [1].

Using duality arguments together with the properties (3.9), one can obtain the following estimate: for any $\alpha \in [0, 1/2)$,

$$\|A_0^\alpha (Z_h - \rho_h z_h)\|_X \le C_0 h^{\theta(1-2\alpha)} \left\| A_{0h}^{1/2} z_h \right\|_h. \tag{3.20}$$
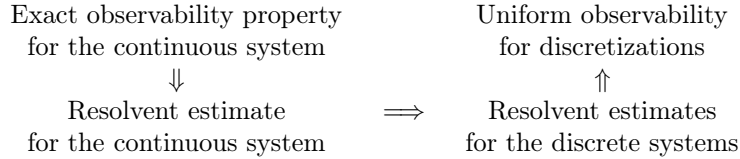
The rest of the proof follows from careful estimates on the error term and the fact that

$$\left\| A_{0h}^{1/2} z_h \right\|_h \le \sqrt{\varepsilon} h^{-\sigma/2} \|z_h\|_h, \quad \forall z_h \in \mathfrak{C}_h(\epsilon/h^\sigma).$$

The complete proof can be found in [13]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2 The wave equation

Sketch of the proof (Wave). The idea used to deal with the Schrödinger equation was to use resolvent characterizations of the observability to prove uniform observability results for the discrete Schrödinger equations. To sum up, the above proof has the following pattern:

Exact observability property for the continuous system		Uniform observability for discretizations
$\Downarrow$		$\Uparrow$
Resolvent estimate for the continuous system	$\Longrightarrow$	Resolvent estimates for the discrete systems

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We should again follow the same pattern. This time, however, since the correspondence between the wave equation (3.2) and its first order description as in (1.1) is more intricate, it is convenient to use a characterization of the observability property specially designed for second order systems:

Theorem 3.2 ([32, 41, 12]). Let A_0 be a self-adjoint positive definite operator on H with compact resolvent and $B_0 \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$. Assume that system (3.2) is admissible.

Then system (3.2) is exactly observable if and only if there exist positive constants m and M such that

$$\omega \|u\|_H^2 \le M^2 \|(A_0 - \omega I)u\|_H^2 + m^2 \omega \|Bu\|_Y^2, \quad \forall u \in \mathcal{D}(A_0), \ \forall \omega \in \mathbb{R}_+. \ (3.21)$$

Besides, the proof is constructive. In particular, if the resolvent estimate (3.21) holds, then the observability estimate (3.4) holds with constants depending explicitly on m, M and the first eigenvalue of A_0.

Note that, although Theorem 3.2 looks very similar to Theorem 2.2, its proof is completely different and requires some more technical estimates, see [12] for the complete proof.

Once Theorem 3.2 has been stated, the proof of the uniform observability property for (3.8) can be done similarly as for the discrete Schrödinger equations (3.7).

The complete proof can be found in [12]. □

3.3 From Schrödinger to the waves

When the Geometric Control Condition is satisfied, that is when the wave equation is exactly observable (see [3, 4]), it is well-known that the corresponding Schrödinger equation has better controllability properties, see e.g. [34]. In particular, it can be controlled in any arbitrary small time $T > 0$.

Similarly, in the space semidiscrete setting, this is also the case:

Theorem 3.3 ([11]). *Let A_0 be a positive definite unbounded operator with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. Assume that the approximations $(\rho_h)_{h>0}$ satisfy property (3.9). Set*

$$\varsigma = \theta \min\{2(1 - 2\kappa), 1\}. \quad (3.22)$$

Then there exists $\varepsilon > 0$ such that for all $T^* > 0$, there exist positive constants $k_*, K_* > 0$ such that, for any $h > 0$, any solution of (3.7) with initial data z_{0h}

$$z_{0h} \in \mathfrak{C}_h \left(\frac{\varepsilon}{h^\varsigma} \right) \quad (3.23)$$

satisfies (3.13).

Note that Theorem 3.3 improves Theorem 3.1 (Schrödinger) in two ways: the filtering scale is larger and the uniform observability property holds in any arbitrary small time $T > 0$.

Sketch of the proof. The proof again mimics the one in [34] in the continuous case to prove that the high-frequency solutions of the discrete Schrödinger equations, or to be more precise, the solutions whose initial data are in $\mathfrak{C}_h(\varepsilon h^{-\varsigma}) \cap \mathfrak{C}_h(K)^\perp$ for K large enough independent of $h > 0$, can be observable in any time $T > 0$.

However, the finite-dimensional argument in [34] used to derive that any solution can be observable in any time $T > 0$ cannot be used, since this is not constructive.

We shall rather use the argument developed in [20] to solve this problem. But this argument strongly uses the fact that, since there are only a finite number of eigenvalues in $[0, K]$, there exists a positive constant γ_h such that

$$\inf_{\lambda_j^h \in [0,K]} \{\lambda_{j+1}^h - \lambda_j^h\} \geq \gamma_h.$$

But there is no reason for γ_h to be bounded from below uniformly in $h > 0$ by a strictly positive constant.

We therefore need to combine the explicit construction in [20] with the convergence of the spectrum in the range $[0, K]$. The idea is the following: The continuous eigenvalues λ_j^0 are well-separated by some $\gamma > 0$. Therefore, fix $\alpha \in (0, \gamma/3)$, then, for $h > 0$ small enough, the spectrum of the operators A_{0h} satisfies:

$$\{\lambda_j^h\} \cap [0, K] \subset \cup_{\lambda_j^0 \leq K} [\lambda_j^0 - \alpha, \lambda_j^0 + \alpha].$$

Now, for any j such that $\lambda_j^0 \leq K$, set

$$X_j^h = \mathrm{Span}\{\Phi_k^h \text{ such that } \lambda_k^h \in [\lambda_j^0 - \alpha, \lambda_j^0 + \alpha]\}.$$

Because of the resolvent estimate, it is easy to check that, taking $\alpha > 0$ small enough, there exists a constant $c > 0$ such that

$$\forall j \text{ such that } \lambda_j^0 \leq K, \ \forall z_h \in X_j^h, \quad c \, \|z_h\|_h \leq \|Bz_h\|_Y.$$

The constructive argument in [20] (see also [28]) can then be adapted. The complete proof can be found in [11]. $\square$

Remark 3.4. According to Section 2.3, all these results can be combined with Theorems 2.1 and 2.3 to obtain uniform observability properties for fully-discrete approximation schemes of (3.1) and (3.2).

For instance, assuming that the continuous wave model (3.2) is admissible and exactly observable, the fully discrete wave equation

$$\begin{cases} \dfrac{u_h^{k+1} + u_h^{k-1} - 2u_h^k}{(\Delta t)^2} + A_{0h}\left(\dfrac{u_h^{k-1} + 2u_h^k + u_h^{k+1}}{4}\right) = 0, \quad k \in \mathbb{Z}, \\ \left(\dfrac{u_h^0 + u_h^1}{2}, \dfrac{u_h^1 - u_h^0}{\Delta t}\right) = (u_{0h}, u_{1h}) \in V_h^2, \end{cases} \tag{3.24}$$

observed through

$$y_h^k = B_{0h}\left(\dfrac{u_h^{k+1} - u_h^k}{\Delta t}\right)$$

is exactly observable, uniformly with respect to both discretization parameters $h > 0$ and $\Delta t > 0$ within the class

$$\left(\mathfrak{C}_h \left(\frac{\varepsilon}{h^\sigma} \right) \cap \mathfrak{C}_h \left(\frac{\delta^2}{(\Delta t)^2} \right) \right)^2 = \left(\mathfrak{C}_h \left(\min \left\{ \frac{\varepsilon}{h^\sigma}, \frac{\delta^2}{(\Delta t)^2} \right\} \right) \right)^2.$$

4   Further comments

1. In this article, we have explained how the resolvent characterizations of the exact observability property yield new results for time semidiscrete, space semidiscrete, and according to Section 2.3, fully discrete approximation schemes of conservative systems. These methods are robust and apply to a wide range of problems, including admissibility properties (that can be derived similarly, see [14, 13, 12]), controllability properties ([13, 12]) and stabilization properties ([16, 17]).

2. A widely open question consists in finding the sharp filtering scale for uniform observability property to hold for space semidiscrete approximation schemes on nonuniform meshes. We think that the works [7, 8], which present a study of the observability properties of the 1d wave equation in highly heterogeneous media, might give some insights to address this issue.

3. In this article, we assumed that the continuous systems are exactly observable. However, there are several important models of vibrations where the energy is only weakly observable. That is the case for instance for networks of vibrating strings [9] or when the *Geometric Control Condition* is not fulfilled (see [3, 30]). It would be interesting to address the observability issues for the space semi-discretizations of such systems. To our knowledge, this issue is widely open.

4. The resolvent estimates are also a well-known tool for dealing with Strichartz estimates, see [35]. Again, we think that this approach can yield new results and uniform dispersive estimates for discrete Schrödinger and wave equations, similarly as what has been done in [23, 22, 24].

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