Spectral conditions for admissibility and observability of Schrödinger systems: Applications to finite element discretizations

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Abstract

In this article, we derive uniform admissibility and observability properties for the finite element space semi-discretizations of $i\dot{z} = A_0 z$, where A_0 is an unbounded self-adjoint positive definite operator with compact resolvent. In order to address this problem, we present several spectral criteria for admissibility and observability of such systems, which will be used to derive several results for space semi-discretizations of $i\dot{z} = A_0 z$. Our approach provides very general results, which stand in any dimension and for any regular mesh (in the sense of finite elements). We also present applications to admissibility and observability for fully discrete approximation schemes, and to controllability and stabilization issues.

Keywords: Observability, Admissibility, Schrödinger equation, Finite element method, Spectral methods, Controllability, Stabilization.

1 Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A_0 : \mathcal{D}(A_0) \subset X \to X$ be an unbounded self-adjoint positive definite operator with compact resolvent. Let us consider the following abstract system:

$$i\dot{z}(t) = A_0 z(t), \quad t \in \mathbb{R}, \qquad z(0) = z_0 \in X. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, and $z = z(t)$ is the *state* of the system. Such systems are often used as models for quantum dynamics (Schrödinger equation).

Note that the system (1.1) is conservative: the energy $\|z(t)\|_X^2$ of solutions of (1.1) is constant.

In the following, we shall consider the spaces $\mathcal{D}(A_0^s)$ for $s \geq 0$, endowed with the norm $\|A_0^s(\cdot)\|_X$.

Assume that Y is another Hilbert space endowed with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X,Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ be an observation operator and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

We assume that the operator $B \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ is admissible for system (1.1) in the following sense:

Definition 1.1. The operator B is an admissible observation operator for system (1.1) if for every $T > 0$ there exists a constant $K_T > 0$ such that

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2, \qquad \forall z_0 \in \mathcal{D}(A_0), \tag{1.3}$$

for every solutions of (1.1).

Note that if B is *bounded* in X, i.e. if it can be extended in such a way that $B \in \mathfrak{L}(X,Y)$, then B is obviously an admissible observation operator, and K_T can be chosen as $K_T = T \|B\|_{\mathfrak{L}(X,Y)}^2$. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property of the solutions of the evolution equation (1.1).

The exact observability property for system (1.1)-(1.2) can be formulated as follows:

Definition 1.2. System (1.1)-(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|Bz(t)\|_Y^2 \, dt, \qquad \forall z_0 \in \mathcal{D}(A_0). \tag{1.4}$$

for every solution of (1.1).

Moreover, system (1.1)-(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that observability and admissibility issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [31]). These links will be made precise later in Section 6.

There is an extensive literature providing observability results for Schrödinger equations, by several different methods including microlocal analysis [6, 29], multipliers and Fourier series [34], etc. Our goal in this paper is to develop a theory allowing to get admissibility and observability results for space semi-discrete systems as a direct consequence of those corresponding to the continuous ones, thus avoiding technical developments in the discrete settings.

Let us now introduce the finite element method for (1.1).

Let $(V_h)_{h>0}$ be a sequence of finite-dimensional subspaces of X such that $\dim(V_h) = n_h$. We denote by π_h the embedding of V_h into X. For each $h > 0$, the inner product $< \cdot, \cdot >_X$ in X induces a structure of Hilbert space on V_h for the scalar product $< \cdot, \cdot >_h = < \pi_h \cdot, \pi_h \cdot >_X$. This obviously makes π_h continuous from V_h to X.

We assume that, for each $h > 0$, the vector space $\pi_h(V_h)$ is a subspace of $\mathcal{D}(A_0^{1/2})$. We thus define the linear operator $A_{0h} : V_h \to V_h$ by

$$< A_{0h}\phi_h, \psi_h >_h = < A_0^{1/2}\pi_h\phi_h, A_0^{1/2}\pi_h\psi_h >_X, \quad \forall (\phi_h, \psi_h) \in V_h^2. \tag{1.5}$$

The operator A_{0h} defined in (1.5) obviously is self-adjoint and positive definite. If we introduce the adjoint π_h^* of π_h, definition (1.5) reads as:

$$A_{0h} = (A_0^{1/2}\pi_h)^*(A_0^{1/2}\pi_h) = \pi_h^* A_0 \pi_h. \tag{1.6}$$

This operator A_{0h} corresponds to the finite element discretization of the operator A_0. We thus consider the following space semi-discretizations of (1.1):

$$i\dot{z}_h = A_{0h} z_h, \quad t \in \mathbb{R}, \qquad z_h(0) = z_{0h} \in V_h. \tag{1.7}$$

In this context, for all $h > 0$, the observation operator naturally becomes $B_h = B\pi_h$. Note that, when $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$, this definition always makes sense. We are thus led to impose $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$.

Note that one could have considered $B \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ and a finite element method such that $\pi_h(V_h) \subset \mathcal{D}(A_0)$. However, even in that case, the results presented below fail when $B \notin \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$, see Section 8.3.

We now make precise the assumptions we have on π_h, and which will be needed in our analysis. One easily checks that

$$\pi_h^* \pi_h = Id_{V_h}. \tag{1.8}$$

The embedding π_h describes the finite element approximation we have chosen. In particular, the vector space $\pi_h(V_h)$ approximates, in the sense given hereafter, the space $\mathcal{D}(A_0^{1/2})$: there exist $\theta > 0$ and $C_0 > 0$, such that for all $h > 0$,

$$\begin{cases} \left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \leq C_0 \left\| A_0^{1/2}\phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0^{1/2}), \\ \left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \leq C_0 h^\theta \left\| A_0\phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0). \end{cases} \tag{1.9}$$

Note that in many applications, and in particular for A_0 the Laplace operator on a bounded domain with Dirichlet boundary conditions, estimates (1.9) are satisfied for $\theta = 1$, when considering regular meshes (see [46] and Section 4).

We will not discuss convergence results for the numerical approximation schemes presented here, which are classical under assumption (1.9), and which can be found for instance in the textbook [46].

In the following, our goal is to obtain uniform observability properties for (1.7) similar to (1.4).

Let us mention that similar questions have already been investigated in [44, 30] for the finite difference approximation schemes of the beam equation, for which we expect the same admissibility and observability properties as for (1.7) to hold. To be more precise, in [30], the authors considered the finite-difference approximation scheme of the 1d beam equation on a uniform mesh, observed through the boundary value. They proved that, in this case, the observability properties do not hold uniformly in the space discretization parameter for any initial data. Though, they proved, similarly as in [24] which dealt with 1d finite difference approximation schemes of the wave equation, that one can recover uniform observability results when filtering the initial data. However, as pointed out in [44], the space semi-discrete beam equations are uniformly observable, without any filtering condition, when observed on a distributed set. But in higher dimension, Otared Kavian in [53] proposed a counterexample which proves that unique continuation properties do not hold anymore in the discrete setting due to the existence of localized high frequency solutions.

Therefore, it is natural to restrict ourselves to classes of conveniently filtered initial data. For all $h > 0$, since A_{0h} is a self-adjoint positive definite matrix, the spectrum of A_{0h} is given by a sequence of positive eigenvalues

$$0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{n_h}^h, \tag{1.10}$$

and normalized (in V_h) eigenvectors $(\Phi_j^h)_{1\leq j\leq n_h}$. For any $s > 0$, we can now define, for any $h > 0$, the filtered space

$$\mathcal{C}_h(s) = \operatorname{span}\Big\{\Phi_j^h \text{ such that the corresponding eigenvalue satisfies } |\lambda_j^h| \leq s\Big\}.$$

We are now in position to state the main results of this article:

Theorem 1.3. *Let A_0 be a self-adjoint positive definite operator with compact resolvent, and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. Assume that the maps $(\pi_h)_{h>0}$ satisfy property* (1.9). *Set*

$$\sigma = \theta \min\Big\{2(1-2\kappa), \frac{2}{3}\Big\}. \tag{1.11}$$

Admissibility: *Assume that system* (1.1)-(1.2) *is admissible.*

Then, for any $\eta > 0$ and $T > 0$, there exists a positive constant $K_{T,\eta} > 0$ such that, for any $h \in (0,1)$, any solution of (1.7) *with initial data*

$$z_{0h} \in \mathcal{C}_h(\eta/h^\sigma) \tag{1.12}$$

satisfies

$$\int_0^T \|B_h z_h(t)\|_Y^2 \, dt \leq K_{T,\eta} \|z_{0h}\|_h^2. \tag{1.13}$$

Observability: *Assume that system* (1.1)-(1.2) *is admissible and exactly observable.*

Then there exist $\epsilon > 0$, a time T^* and a positive constant $k_* > 0$ such that, for any $h \in (0,1)$, any solution of (1.7) with initial data

$$z_{0h} \in \mathcal{C}_h(\epsilon/h^\sigma) \tag{1.14}$$

satisfies

$$k_* \|z_{0h}\|_h^2 \le \int_0^{T^*} \|B_h z_h(t)\|_Y^2 \, dt. \tag{1.15}$$

This theorem is based on spectral characterizations of admissibility and exact observability for (1.1)-(1.2). In this sense, our approach is close to the frequency domain approach developed in [45] in the context of the stabilization of abstract wave equations.

For characterizing the admissibility property, we use the results in [12] to obtain an explicit characterization based on a resolvent estimate, which can also be deduced from [19] (see also [54] for a similar result in a more general setting).

For the exact observability property, we use the resolvent estimate criterion proposed in [6, 37].

The main idea, then, consists in proving uniform (in h) resolvent estimates for the operators A_{0h} and B_h, in order to recover uniform (in h) admissibility and observability estimates. This idea is completely natural since the operators A_{0h} and B_h correspond to discrete versions of A_0 and B, respectively.

Note that, in a earlier version of that work, interpreting resolvent estimates as interpolation inequalities, we found the same result but with $\sigma = \theta \min\{2(1-2\kappa), 2/5\}$ instead of (1.11). This improvement is due to a remark of Miller [38].

Theorem 1.3 has several important applications. As a straightforward corollary of the results in [12], one can thus derive observability properties for general fully discrete approximation schemes based on (1.7). Precise statements will be given in Section 5.

Besides, it also has relevant applications in control theory. Indeed, it implies that the Hilbert Uniqueness Method (see [31]) can be adapted in the discrete setting to provide efficient algorithms to compute approximations of exact controls for the continuous systems. This will be clarified in Section 6.

We will also present consequences of Theorem 1.3 to stabilization issues for space semi-discrete models based on (1.7), using the results [15].

Let us briefly comment some related works. Similar problems have been extensively studied in the last decade for various space semi-discretizations of the 1d wave equation, see for instance the review article [53] and the references therein. The numerical schemes on uniform meshes provided by finite difference and finite element methods do not have uniform observability properties, whatever the time T is, see [24] (see also [44, 30] for the beam equation). This is due to high frequency waves which do not propagate, see [50, 35]. In other words, these numerical schemes create some spurious high-frequency wave solutions which do not travel.

In this context, filtering techniques have been extensively developed. It has been proved in [24, 51] (or [30, 44] for the beam equation) that filtering the initial data removes these spurious waves, and makes possible uniform observability properties to hold. Other ways to filter these spurious waves exist, for instance using wavelet filtering approaches as in [41] or bi-grids techniques [16, 42]. However, to the best of our knowledge, these methods have been analyzed only for uniform grids in small dimensions (namely in 1d or 2d). Also note that these results prove uniform observability properties for larger classes of initial data than the ones stated here, but in more particular cases. In particular, we emphasize that Theorem 1.3 holds in any dimension and for any regular mesh.

Let us also mention that observability properties are equivalent to stabilization properties (see [20]), at least when the observation operator is bounded. Therefore, observability properties can be deduced from the literature in stabilization theory. In particular, we refer to the works [48, 47, 40, 14], which prove uniform exponential decay results for damped space semi-discrete wave equations in 1d and 2d, discretized on uniform meshes using finite difference methods, in which a numerical viscosity term has been added. Again, these results are better than the ones derived here, but apply in the more restrictive context of 1d or 2d wave equations on uniform meshes. Similar results have also been proved in [45] in a general context close to ours, but for bounded observation operators. Besides, in [45], a non trivial spectral condition on A_0 is needed, which reduces the scope of applications mainly to 1d equations.

To the best of our knowledge, there are very few papers dealing with nonuniform meshes. A first step in this direction can be found in the context of the stabilization of the 1d wave equation in [45]: indeed, stabilization properties are equivalent (see [20]) to observability properties for the corresponding conservative systems. The results in [45] can therefore be applied to 1d wave equation on nonuniform meshes to derive uniform observability results within the class of data filtered at the scale $h^{-\theta}$. Though, they strongly use a spectral gap condition on the eigenvalues of the operator, which does not hold for the wave equation in higher dimension. Another result in this direction is presented in [10], again in the context of the 1d wave equation, but discretized using a mixed finite element method as in [2, 7]. However, in [10], the results are based on a precise description of the spectrum, and in particular on a spectral gap condition on the eigenvalues.

We shall also mention recent works on spectral characterizations of the exact observability property for abstract conservative systems. We refer to [6, 37] for a very general approach for linear conservative systems, which yields a necessary and sufficient spectral condition for the exact observability property. Let us also mention the article [43], in which a spectral characterization of the exact observability property based on wave packets is given. We also point out the recent article [4], which considers several (weak) observability properties given as interpolation properties, which are close to the ones that we will prove in the present work.

We also mention the recent work [12] which proved admissibility and observability estimates for general time semi-discrete conservative linear systems. In

[12], a very general approach is given, which allows to deal with a large class of time-discrete approximation schemes. This approach is based, as here, on a spectral characterization of the exact observability property for conservative linear systems (namely the one in [6, 37]). Later on in [15] (see also [13]), the stabilization properties of time-discrete approximation schemes of damped systems were studied. In particular, [15] introduces time-discretizations which are guaranteed to enjoy uniform stabilization properties.

Let us also emphasize that the results in Theorem 1.3 may not be sharp, in view of the results in [44, 30], which can be adapted to the finite element space semi-discretization of the 1d Schrödinger equation to prove that the sharp filtering scale, in 1d and on uniform meshes, is h^{-2}. In the general setting presented here, we do not have any conjecture on the sharp filtering scale, although the counterexample of Kavian in [53] shows that filtering the data is necessary in general. This question deserves further work.

We shall finally mention the works [27, 28] which study the control properties of discrete approximations of abstract *parabolic* equations.

This article is organized as follows:

In Section 2, we present several spectral conditions which are equivalent to the admissibility and exact observability properties for abstract systems taking the form (1.1)-(1.2). In Section 3, we prove Theorem 1.3. In Section 4, we provide some examples of applications of Theorem 1.3. In Section 5, we consider admissibility and exact observability properties for fully discrete approximation schemes of (1.7). In Section 6, some applications of Theorem 1.3 in controllability theory are indicated. In Section 7, we also present applications to stabilization theory. We finally present some further comments and open questions.

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2 Spectral methods

This section recalls and presents various spectral characterizations of the admissibility and exact observability properties for abstract systems such as (1.1)-(1.2). Here, we do not deal with the discrete approximation schemes (1.7).

To state our results properly, we introduce some notations.

When dealing with the abstract system (1.1)-(1.2), it is convenient to introduce the spectrum of the operator A_0. Since A_0 is self-adjoint and positive definite, its spectrum is given by a sequence of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty, \tag{2.1}$$

and normalized (in X) eigenvectors $(\Phi_j)_{j\in\mathbb{N}^*}$.

Since some of the results below extend to a larger class of systems than (1.1), we introduce the following abstract system

$$\begin{cases} \dot{z} = Az, \quad t \geq 0, \\ z(0) = z_0 \in X, \end{cases} \qquad y(t) = Bz(t), \tag{2.2}$$

where $A : \mathcal{D}(A) \subset X \to X$ is an unbounded skew-adjoint operator with compact resolvent. In particular, its spectrum is given by a sequence $(i\mu_j)_j$, where the constants μ_j are real and $|\mu_j| \to \infty$ when $j \to \infty$, and the corresponding eigenvectors $(\Psi_j)_j$ (normalized in X) constitute an orthonormal basis of X. Note that systems of the form (1.1)-(1.2) indeed are particular instances of (2.2).

This section is organized as follows.

First, we present spectral characterizations of the admissibility properties of systems (1.1)-(1.2), based on the results in [12], which we recall. Then, we present spectral characterizations of the exact observability property of systems (1.1)-(1.2), based on the articles [6, 37].

2.1 Characterizations of admissibility

2.1.1 Wave packet characterization

First, we consider the general abstract conservative equation (2.2), and recall the results in [12, Section 6]. Note that the admissibility inequality for (2.2) consists in the existence, for any $T > 0$, of a positive constant K_T such that any solution z of (2.2) satisfies

$$\int_0^T \|Bz(t)\|_Y^2 \ dt \leq K_T \|z_0\|_X^2, \quad \forall z_0 \in \mathcal{D}(A). \tag{2.3}$$

Theorem 2.1 ([12]). *Let A be a skew-adjoint unbounded operator on X with compact resolvent, and B be in $\mathfrak{L}(\mathcal{D}(A), Y)$.*

System (2.2) is admissible in the sense of (2.3) if and only if

$$\begin{cases} \textit{There exist } r > 0 \textit{ and } D > 0 \textit{ such that} \\ \textit{for all } \mu \in \mathbb{R} \textit{ and for all } z = \displaystyle\sum_{l \in J_r(\mu)} c_l \Psi_l : \quad \|Bz\|_Y \leq D \|z\|_X, \end{cases} \tag{2.4}$$

where

$$J_r(\mu) = \{l \in \mathbb{N}, \textit{ such that } |\mu_l - \mu| \leq r\}. \tag{2.5}$$

Besides, if (2.4) holds, then the constant K_T in (2.3) can be chosen as follows:

$$K_T = K_{\pi/2r} \left\lceil \frac{2rT}{\pi} \right\rceil, \quad \textit{with } K_{\pi/2r} = \frac{3\pi^4 D}{4r}. \tag{2.6}$$

To be more precise, in [12, Section 6], the estimates (2.6) are not given explicitly, but directly come from the proof of Theorem 6.1 in [12], which yields the constant

$$K_{\pi/2r} = 3D\hat{M}_r(0),$$

where $\hat{M}_r(0)$ is the Fourier transform at 0 of the function

$$M_r(t) = \frac{\pi^2}{8}\Big(\frac{\sin(rt)}{rt}\Big)^2.$$

This makes precise the constant $K_{\pi/2r}$, and the constant K_T for $T > 0$ can be obtained as a simple consequence of the semi-group property and the conservation of the energy for solutions of (2.2). In particular, for $T < \pi/2r$, one can take $K_T = K_{\pi/2r}$ (see (2.3)).

2.1.2 Resolvent characterization

In practice, when dealing with sequences of operators, whose eigenvectors may change, Theorem 2.1 is not easy to use. We therefore introduce other characterizations of admissibility of (2.2), which yield more convenient criteria.

Theorem 2.2. *Let A be a skew-adjoint (possibly) unbounded operator on X with compact resolvent, and B be in $\mathfrak{L}(\mathcal{D}(A), Y)$.*

System (2.2) is admissible in the sense of (2.3) if and only if there exist positive constants m and M such that

$$\|Bz\|_Y^2 \leq M^2 \|(A - i\omega I)z\|_X^2 + m^2 \|z\|_X^2, \quad \forall z \in \mathcal{D}(A), \forall \omega \in \mathbb{R}, \tag{2.7}$$

or, equivalently,

$$\|Bz\|_Y^2 \leq M^2 \|(A - i\omega I)z\|_X^2 + m^2 \|z\|_X^2, \quad \forall z \in \mathcal{D}(A),\ \forall \omega \in I(A), \tag{2.8}$$

where $I(A) \subset \mathbb{R}$ denotes the convex hull of $-i\Lambda(A)$, where $\Lambda(A)$ is the spectrum of A.

Besides, if (2.7) holds, then the constant K_T in (2.3) can be chosen as follows:

$$K_T = K_1 \lceil T \rceil \quad \text{with } K_1 = \frac{3\pi^3}{2}\sqrt{m^2 + M^2\frac{\pi^2}{4}}. \tag{2.9}$$

Note that a similar result can also be deduced from [19], but without stating the explicit dependences (2.9) (see also [54]).

Proof. The equivalence of (2.7) and (2.8) is due to a remark of Luc Miller [38]. For $z \in \mathcal{D}(A)$ expanded on the basis of eigenfunctions of A as $\sum_j a_j \Psi_j$, one can study the quadratic form

$$\omega \mapsto \|(A - i\omega I)z\|_X^2 = \sum_j |a_j|^2 (\omega - \mu_j)^2.$$

In particular, it has only one critical point ω_z defined by

$$\omega_z = \sum_j \frac{|a_j|^2}{\sum_k |a_k|^2} \mu_j.$$

This obviously implies that for all $z \in \mathcal{D}(A)$, $\omega_z \in I(A)$.

In particular, for $\omega \in \mathbb{R}$ and $z \in \mathcal{D}(A)$,

$$\|(A - i\omega_z I)z\|_X^2 \leq \|(A - i\omega I)z\|_X^2 , \tag{2.10}$$

and then (2.8) implies (2.7).

Assume that system (2.2) is admissible in the sense of (2.3). Then Theorem 2.1 proves the existence of constants r and D such that (2.4) holds.

We now recall the following result, which is inspired by [43], and precisely stated in [12, Lemma 6.2]:

Lemma 2.3. *Under the hypotheses of Theorem 2.2, assume that system* (2.2) *is admissible. For $\varepsilon > 0$, denote by*

$$V(\omega, \varepsilon) = \operatorname{span}\{\Psi_j \textit{ such that } |\mu_j - \omega| \leq \varepsilon\}.$$

Let us define $K(\omega, \varepsilon)$ as

$$K(\omega, \varepsilon) = \left\| B(A - i\omega I)^{-1} \right\|_{\mathfrak{L}(V(\omega,\varepsilon)^\perp, Y)} .$$

Then, for any $\varepsilon > 0$, $K(\omega, \varepsilon)$ is uniformly bounded in ω, that is

$$K(\varepsilon) = \sup_{\omega \in \mathbb{R}} K(\omega, \varepsilon) < \infty. \tag{2.11}$$

Besides, the following estimate holds

$$K(\varepsilon) \leq \sqrt{\frac{K_1}{1 - \exp(-1)}} \Big(1 + \frac{1}{\varepsilon}\Big), \tag{2.12}$$

where K_1 is the admissibility constant in (2.3) *for $T = 1$.*

Let $z \in \mathcal{D}(A)$ and $\omega \in \mathbb{R}$. Write $z = z_\omega + z_{\omega^\perp}$, with $z_\omega \in V(\omega, r)$ and $z_{\omega^\perp} \in V(\omega, r)^\perp$. Note that this decomposition is unique and that z_ω and $z_{\omega^\perp}$ are orthogonal in X, and with respect to the scalar product $< (A - i\omega I)\cdot, (A - i\omega I)\cdot >_X$. Then we have

$$\begin{aligned}
\|Bz\|_Y^2 &\leq 2\|Bz_\omega\|_Y^2 + 2\|Bz_{\omega^\perp}\|_Y^2 \\
&\leq 2D^2 \|z_\omega\|_X^2 + 2K(r)^2 \|(A - i\omega I)z_{\omega^\perp}\|_X^2 \\
&\leq 2D^2 \|z\|_X^2 + 2K(r)^2 \|(A - i\omega I)z\|_X^2 ,
\end{aligned}$$

and (2.7) is proved.

Conversely, assume that (2.7) holds. Let ε be a positive constant. Then, for all $\omega \in \mathbb{R}$, for all $z \in V(\omega, \varepsilon)$,

$$\|(A - i\omega I)z\|_X^2 \leq \varepsilon^2 \|z\|_X^2,$$

and thus we get

$$\|Bz\|_Y^2 \leq (m^2 + M^2\varepsilon^2) \|z\|_X^2.$$

Estimate (2.4) follows with $r = \varepsilon$ and $D = \sqrt{m^2 + M^2\varepsilon^2}$, and, by Theorem 2.1, this implies the admissibility of system (2.2). Taking $\varepsilon = \pi/2$, we obtain the estimate (2.9). $\square$

2.2 Characterizations of observability

We recall the results in [6, 37] concerning the observability properties for (2.2), which consist in the existence of a time T^* and a constant k_{T^*} such that any solution of (2.2) with initial date $z_0 \in \mathcal{D}(A)$ satisfies

$$k_{T^*} \|z_0\|_X^2 \leq \int_0^{T^*} \|Bz(t)\|_Y^2 \; dt. \tag{2.13}$$

Theorem 2.4 ([6, 37])**.** *Let A be a skew-adjoint unbounded operator on X with compact resolvent, and $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.*

If system (2.2) is admissible and exactly observable in time T^*, then there exist positive constants m and M such that

$$\|z\|_X^2 \leq M^2 \|(A - i\omega I)z\|_X^2 + m^2 \|Bz\|_Y^2, \quad \forall z \in \mathcal{D}(A),\ \forall \omega \in \mathbb{R}, \tag{2.14}$$

or, equivalently,

$$\|z\|_X^2 \leq M^2 \|(A - i\omega I)z\|_X^2 + m^2 \|Bz\|_Y^2, \quad \forall z \in \mathcal{D}(A),\ \forall \omega \in I(A), \tag{2.15}$$

where $I(A) \subset \mathbb{R}$ denotes the convex hull of $-i\Lambda(A)$, where $\Lambda(A)$ is the spectrum of A.

Besides, in (2.14), one can choose $m = \sqrt{2T^*/k_{T^*}}$ and $M = T^*\sqrt{K_{T^*}/k_{T^*}}$ where the constants k_{T^*} and K_{T^*} are the ones in (2.13) and (2.3) respectively.

Conversely, if (2.14) holds, then for any time $T > \pi M$, system (2.2) is exactly observable, and the constant k_T in (1.4) can be chosen as

$$k_T = \frac{1}{2m^2 T}(T^2 - \pi^2 M^2). \tag{2.16}$$

The equivalence of (2.14) and (2.15) is due to a remark of Luc Miller [38] and follows from (2.10).

2.3 Interpolation inequalities

This subsection aims at writing the above resolvent estimates (2.7) and (2.14) as interpolation inequalities when considering the abstract system (1.1)-(1.2). Although these characterizations are of no use in the proof of Theorem 1.3, they are of independent interest.

Theorem 2.5. *Assume that $A_0 : \mathcal{D}(A_0) \subset X \to X$ is an unbounded self-adjoint positive definite operator with compact resolvent, and that $B \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ for some Hilbert space Y.*

1. System (1.1)-(1.2) is admissible in the sense of (1.3) if and only if there exist positive constants α and β such that

$$\left\|A_0^{1/2} z\right\|_X^4 \leq \|z\|_X^2 \left(\|A_0 z\|_X^2 + \alpha^2 \|z\|_X^2 - \beta^2 \|Bz\|_Y^2 \right), \quad \forall z \in \mathcal{D}(A_0). \tag{2.17}$$

2. If system (1.1)-(1.2) is admissible and exactly observable, then there exist positive constants α and β such that

$$\left\|A_0^{1/2} z\right\|_X^4 \leq \|z\|_X^2 \left(\|A_0 z\|_X^2 + \alpha^2 \|Bz\|_Y^2 - \beta^2 \|z\|_X^2 \right), \quad \forall z \in \mathcal{D}(A_0). \tag{2.18}$$

Proof. These results are based on Theorems 2.2 and 2.4. The idea consists in writing the resolvent conditions (2.7) and (2.14) as the non-negativity of a polynomial of degree two in $\omega \in \mathbb{R}$.

The proofs of items 1 and 2 are very similar. We prove the first statement the other one is left to the reader.

Condition (2.7) for (1.1)-(1.2) reads as follows: there exist positive constants m and M such that

$$\|Bz\|_Y^2 \leq M^2 \|(A_0 - \omega I)z\|_X^2 + m^2 \|z\|_X^2, \quad \forall z \in \mathcal{D}(A), \forall \omega \in \mathbb{R}. \tag{2.19}$$

This is equivalent to say that, for all $z \in \mathcal{D}(A)$, the quadratic form in ω

$$\omega^2 \|z\|_X^2 - 2\omega \left\|A_0^{1/2} z\right\|_X^2 + \|A_0 z\|_X^2 + \frac{m^2}{M^2} \|z\|_X^2 - \frac{1}{M^2} \|Bz\|_Y^2$$

is nonnegative or, equivalently, that its discriminant is nonpositive, i.e.

$$\left\|A_0^{1/2} z\right\|_X^4 \leq \|z\|_X^2 \left(\|A_0 z\|_X^2 + \frac{m^2}{M^2} \|z\|_X^2 - \frac{1}{M^2} \|Bz\|_Y^2 \right).$$

This coincides with (2.17) after the identification $\alpha = m/M$ and $\beta = 1/M$. The equivalence of (2.7) and (2.17) is then straightforward. $\square$

3 Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3. To this end, we consider an unbounded self-adjoint positive definite operator A_0 with compact resolvent, and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. We also assume (1.9).

For convenience, since B is assumed to belong to $\mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, we introduce a constant K_B such that

$$\|B\phi\|_Y \leq K_B \|A_0^\kappa \phi\|_X, \quad \forall \phi \in \mathcal{D}(A_0^\kappa). \tag{3.1}$$

The proof is divided into two major parts, one analyzing the admissibility properties (1.13), and the other one the observability properties (1.15).

3.1 Admissibility

Proof of Theorem 1.3: Admissibility. Assume that system (1.1)-(1.2) is admissible. Then, from Theorem 2.2, (2.19) holds for some positive constants m and M.

The admissibility properties (1.13) are the ones corresponding to the operator $A_{0h}|_{\mathcal{C}_h(\eta/h^\sigma)}$. In view of Theorem 2.2, they are thus equivalent to the existence of positive constants M_* and m_* such that

$$\|B_h z_h\|_Y^2 \leq M_*^2 \|(A_{0h} - \omega I) z_h\|_h^2 + m_*^2 \|z_h\|_h^2, \qquad \forall z \in \mathcal{D}(A), \forall \omega \in [0, \eta/h^\sigma]. \tag{3.2}$$

To prove inequality (3.2), a natural idea would have been to choose $z = \pi_h z_h$ in (2.19). However, since we did not assume that $\pi_h(V_h) \subset \mathcal{D}(A_0)$, this cannot be done. For instance, in the case of P1 finite elements for A_0 the Laplace operator (say on $(0,1)$) with Dirichlet boundary conditions, we have $\pi_h(V_h) \cap \mathcal{D}(A_0) = \{0\}$. Actually, even if we assume $\pi_h(V_h) \subset \mathcal{D}(A_0)$, for z_h lying in a filtered class, it is not clear that the quantities $\|A_{0h} z_h\|_h$ and $\|A_0 \pi_h z_h\|_X$ are close.

Therefore, in the sequel, we fix $h > 0$, $\omega \in [0, \eta/h^\sigma]$, and $z_h \in \mathcal{C}_h(\eta/h^\sigma)$, where η is an arbitrary positive number independent of $h > 0$. We then define $Z_h \in X$ by the relation

$$A_0 Z_h = \pi_h A_{0h} z_h = \pi_h \pi_h^* A_0 \pi_h z_h. \tag{3.3}$$

Note that (3.3) defines Z_h properly, since A_0 is invertible.

Besides, $Z_h \in \mathcal{D}(A_0)$, since $A_0 Z_h$ belongs to X by (3.3). It follows that (2.17) applies and gives

$$\|B Z_h\|_Y^2 \leq M^2 \|(A_0 - \omega I) Z_h\|_X^2 + m^2 \|Z_h\|_X^2. \tag{3.4}$$

Below, we will deduce (3.2) from (3.4), by comparing each term carefully.

From the definition (3.3) of Z_h, we have

$$(A_0 - \omega) Z_h = \pi_h (A_{0h} - \omega) z_h + \omega(\pi_h z_h - Z_h). \tag{3.5}$$

We thus estimate $Z_h - \pi_h z_h$. Using (1.6) and (3.3), for all $\phi \in \mathcal{D}(A_0)$, we have:

$$\begin{aligned} < Z_h, A_0 \phi >_X &= < A_0 Z_h, \phi >_X = < \pi_h A_{0h} z_h, \phi >_X \\ &= < \pi_h \pi_h^* A_0 \pi_h z_h, \phi >_X = < A_0^{1/2} \pi_h z_h, A_0^{1/2} \pi_h \pi_h^* \phi >_X. \end{aligned} \tag{3.6}$$

In particular, this implies that

$$\begin{aligned} < (Z_h - \pi_h z_h), A_0 \phi >_X &= < Z_h, A_0 \phi >_X - < A_0^{1/2} \pi_h z_h, A_0^{1/2} \phi >_X \\ &= < A_0^{1/2} \pi_h z_h, A_0^{1/2} (\pi_h \pi_h^* - I) \phi >_X . \end{aligned}$$

Using (1.9) and the invertibility of A_0, we obtain

$$\begin{aligned} \| Z_h - \pi_h z_h \|_X &= \sup_{\substack{\phi \in \mathcal{D}(A_0), \\ \|A_0 \phi\|_X = 1}} \left\{ < (Z_h - \pi_h z_h), A_0 \phi >_X \right\} \\ &\leq \left\| A_0^{1/2} \pi_h z_h \right\|_X \sup_{\substack{\phi \in \mathcal{D}(A_0), \\ \|A_0 \phi\|_X = 1}} \left\| A_0^{1/2} (\pi_h \pi_h^* - I) \phi \right\|_X \\ &\leq C_0 h^\theta \left\| A_0^{1/2} \pi_h z_h \right\|_X . \end{aligned}$$

Besides, for any $\gamma \in [0, 1]$, in view of (1.9), interpolation properties yield

$$\left\| A_0^{1/2} (\pi_h \pi_h^* - I) \phi \right\|_X \leq C_0 h^{\theta(1-\gamma)} \left\| A_0^{1-\gamma/2} \phi \right\|_X , \quad \forall \phi \in \mathcal{D}(A_0^{1-\gamma/2}),$$

and thus, as above,

$$\begin{aligned} \left\| A_0^{\gamma/2} (Z_h - \pi_h z_h) \right\|_X &= \sup_{\substack{\phi \in \mathcal{D}(A_0^{1-\gamma/2}), \\ \left\|A_0^{1-\gamma/2} \phi\right\|_X = 1}} \{ < A_0^{\gamma/2} (Z_h - \pi_h z_h), A_0^{1-\gamma/2} \phi >_X \} \\ &\leq \left\| A_0^{1/2} \pi_h z_h \right\|_X \sup_{\substack{\phi \in \mathcal{D}(A_0^{1-\gamma/2}), \\ \left\|A_0^{1-\gamma/2} \phi\right\|_X = 1}} \left\| A_0^{1/2} (\pi_h \pi_h^* - I) \phi \right\|_X \\ &\leq C_0 h^{\theta(1-\gamma)} \left\| A_0^{1/2} \pi_h z_h \right\|_X . \end{aligned}$$

For $\gamma = 2\kappa$, we obtain

$$\| A_0^\kappa (Z_h - \pi_h z_h) \|_X \leq C_0 h^{\theta(1-2\kappa)} \left\| A_0^{1/2} \pi_h z_h \right\|_X .$$

Besides, using the definition (1.5) of A_{0h}, one easily gets that

$$\left\| A_{0h}^{1/2} \phi_h \right\|_h = \left\| A_0^{1/2} \pi_h \phi_h \right\|_X , \quad \forall \phi_h \in V_h. \tag{3.7}$$

It follows that

$$\begin{cases} \| Z_h - \pi_h z_h \|_X \leq C_0 h^\theta \left\| A_{0h}^{1/2} z_h \right\|_h , \\ \| A_0^\kappa (Z_h - \pi_h z_h) \|_X \leq C_0 h^{\theta(1-2\kappa)} \left\| A_{0h}^{1/2} z_h \right\|_h . \end{cases} \tag{3.8}$$

In particular, this implies

$$| \, \| Z_h \|_X - \| z_h \|_h | \leq C_0 h^\theta \left\| A_{0h}^{1/2} z_h \right\|_h , \tag{3.9}$$

and

$$| \, \|(A_0-\omega)Z_h\|_X - \|(A_{0h}-\omega)z_h)\|_h | \leq C_0 \omega h^\theta \left\| A_{0h}^{1/2} z_h \right\|_h , \tag{3.10}$$

Using (3.1) and the estimate (3.8), we also obtain

$$| \, \|BZ_h\|_Y - \|B_h z_h\|_Y \, | \leq K_B C_0 h^{\theta(1-2\kappa)} \left\| A_{0h}^{1/2} z_h \right\|_h . \tag{3.11}$$

Using $z_h \in \mathcal{C}_h(\eta/h^\sigma)$ and $\omega \in [0,\eta/h^\sigma]$, we deduce from (3.8)-(3.10)-(3.11) that

$$\begin{cases} \|Z_h\|_X \leq \|z_h\|_h + C_0 h^{\theta-\sigma/2}\sqrt{\eta}\, \|z_h\|_h , \\ \|(A_0-\omega)Z_h\|_X \leq \|(A_{0h}-\omega)z_h)\|_h + C_0 \eta^{3/2} h^{\theta-3\sigma/2} \|z_h\|_h \\ \|BZ_h\|_Y \geq \|B_h z_h\|_Y - K_B C_0 \sqrt{\eta} h^{\theta(1-2\kappa)-\sigma/2} \|z_h\|_h . \end{cases}$$

This yields

$$\begin{cases} \|Z_h\|_X^2 \leq 2\|z_h\|_h^2 + 2C_0^2 h^{2\theta-\sigma}\eta \|z_h\|_h^2 , \\ \|(A_0-\omega)Z_h\|_X^2 \leq \|(A_{0h}-\omega)z_h)\|_h^2 + 2C_0 \eta^3 h^{2\theta-3\sigma} \|z_h\|_h^2 , \\ \|BZ_h\|_Y^2 \geq \dfrac{1}{2}\|B_h z_h\|_Y^2 - K_B^2 C_0^2 \eta h^{2\theta(1-2\kappa)-\sigma} \|z_h\|_h^2 . \end{cases}$$

Plugging these estimates in (3.4), we obtain

$$\begin{aligned} &\frac{1}{2}\|B_h z_h\|_Y^2 \leq 2M^2 \|(A_{0h}-\omega I)z_h\|_h^2 \\ &\quad + \|z_h\|_h^2 \Big(2m^2(1+C_0^2\eta h^{2\theta-\sigma}) + 2M^2C_0^2\eta^3 h^{2\theta-3\sigma} + K_B^2 C_0^2 \eta h^{2\theta(1-2\kappa)-\sigma} \Big). \end{aligned}$$

In particular, with σ as in (1.11), from (3.4) we deduce (3.2) with

$$M_*^2 = 4M^2, \quad m_*^2 = 4m^2(1+C_0^2\eta) + 4M^2C_0^2\eta^3 + 2K_B^2C_0^2\eta,$$

uniformly with respect to $h \leq 1$.

This completes the proof of the first statement in Theorem 1.3. Also note that, using Theorem 2.2, one can get explicit estimates on the constant $K_{T,\eta}$ in (1.13). □

3.2 Observability

Proof of Theorem 1.3: Observability. Assume that system (1.1)-(1.2) is admissible and exactly observable. Then, from Theorem 2.4, there exist positive constants m and M such that (2.14) holds.

In view of Theorem 2.4, our goal is to prove that there exist positive constants m_* and M_* such that for any $h>0$, the following inequality holds:

$$\|z_h\|_h^2 \leq M_*^2 \|(A_{0h}-\omega I)z_h\|_h^2 + m_*^2 \|B_h z_h\|_Y^2 , \qquad \forall z \in \mathcal{D}(A), \forall \omega \in [0,\epsilon/h^\sigma]. \tag{3.12}$$

To prove inequality (3.12), as before, we fix $h > 0$, $\omega \in [0, \epsilon/h^\sigma]$ and $z_h \in \mathcal{C}_h(\epsilon/h^\sigma)$, where ϵ is a positive parameter independent of $h > 0$ that we will choose later on, and we introduce the element $Z_h \in X$ defined by (3.3). Again, since $A_0 Z_h$ belongs to X by (3.3), $Z_h \in \mathcal{D}(A_0)$. Then (2.18) applies and yields

$$M^2 \left\|(A_0 - \omega I) Z_h\right\|_X^2 + m^2 \left\|B Z_h\right\|_Y^2 \geq \left\|Z_h\right\|_X^2 . \tag{3.13}$$

Using $z_h \in \mathcal{C}_h(\epsilon/h^\sigma)$ and $\omega \in [0, \epsilon/h^\sigma]$, we deduce from (3.8)-(3.10)-(3.11) that

$$\begin{cases} \left\|Z_h\right\|_X \geq \left\|z_h\right\|_h - C_0 h^{\theta - \sigma/2} \sqrt{\epsilon} \left\|z_h\right\|_h , \\ \left\|(A_0 - \omega) Z_h\right\|_X \leq \left\|(A_{0h} - \omega) z_h)\right\|_h + C_0 \epsilon^{3/2} h^{\theta - 3\sigma/2} \left\|z_h\right\|_h \\ \left\|B Z_h\right\|_Y \leq \left\|B_h z_h\right\|_Y + K_B C_0 \sqrt{\eta} h^{\theta(1-2\kappa) - \sigma/2} \left\|z_h\right\|_h . \end{cases} \tag{3.14}$$

In particular, from (3.13), we deduce

$$\begin{aligned} \left\|z_h\right\|_h^2 \Big(\frac{1}{2} - C_0^2 \epsilon h^{2\theta - \sigma} - 2M^2 C_0^2 \epsilon^3 h^{2\theta - 3\sigma} - 2m^2 C_0^2 K_B^2 \epsilon h^{2\theta(1-2\kappa) - \sigma}\Big) \\ \leq 2M_*^2 \left\|(A_{0h} - \omega I) z_h\right\|_h^2 + 2m_*^2 \left\|B_h z_h\right\|_Y^2 . \end{aligned}$$

With σ as in (1.11), choosing $\epsilon > 0$ small enough such that

$$C_0^2 \epsilon + 2M^2 C_0^2 \epsilon^3 + 2m^2 C_0^2 K_B^2 \epsilon = \frac{1}{4},$$

for all $h \leq 1$, we get (3.12) with

$$M_* = 2M, \quad m_* = 2m,$$

which completes the proof of Theorem 1.3.

Also remark that Theorem 2.5 provides explicit estimates on the constants T^* and k_* in Theorem 2.4. □

Remark 3.1. In an earlier version of this work, we use the interpolation properties discussed in Theorem 2.5 to derive uniform admissibility and observability properties for (1.7), the advantage being that ω does not appear anymore in the spectral conditions (2.17) and (2.18). However, when using these criteria, one needs to estimate the difference

$$\left| \left\| A_0^{1/2} Z_h \right\|_X^2 - \left\| A_{0h}^{1/2} z_h \right\|_h^2 \right| ,$$

which makes the results obtained this way less precise. Actually, using these interpolation properties, we only managed to prove Theorem 1.3 with $\sigma = \theta \min\{2(1-2\kappa), 2/5\}$ instead of (1.11).

The remark that our proof can also be used directly on the resolvent estimates (2.7) and (2.14) is due to Miller [38].

Remark 3.2. Similar results hold when the operator A_0 only is nonnegative. This can be done without restriction with the following argument.

The function z is solution of (1.1) if and only if $z_* = z\exp(-it)$ is solution of

$$\begin{cases} i\dot{z}_* = (A_0 + Id)z_*, \quad t \geq 0, \\ z_*(0) = z_0. \end{cases} \tag{3.15}$$

The observation y in (1.2) now reads on (3.15) as $y(t) = \exp(it)Bz_*(t)$.

Thus the admissibility and observability properties for (1.1)-(1.2) are equivalent to the corresponding ones for (3.15). Also remark that $A_* = A_0 + Id$ has exactly the same domain as A_0, with equivalent norms, but now, A_* is positive definite.

Besides, when discretizing (3.15) using a finite element method, the discretized version of A_* simply is $A_{*h} = A_{0h} + Id_{V_h}$, and again, the admissibility and observability properties for (1.7) and for

$$\begin{cases} \dot{z}_{*h} = A_{0h}z_{*h} + z_{*h}, \quad t \geq 0, \\ z_{*h}(0) = z_{0h} \in V_h, \end{cases} \qquad y_h(t) = e^{it}B_h z_{*h}(t), \quad t \geq 0,$$

are equivalent.

Note that this argument can also be applied to deal with self-adjoint operators A_0 that are only bounded from below in the sense of quadratic forms.

4 Examples of applications

This section is dedicated to present some applications to Theorem 1.3, and to compare our results with the existing ones in the literature.

4.1 The 1-d case

Let us consider the classical 1d Schrödinger equation:

$$\begin{cases} i\partial_t z + \partial_{xx}^2 z = 0, & (t,x) \in \mathbb{R} \times (0,1), \\ z(t,0) = z(t,1) = 0, & t \in \mathbb{R}, \\ z(0,x) = z_0(x), & x \in (0,1). \end{cases} \tag{4.1}$$

For (a,b) a subset of $(0,1)$, we observe system (4.1) through

$$y(t,x) = z(t,x)\chi_{(a,b)}(x), \tag{4.2}$$

where $\chi_{(a,b)}$ is the characteristic function of (a,b).

This models indeed enters in the abstract framework considered in this article, by setting $A_0 = -\partial_{xx}^2$ with Dirichlet boundary conditions, and $B = \chi_{(a,b)}$. Indeed, A_0 is a self-adjoint positive definite operator with compact resolvent in $L^2(0,1)$ and of domain $H^2(0,1) \cap H_0^1(0,1)$. The operator B obviously is continuous on $L^2(0,1)$ with values in $L^2(0,1)$. The admissibility property for (4.1)-(4.2) is then straightforward.

The observability property for (4.1)-(4.2) is well-known to hold in any time $T > 0$ in 1d. This can be seen for instance using multipliers techniques [33].

To construct the space V_h, we use P1 finite elements. More precisely, for $n_h \in \mathbb{N}$, set $h = 1/(n_h+1) > 0$ and define the points $x_j = jh$ for $j \in \{0, \cdots, n_h+1\}$. We define the basis functions

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h}\right]^+, \quad \forall j \in \{1, \cdots, n_h\}.$$

Now, $V_h = \mathbb{C}^{n_h}$, and the embedding π_h simply is

$$\begin{array}{rcl} \pi_h : V_h = \mathbb{C}^{n_h} & \rightarrow & L^2(0,1) \\ z_h = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n_h} \end{pmatrix} & \mapsto & \pi_h z_h(x) = \displaystyle\sum_{j=1}^{n_h} z_j e_j(x). \end{array}$$

Usually, the resulting schemes are written as

$$\begin{cases} iM_h \dot{z}_h(t) + K_h z_h(t) = 0, & t \in \mathbb{R}, \\ z_h(0) = z_{0h}, \end{cases} \qquad y_h(t) = B\pi_h z_h(t), \quad t \in \mathbb{R}, \quad (4.3)$$

where M_h and K_h are $n_h \times n_h$ matrices defined by $(M_h)_{i,j} = \int_0^1 e_i(x) e_j(x)\, dx$ and $(K_h)_{i,j} = \int_0^1 \partial_x e_i(x) \partial_x e_j(x)\, dx$. Note that, since M_h is a Gram matrix associated to a basis, it is invertible, self-adjoint and positive definite, and thus the following defines a scalar product:

$$< \phi_h, \psi_h >_h = \phi_h^* M_h \psi_h, \quad (\phi_h, \psi_h) \in V_h^2. \qquad (4.4)$$

Besides, from the definition of M_h, one easily checks that

$$< \phi_h, \psi_h >_h = \int_0^1 \overline{\pi_h(\phi_h)(x)} \pi_h(\psi_h)(x)\, dx, \quad \forall (\phi_h, \psi_h) \in V_h^2,$$

as presented in the introduction.

Similarly, one obtains that, for all $(\phi_h, \psi_h) \in V_h^2$,

$$\begin{aligned} \phi_h^* K_h \psi_h = \phi_h^* M_h M_h^{-1} K_h \psi_h &= < \phi_h, M_h^{-1} K_h \psi_h >_h = \phi_h^* K_h M_h^{-1} M_h \psi_h \\ &= < M_h^{-1} K_h \phi_h, \psi_h >_h = \int_0^1 \overline{\partial_x(\pi_h \phi_h)(x)} \partial_x(\pi_h \psi_h)(x)\, dx, \end{aligned}$$

In other words, the operator $M_h^{-1} K_h$ coincides with the operator A_{0h} of our framework. Note that this operator indeed is self-adjoint, as expected, but with respect to the scalar product (4.4) and not with the usual hilbertian norm of $\mathbb{C}^{n_h}$.

It is by now a common feature of finite element techniques (see for instance [46]) that, in this case, estimates (1.9) hold for $\theta = 1$. We can thus apply Theorem 1.3 to systems (4.3):

Theorem 4.1. *There exist $\epsilon > 0$, a time T^* and a constant k_* such that for any $h > 0$, any solution z_h of* (4.3) *with initial data $z_{0h} \in \mathcal{C}_h(\epsilon/h^{2/3})$ satisfies* (1.15)*.*

This result is to be compared with the ones in [44, 30]: In [44], it is proved that, for the finite difference approximation schemes of the 1d beam equation observed on a subset of $(0, 1)$, uniform observability properties hold without filtering. However, as mentioned in [30], when considering boundary observation (which does not fit our setting), observability properties hold uniformly only in the filtered classes $\mathcal{C}_h(\alpha/h^2)$ for $\alpha < 4$. Though not stated in [44, 30], the same results hold for 1d space semi-discrete Schrödinger equation when discretizing on uniform meshes, thus providing better results than our approach.

Though, as we will see hereafter, we can tackle more general cases, even in 1d, for instance taking sequence of nonuniform meshes.

4.2 More general cases

Let us mention that our results also apply in more intricate cases. Let Ω be a smooth bounded domain of $\mathbb{R}^N$ for $N \in \mathbb{N}^*$, and consider

$$\begin{cases} i\partial_t z + \mathrm{div}_x(\sigma(x)\nabla_x z) = V(x)z, & (t,x) \in \mathbb{R}\times\Omega, \\ z(t,x) = 0, & (t,x) \in \mathbb{R}\times\partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{4.5}$$

where σ is a C^1 positive real valued function on $\bar{\Omega}$, and V is a real-valued nonnegative bounded function in Ω. This indeed enters in the abstract setting of (1.1) by setting $A_0 = -\mathrm{div}_x(\sigma(x)\nabla_x\cdot) + V(x)$ with Dirichlet boundary condition, which is a self-adjoint positive definite operator with compact resolvent in $L^2(\Omega)$ and of domain $H^2(\Omega) \cap H_0^1(\Omega)$.

Let ω be an open subdomain of Ω and consider the observation operator

$$y(t,x) = \chi_\omega(x)z(t,x), \quad t \in \mathbb{R}. \tag{4.6}$$

Assume that system (4.5)-(4.6) is exactly observable.

To guarantee this property to hold, one can assume for instance that the *Geometric Control Condition* (see [29, 3]) is satisfied. This condition, roughly speaking, asserts the existence of a time T^* such that all the rays of Geometric Optics enters in the observation domain in a time smaller than T^*.

But, in fact, the Schrödinger equation behaves slightly better than a wave equation from the observability point of view because of the infinite velocity of propagation [6, 29, 36]. The *Geometric Control Condition* is sufficient but not always necessary. For instance, in [25], it has been proved that when the domain Ω is a square, for any non-empty bounded open subset ω, the observability property (1.4) holds for system (1.1). Other geometries have been also dealt with, see for instance [5, 1, 6, 49].

We consider P1 finite elements on meshes $\mathcal{T}_h$. We furthermore assume that the meshes $\mathcal{T}_h$ of the domain Ω are regular in the sense of [46, Section 5].

Roughly speaking, this assumption imposes that the polyhedra in $(\mathcal{T}_h)$ are not too flat:

Definition 4.2. Let $\mathcal{T} = \cup_{K\in\mathcal{T}} K$ be a mesh of a bounded domain Ω. For each polyhedron $K \in \mathcal{T}$, we define h_K as the diameter of K and ρ_K as the maximum diameter of the spheres $S \subset K$. We then define the regularity of $\mathcal{T}$ as

$$\operatorname{Reg}(\mathcal{T}) = \sup_{K\in\mathcal{T}} \left\{ \frac{h_K}{\rho_K} \right\}.$$

A sequence of meshes $(\mathcal{T}_h)_{h>0}$ is said to be uniformly regular if

$$\sup_h \operatorname{Reg}(\mathcal{T}_h) < \infty.$$

In this case, see [46, Section 5], estimates (1.9) again hold for $\theta = 1$, and Theorem 1.3 implies:

Theorem 4.3. *Assume that system* (4.5)-(4.6) *is exactly observable. Given a sequence of meshes* $(\mathcal{T}_h)_{h>0}$ *which is uniformly regular, there exist* $\epsilon > 0$, *a time* T^* *and a constant* k_* *such that for any* $h > 0$, *any solution* z_h *of the P1 finite element approximation scheme of* (4.5) *corresponding to the mesh* $\mathcal{T}_h$ *with initial data* $z_{0h} \in \mathcal{C}_h(\epsilon/h^{2/3})$ *satisfies* (1.15).

To our knowledge, this is the first time that observability properties for space semi-discretizations of (4.5) are derived in such generality. In particular, we emphasize that the only non-trivial assumption we used is (1.9), which is needed anyway to guarantee the convergence of the numerical schemes under consideration.

5 Fully discrete approximation schemes

This section is based on the article [12], which studied observability properties of time discrete conservative linear systems. As said in [12, Section 5], this study can be combined with observability results on space semi-discrete systems to deduce observability properties for fully discrete systems. Below, we present some applications of the results in [12].

Let us consider time discretizations of (1.7) which takes the form

$$z_h^{k+1} = \mathbb{T}_{\triangle t,h} z_h^k, \quad k \in \mathbb{N}, \qquad z_h^0 = z_{0h} \in V_h. \tag{5.1}$$

Here $\triangle t > 0$ denotes the time discretization parameter, and z_h^k corresponds to an approximation of the solution z_h of (1.7) at time $t_k = k\triangle t$. The operator $\mathbb{T}_{\triangle t,h} : V_h \to V_h$ is an approximation of $\exp(-i(\triangle t)A_{0h})$.

To be more precise, we assume that there exists a smooth strictly increasing function ζ defined on an interval $(-R, R)$ (with $R \in (0, \infty]$) with values in $(-\pi, \pi)$, and such that

$$\mathbb{T}_{\triangle t,h} = \exp(-i\zeta((\triangle t)A_{0h})). \tag{5.2}$$

In particular, this assumption implies that the operator $\mathbb{T}_{\triangle t,h}$ is unitary, and then the solutions of (5.1) have constant norms. The parameter R corresponds to a frequency limit $R/\triangle t$ imposed by the time discretization method under consideration. The fact that the range of ζ is included in $(-\pi,\pi)$ reflects that one cannot measure frequencies higher than $\pi/\triangle t$ in a mesh of size $\triangle t$. The hypothesis on the strict monotonicity of ζ is a non-degeneracy condition on the group velocity (see for instance [50] and [12, Remark 4.9]) of solutions of (5.1) which is necessary to guarantee the propagation of solutions required for observability properties to hold.

We also assume

$$\lim_{\eta\to 0}\frac{\zeta(\eta)}{\eta}=1,$$

which guarantees the consistency of the time discrete schemes (5.1) with the time continuous models (1.7).

Remark that these hypotheses are usually satisfied for conservative time-discrete approximation schemes such as the midpoint discretization or the so-called fourth order Gauss method (see for instance [18] or [12, Subsection 4.2]).

Then, from [12], we get:

Theorem 5.1. *Let A_0 be an unbounded self-adjoint positive definite operator with compact resolvent on X, and $B\in\mathfrak{L}(\mathcal{D}(A_0^\kappa),Y)$, with $\kappa<1/2$.*

Assume that the maps $(\pi_h)_{h>0}$ satisfy property (1.9). *Set σ as in* (1.11).

Consider a time discrete approximation scheme characterized by a function ζ as above, and let $\delta\in(0,R)$.

Admissibility: *Assume that system* (1.1)-(1.2) *is admissible.*

Then, for any $\eta>0$ and $T>0$, there exists a positive constant $K_{T,\eta,\delta}>0$ such that, for any $h>0$ and $\triangle t>0$, any solution of (5.1) *with initial data*

$$z_{0h}\in\mathcal{C}_h(\eta/h^\sigma)\cap\mathcal{C}_h(\delta/\triangle t) \tag{5.3}$$

satisfies

$$\triangle t\sum_{k\triangle t\in[0,T]}\left\|B_h z_h^k\right\|_Y^2\le K_{T,\eta,\delta}\left\|z_{0h}\right\|_h^2. \tag{5.4}$$

Observability: *Assume that system* (1.1)-(1.2) *is admissible and exactly observable.*

Then there exist $\epsilon>0$, a time T^* and a positive constant $k_*>0$ such that, for any $h>0$ and $\triangle t>0$, any solution of (5.1) *with initial data*

$$z_{0h}\in\mathcal{C}_h(\epsilon/h^\sigma)\cap\mathcal{C}_h(\delta/\triangle t) \tag{5.5}$$

satisfies

$$k_*\left\|z_{0h}\right\|_h^2\le\triangle t\sum_{k\triangle t\in[0,T^*]}\left\|B_h z_h^k\right\|_Y^2. \tag{5.6}$$

Obviously, inequalities (5.4)-(5.6) are time discrete counterparts of (1.13)-(1.15). Remark that, as in Theorem 1.3, a filtering condition is needed, but which now depends on both time and space discretization parameters.

Also remark that if $(\triangle t)h^{-\sigma}$ is small enough, then $\mathcal{C}_h(\epsilon/h^\sigma)\cap\mathcal{C}_h(\delta/\triangle t) = \mathcal{C}_h(\epsilon/h^\sigma)$. Roughly speaking, this indicates that under the CFL type condition $(\triangle t)h^{-\sigma}\leq \delta/\epsilon$, then system (5.1) behaves, with respect to the admissibility and observability properties, similarly as the space semi-discrete equations (1.7).

6 Controllability properties

In this section, we present applications of Theorem 1.3 to controllability properties. In the sequel, we thus assume that the continuous system (1.1)-(1.2) is admissible and exactly observable.

6.1 The continuous setting

We consider the following control problem: Given $T>0$, for any $y_0\in X$, find a control $v\in L^2(0,T;Y)$ such that the solution y of

$$\dot{y} = -iA_0 y + B^* v(t), \quad t\in[0,T], \qquad y(0)=y_0, \tag{6.1}$$

satisfies

$$y(T)=0. \tag{6.2}$$

It is well-known (see for instance [31]) that the controllability issue in time T for (6.1) is equivalent to the exact observability property for (1.1)-(1.2) in time T. Indeed, these two properties are dual, and this duality can be made precise using the Hilbert Uniqueness Method (HUM in short), see [31].

Roughly speaking, the idea of HUM is to consider the set of all functions $v\in L^2(0,T;Y)$ such that the corresponding solution of (6.1) satisfies (6.2), which we will call in the sequel admissible controls for (6.1), and to select the one of minimal $L^2(0,T;Y)$ norm.

This control of minimal $L^2(0,T;Y)$ norm for (6.1), which we will denote by v_{HUM}, is characterized through the minimizer of the functional $\mathcal{J}$ defined on X by

$$\mathcal{J}(z_T) = \frac{1}{2}\int_0^T \|Bz(t)\|_Y^2 \ dt + \Re(< y_0, z(0) >_X), \tag{6.3}$$

where $\Re(\cdot)$ denotes the real part and z is the solution of

$$\dot{z} = -iA_0 z, \quad t\in[0,T], \qquad z(T)=z_T. \tag{6.4}$$

Indeed, if z_T^* is the minimizer of $\mathcal{J}$, then $v_{HUM}(t)=Bz^*(t)$, where z^* is the solution of (6.4) with initial data z_T^*.

Besides, the only admissible control v for (6.1) that can be written as $v=Bz$ for a solution z of (6.4) is the HUM control v_{HUM}. This characterization will be used in the sequel.

Note that the observability property for (1.1)-(1.2) implies the strict convexity and the coercivity of $\mathcal{J}$ and therefore guarantees the existence of a unique minimizer for $\mathcal{J}$.

6.2 The space semi-discrete setting

We are in the setting of Theorem 1.3. Therefore there exists a time T^* such that (1.15) holds for any solution of (1.7) with initial data in the filtered space $\mathcal{C}_h(\epsilon/h^\sigma)$.

Now, if we try to compute an approximation of the control v_{HUM}, a natural idea consists in computing the discrete HUM controls for discrete versions of (6.1), which provides a sequence of controls that shall converge to the HUM control v_{HUM} for (6.1). However, this method may fail due to high-frequency spurious waves created by the discretization process. We refer for instance to [53] for a detailed presentation of this fact in the context of the 1d wave equation. It is then natural to develop filtering techniques to overcome this difficulty. This is precisely the object of several articles, see for instance [42, 52, 53, 41, 17], and the methods presented below follow and adapt their approach.

We now fix $T \geq T^*$.

Following the strategy of HUM, we will introduce the adjoint problem:

$$\dot{z}_h = -iA_{0h}z_h, \quad t \in [0,T], \qquad z_h(T) = z_{Th}. \tag{6.5}$$

6.2.1 Method I

For any $h > 0$, we consider the following control problem: For any $y_{0h} \in V_h$ find $v_h \in L^2(0,T;Y)$ of minimal $L^2(0,T;Y)$ such that the solution y_h of

$$\dot{y}_h = -iA_{0h}y_h + B_h^* v_h(t), \quad t \in [0,T], \qquad y_h(0) = y_{0h}, \tag{6.6}$$

satisfies

$$P_h y_h(T) = 0, \tag{6.7}$$

where P_h is the orthogonal projection in V_h on $\mathcal{C}_h(\epsilon/h^\sigma)$.

To deal with this problem, we introduce the functional $\mathcal{J}_h$ defined for $z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma)$ by

$$\mathcal{J}_h(z_{Th}) = \frac{1}{2}\int_0^T \|B_h z_h(t)\|_Y^2 \, dt + \Re(< y_{0h}, z_h(0) >_h), \tag{6.8}$$

where z_h is the solution of (6.5) with initial data $z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma)$.

For each $h > 0$, the functional $\mathcal{J}_h$ is strictly convex and coercive (see (1.15)), and thus has a unique minimizer $z_{Th}^* \in \mathcal{C}_h(\epsilon/h^\sigma)$. Besides, we have:

Lemma 6.1. *For all $h > 0$, let $z_{Th}^* \in \mathcal{C}_h(\epsilon/h^\sigma)$ be the unique minimizer of $\mathcal{J}_h$, and denote by z_h^* the corresponding solution of* (6.5).

Then the solution of (6.6) *with $v_h = B_h z_h^*$ satisfies* (6.7).

Sketch of the proof. We present briefly the proof, which is standard (see for instance [31]).

On one hand, multiplying (6.6) by z_h solution of (6.5) with initial data z_{Th}, we get that, for all $z_{Th} \in V_h$,

$$\int_0^T < v_h(t), B_h z_h(t) >_Y \ dt + < y_{0h}, z_h(0) >_h - < y_h(T), z_h(T) >_h = 0. \quad (6.9)$$

On the other hand, the Fréchet derivative of the functional $\mathcal{J}_h$ at z_{Th}^* yields:

$$\Re\Big(\int_0^T < B_h z_h^*(t), B_h z_h(t) >_Y \ dt \Big) + \Re(< y_{0h}, z_h(0) >_h) = 0,$$
$$\forall z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma). \quad (6.10)$$

Therefore, setting $v_h = B_h z_h^*$, taking the real part of (6.9) and subtracting it to (6.10), we obtain

$$\Re(< y_h(T), z_{Th} >_h) = 0, \quad \forall z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma),$$

or, equivalently, (6.7). □

We then investigate the convergence of the discrete controls v_h obtained in Lemma 6.1.

Theorem 6.2. *Assume that the the continuous system* (1.1)-(1.2) *is admissible and exactly observable, and that $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$ with $\kappa < 1/2$. Also assume that*

$$Y_X = \Big\{ v \in Y, \ \textit{such that } B^* v \in X \Big\} \quad (6.11)$$

is dense in Y.

Let $y_0 \in X$, and consider a sequence $(y_{0h})_{h>0}$ such that y_{0h} belongs to V_h for any $h > 0$ and

$$\pi_h y_{0h} \to y_0 \quad \text{in } X. \quad (6.12)$$

Then the sequence $(v_h)_{h>0}$ of discrete controls given by Lemma 6.1 converges in $L^2(0,T;Y)$ to the HUM control v_{HUM} of (6.1).

Remark that, for $y_0 \in \mathcal{D}(A_0)$, in view of (1.9), the sequence $(y_{0h})_h = (\pi_h^* y_0)$ converges to y_0 in X in the sense of (6.12). For $y_0 \in X$, one can then find a sequence $(y_{0h})_{h>0}$ satisfying (6.12) and $y_{0h} \in V_h$ for any $h > 0$ by using the density of $\mathcal{D}(A_0)$ into X.

The technical assumption $\bar{Y}_X = Y$ on B is usually satisfied, and thus does not limit the range of applications of Theorem 6.2. Also note that when B is bounded from X to Y, the space Y_X coincides with Y and this condition is then automatically satisfied.

Proof. The proof is divided into several parts: First, we prove that the sequence $(v_h)_{h>0}$ is bounded in $L^2(0,T;Y)$. Then, we show that any weak accumulation point v of $(v_h)_{h>0}$ is an admissible control for (6.1). We then prove that v coincides with the HUM control v_{HUM} of (6.1), which also proves that there is only one accumulation point for the sequence (v_h). Finally, we prove the strong convergence of the sequence (v_h) to $v = v_{HUM}$ in $L^2(0,T;Y)$.

The discrete controls are bounded Using that z_{Th}^* minimizes $\mathcal{J}_h$, we obviously have that $\mathcal{J}_h(z_{Th}^*) \leq J_h(0) = 0$, and therefore

$$\int_0^T \|B_h z_h^*(t)\|_Y^2 \; dt \leq -2\Re(< y_{0h}, z_h^*(0) >_h) \leq 2 \|\pi_h y_{0h}\|_X \|z_h^*(0)\|_h .$$

Since T has been chosen such that the observability inequality (1.15) holds for any solution of (1.7) -or equivalently (6.5)- with initial data in $\mathcal{C}_h(\epsilon/h^\sigma)$ with a constant k_* independent of h, we get the following two inequalities:

$$k_* \|z_h^*(0)\|_h \leq 2 \|\pi_h y_{0h}\|_X , \qquad \int_0^T \|B_h z_h^*(t)\|_Y^2 \; dt \leq \frac{4}{k_*} \|\pi_h y_{0h}\|_X^2 . \tag{6.13}$$

Since $v_h = B_h z_h^*$ and the sequence $(\pi_h y_{0h})$ is convergent in X, we deduce from (6.13) that the sequence $(v_h)_{h>0}$ is bounded in $L^2(0,T;Y)$. Therefore we can extract subsequences such that the sequence $(v_h)_{h>0}$ weakly converges in $L^2(0,T;Y)$. From now on, we assume that

$$v_h \rightharpoonup v \quad \text{in } L^2(0,T;Y). \tag{6.14}$$

The weak accumulation point v is an admissible control for (6.1) Using the same duality as in (6.9), v is an admissible control for (6.1) if and only if for any solution z of (6.4), we have

$$\Re\Big(\int_0^T < v(t), Bz(t) >_Y \; dt \Big) + \Re(< y_0, z(0) >_X) = 0. \tag{6.15}$$

Since we already get from (6.10) that any solution of (6.5) with initial data $z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma)$ satisfies

$$\Re\Big(\int_0^T < v_h(t), B_h z_h(t) >_Y \; dt \Big) + \Re(< y_{0h}, z_h(0) >_h) = 0, \tag{6.16}$$

the proof of (6.15) is based on the convergence of the solutions of (6.5) to the solutions of (6.4):

Lemma 6.3. *[46, Chapter 8] Assume that $z_T \in \mathcal{D}(A_0)$, and consider a sequence $(\pi_h z_{Th})_{h>0}$ which weakly converges to z_T in $\mathcal{D}(A_0^{1/2})$.*

Then the sequence of solutions $(z_h)_{h>0}$ of (6.5) with initial data z_{Th} converges to the solution z of (6.4) with initial data z_T in the following sense:

$$\begin{aligned} &\pi_h z_h \to z \quad \text{in } C([0,T];X), \\ &\pi_h z_h \to z \quad \text{in } L^\infty(0,T;\mathcal{D}(A_0^{1/2})) \; w-*. \end{aligned} \tag{6.17}$$

Strictly speaking, the proof in [46] is dealing with the convergence of wave type equations, but it can be easily adapted to our case.

Therefore, taking $z_T \in \mathcal{D}(A_0)$, we only have to choose $z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma)$ such that $(\pi_h z_{Th}) \to z_T$ in $\mathcal{D}(A_0^{1/2})$. This can be done by choosing

$$z_{Th} = P_h \pi_h^* z_T.$$

Indeed, with this choice, we have

$$\begin{aligned}
\|\pi_h z_{Th} - z_T\|_X &\leq \|(P_h - I)\pi_h^* z_T\|_h + \|(\pi_h\pi_h^* - I)z_T\|_X \\
&\leq \frac{h^{\sigma/2}}{\sqrt{\epsilon}} \left\| A_{0h}^{1/2} \pi_h^* z_T \right\|_h + \|(\pi_h\pi_h^* - I)z_T\|_X \\
&\leq \frac{h^{\sigma/2}}{\sqrt{\epsilon}} \left\| A_0^{1/2} \pi_h \pi_h^* z_T \right\|_X + \|(\pi_h\pi_h^* - I)z_T\|_X \\
&\leq \frac{h^{\sigma/2}}{\sqrt{\epsilon}} \Big(\left\| A_0^{1/2} z_T \right\|_X + \left\| A_0^{1/2} (\pi_h\pi_h^* - I) z_T \right\|_X \Big) + \|(\pi_h\pi_h^* - I)z_T\|_X ,
\end{aligned}$$

and therefore the strong convergence of $(\pi_h z_{Th})_{h>0}$ to z_T in X follows from (1.9). Besides, using (3.7), we have that

$$\begin{aligned}
\left\| A_0^{1/2}(\pi_h z_{Th} - \pi_h\pi_h^* z_T) \right\|_X &= \left\| A_0^{1/2}\pi_h (P_h - Id_{V_h})\pi_h^* z_T \right\|_X \\
&= \left\| A_{0h}^{1/2} (P_h - Id_{V_h})\pi_h^* z_T \right\|_h \leq \left\| A_{0h}^{1/2} \pi_h^* z_T \right\|_h \leq \left\| A_0^{1/2} \pi_h\pi_h^* z_T \right\|_X .
\end{aligned}$$

Combined with (1.9), this indicates that the sequence $(\pi_h z_{Th})_{h>0}$ is bounded in $\mathcal{D}(A_0^{1/2})$. Since it converges strongly to z_T in X, the sequence $(\pi_h z_{Th})_{h>0}$ converges weakly to z_T in $\mathcal{D}(A_0^{1/2})$.

Applying Lemma 6.3 to this particular sequence $(z_{Th})_{h>0}$, the corresponding sequence $(z_h)_{h>0}$ of solutions of (6.5) satisfies (6.17), and for all $h > 0$, $z_{Th} \in \mathcal{C}_h(\epsilon/h^\sigma)$. In particular, the convergences (6.17) imply that the sequence $(\pi_h z_h)_{h>0}$ converges strongly to z in $C([0,T]; \mathcal{D}(A_0^\kappa))$.

Thus, for $z_T \in \mathcal{D}(A_0)$, passing to the limit when $h \to 0$ in (6.16), we obtain that (6.15) holds for solutions of (6.4) for any initial data $z_T \in \mathcal{D}(A_0)$. By density of $\mathcal{D}(A_0)$ in X, we obtain that (6.15) actually holds for any solutions of (6.4) with any initial data $z_T \in X$, and thus v is an admissible control for (6.1).

The weak limit v is the HUM control of (6.1) Here we use that the HUM control v_{HUM} is the only admissible control that can be written as $Bz(t)$ for a solution z of (6.4). Since for all $h > 0$, $v_h(t) = B\pi_h z_h^*(t)$, a natural candidate for z is the limit (in a sense we will make precise below) of the sequence z_h^*.

Here again, we will use a classical lemma on the convergence of the finite element approximation schemes:

Lemma 6.4. *[46, Chapter 8] Let z_T be in X, and consider a sequence $(z_{Th})_{h>0}$ of elements of V_h which weakly converges to z_T in X, in the sense that $(\pi_h z_{Th}) \rightharpoonup z_T$ in X.*

Then the sequence of solutions z_h of (6.5) *with initial data z_{Th} weakly converges in $L^2(0,T;X)$ to the solution z of* (6.4) *with initial data z_T. Besides, for all time $t \in [0,T]$, the sequence $(\pi_h z_h(t))_{h>0}$ weakly converges in X to $z(t)$.*

Lemma 6.4 obviously is a refined version of Lemma 6.3. Actually, it can be deduced directly from Lemma 6.3 by a duality argument.

We now apply Lemma 6.4 to z^*_{Th}: Indeed, since system (6.5) is conservative, estimate (6.13) implies that

$$\|\pi_h z^*_{Th}\|_X = \|z^*_{Th}\|_h = \|z^*_h(0)\|_h$$

is bounded, and thus, up to an extracting process, that the sequence $(\pi_h z^*_{Th})_{h>0}$ weakly converges to some $\tilde{z}^*_T$ in X.

It follows that

$$\pi_h z^*_h \rightharpoonup \tilde{z}^* \quad \text{in } L^2(0,T;X),$$

where $\tilde{z}^*$ denotes the solution of (6.4) with initial data $\tilde{z}^*_T$. Using $\bar{Y}_X = Y$, we thus obtain that

$$v_h = B\pi_h z^*_h \rightharpoonup B\tilde{z}^* \quad \text{in } L^2(0,T;Y).$$

Therefore we obtain that

$$v_h \rightharpoonup v = v_{HUM} \quad \text{in } L^2(0,T;Y), \qquad \pi_h z_h \rightharpoonup \tilde{z}^* = z^* \quad \text{in } L^2(0,T;X), \quad (6.18)$$

where z^* is the solution of (6.4) with initial data z^*_T defined as the unique minimizer of the functional $\mathcal{J}$ in (6.3).

Strong convergence Since the sequence $(v_h)_{h>0}$ weakly converges to $v = v_{HUM}$ in $L^2(0,T;Y)$, we only have to check the convergence of the $L^2(0,T;Y)$ norms.

On one hand, applying (6.15) to z^*, and recalling that $v = v_{HUM} = Bz^*$, we obtain

$$\int_0^T \|v(t)\|_Y^2 \ dt + \Re(< y_0, z^*(0) >_X) = 0.$$

On the other hand, applying (6.16) to z^*_{Th}, and recalling that $v_h = B_h z^*_h$, we obtain

$$\int_0^T \|v_h(t)\|_Y^2 \ dt + \Re(< \pi_h y_{0h}, \pi_h z^*_h(0) >_X) = 0.$$

From Lemma 6.4, the sequence $(\pi_h z^*_h(0))$ weakly converges in X to $z^*(0)$. Since the sequence $(\pi_h y_{0h})_{h>0}$ is assumed to be strongly convergent in X to y_0, we get that

$$\lim_{h\to 0} \int_0^T \|v_h(t)\|_Y^2 \ dt = \int_0^T \|v(t)\|_Y^2 \ dt,$$

and the strong convergence $v_h \to v = v_{HUM}$ in $L^2(0,T;Y)$ is proved. $\square$

6.2.2 Method II

It might seem hard to implement in practice an efficient algorithm to filter the data. We therefore recall the works [17, 53] where an alternate process is given, which uses a Tychonoff regularization of the functionals $\mathcal{J}_h$. Roughly speaking, it consists in the addition of an extra term in the functionals $\mathcal{J}_h$ which makes them coercive on the whole space V_h, uniformly with respect to h. However, for the proofs, we will require the more restrictive condition $B \in \mathfrak{L}(X, Y)$.

Let us introduce, for $h > 0$, the functional $\mathcal{J}_h^*$, defined for $z_{Th} \in V_h$ by

$$\mathcal{J}_h^*(z_{Th}) = \frac{1}{2}\int_0^T \|B_h z_h(t)\|_Y^2 \ dt + \frac{h^\sigma}{2} < A_{0h}\tilde{z}_{Th}, z_{Th} >_h \\ + \Re(< y_{0h}, z_h(0) >_h), \quad (6.19)$$

where z_h is the solution of (6.5) and $\tilde{z}_{Th}$ is the solution of

$$(Id_{V_h} + h^\sigma A_{0h})\tilde{z}_{Th} = z_{Th}. \quad (6.20)$$

This equation simply consists in an elliptic regularization of z_{Th}. The variational formulation of (6.20) is given by

$$< \pi_h\tilde{z}_{Th}, \pi_h\phi_h >_X + h^\sigma < A_0^{1/2}\pi_h\tilde{z}_{Th}, A_0^{1/2}\pi_h\phi_h >_X = < \pi_h z_{Th}, \pi_h\phi_h >_X, \\ \forall \phi_h \in V_h,$$

and thus $\tilde{z}_{Th}$ can be computed directly. To simplify the presentation, it is convenient to introduce the operator

$$\tilde{A}_{0h} = A_{0h}\Big(Id_{V_h} + h^\sigma A_{0h}\Big)^{-1}, \quad (6.21)$$

which satisfies

$$< \tilde{A}_{0h} z_{Th}, z_{Th} >_h = < A_{0h}\tilde{z}_{Th}, z_{Th} >_h = \left\|\tilde{A}_{0h}^{1/2} z_{Th}\right\|_h^2,$$

and the following two properties:

$$\begin{aligned} &\left\|h^{\sigma/2}\tilde{A}_{0h}^{1/2}\psi_h\right\|_h^2 \leq \|\psi_h\|_h^2, \quad \forall \psi_h \in V_h, \\ &\left\|h^{\sigma/2}\tilde{A}_{0h}^{1/2}\psi_h\right\|_h^2 \geq \frac{\delta}{1+\delta}\|\psi_h\|_h^2, \quad \forall \psi_h \in \mathcal{C}_h(\delta/h^\sigma)^\perp, \quad \forall \delta \geq 0. \end{aligned} \quad (6.22)$$

Note in particular, that the operator $h^\sigma \tilde{A}_{0h}$ is bounded on V_h uniformly with respect to $h > 0$. This guarantees uniform continuity properties for $\mathcal{J}_h^*$.

We now check that, for $B \in \mathfrak{L}(X, Y)$, the functionals $\mathcal{J}_h^*$ are strictly convex and uniformly coercive on V_h: Indeed, for $z_{Th} \in V_h$, Theorem 1.3 implies that any solution of (6.5) satisfies

$$k_T \|P_h z_{Th}\|_h^2 \leq \int_0^T \|B_h P_h z_h(t)\|_Y^2 \ dt.$$

It follows that

$$
\begin{aligned}
&\int_0^T \|B_h z_h(t)\|_Y^2 \ dt \\
&\geq \frac{1}{2}\int_0^T \|B_h P_h z_h(t)\|_Y^2 \ dt - \int_0^T \left\|B_h\Big(P_h - Id_{V_h}\Big)z_h(t)\right\|_Y^2 \ dt \\
&\geq \frac{k_T}{2}\|P_h z_{Th}\|_h^2 - T\|B\|_{\mathfrak{L}(X,Y)}^2 \|(P_h - Id_{V_h})z_{Th}\|_h^2 \\
&\geq \frac{k_T}{2}\|z_{Th}\|_h^2 - \Big(T\|B\|_{\mathfrak{L}(X,Y)}^2 + \frac{k_T}{2}\Big)\|(P_h - Id_{V_h})z_{Th}\|_h^2 \\
&\geq \frac{k_T}{2}\|z_{Th}\|_h^2 - \Big(T\|B\|_{\mathfrak{L}(X,Y)}^2 + \frac{k_T}{2}\Big)\frac{1+\epsilon}{\epsilon}\left\|h^{\sigma/2}\tilde{A}_{0h}^{1/2}\Big(Id_{V_h} - P_h\Big)z_{Th}\right\|_h^2 \\
&\geq \frac{k_T}{2}\|z_{Th}\|_h^2 - \Big(T\|B\|_{\mathfrak{L}(X,Y)}^2 + \frac{k_T}{2}\Big)\frac{1+\epsilon}{\epsilon}\left\|h^{\sigma/2}\tilde{A}_{0h}^{1/2}z_{Th}\right\|_h^2.
\end{aligned}
$$

This proves the uniform coercivity of the functionals $\mathcal{J}_h^*$.

Thus, for each $h>0$, $\mathcal{J}_h^*$ has a unique minimizer $Z_{Th} \in V_h$, and the uniform coercivity implies the existence of two constants C_1 and C_2 independent of $h>0$ such that, setting Z_h the solution of (6.5) with initial data Z_{Th},

$$
\|Z_h(0)\|_h^2 \leq C_1\Big(\int_0^T \|B_h Z_h(t)\|_Y^2 \ dt + h^\sigma \left\|\tilde{A}_{0h}^{1/2}Z_{Th}\right\|_h^2\Big) \leq C_2\|y_{0h}\|_h^2.
$$

Besides, setting $v_h = B_h Z_h$, the solution y_h of (6.1) satisfies

$$
y_h(T) = -h^\sigma A_{0h}\tilde{Z}_{Th} = -h^\sigma \tilde{A}_{0h} Z_{Th}.
$$

In particular, if the sequence $(\pi_h y_{0h})_{h>0}$ strongly converge to $y_0 \in X$, the same arguments as before, combined with the uniform coercivity of the functional $\mathcal{J}_h^*$, prove that the sequence (v_h) converges to v_{HUM} strongly in $L^2(0,T;Y)$.

To sum up, the following statement holds:

Theorem 6.5. *Assume that the continuous system* (1.1)-(1.2) *is admissible and exactly observable, and that $B \in \mathfrak{L}(X,Y)$.*

Let $y_0 \in X$, and consider a sequence $(y_{0h})_{h>0}$ such that y_{0h} belongs to V_h for any $h>0$ and $(\pi_h y_{0h}) \to y_0$ in X.

Then the sequence $(v_h)_{h>0}$ of discrete controls given by $v_h = B_h Z_h$, where Z_h is the solution of (6.5) *associated to the minimizer Z_{Th} of $\mathcal{J}_h^*$ (defined in* (6.19)*), converges in $L^2(0,T;Y)$ to the HUM control v_{HUM} of* (6.1)*.*

Remark 6.6. Similar results can be obtained for fully discrete approximation schemes obtained by discretizing equations (1.7) in time. In this case, the proof is based on the observability inequality (5.6) and on convergence results for the fully discrete approximation schemes, which can be found for instance in [46]. We intentionally choose to present the proof in the simpler case of the time continuous setting for simplifying the presentation.

7 Stabilization properties

This section is mainly based on the articles [15, 13], in which stabilization properties are derived for abstract linear damped systems. In this section, we assume $B \in \mathfrak{L}(X, Y)$.

7.1 The continuous setting

Consider the following damped Schr̈odinger type equations:

$$i\dot{z} = A_0 z - iB^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{7.1}$$

The energy of solutions of (7.1), defined by $E(t) = \|z(t)\|_X^2 /2$, satisfies the dissipation law

$$\frac{dE}{dt}(t) = -\|Bz(t)\|_Y^2, \quad t \geq 0. \tag{7.2}$$

System (7.1) is said to be exponentially stable if there exist two positive constants μ and ν such that

$$E(t) \leq \mu\, E(0) \exp(-\nu t), \quad t \geq 0. \tag{7.3}$$

It is by now classical (see [34, 20, 32]) that the exponential decay of the energy of solutions of (7.1) is equivalent (here the operator B is bounded on X) to the observability inequality (1.4) for solutions of (1.1)-(1.2).

7.2 The space semi-discrete setting

We now assume that system (1.1)-(1.2) is exactly observable in the sense of (1.4), or, equivalently (see [34, 20, 32]), that system (7.1) is exponentially stable.

Then, combining Theorem 1.3 and [15], we get:

Theorem 7.1. *Let A_0 be a unbounded self-adjoint with compact resolvent in X, and B be a bounded operator in $\mathfrak{L}(X, Y)$. Assume that system* (7.1) *is exponentially stable in the sense of* (7.3)*. Also assume that the hypotheses of Theorem 1.3 are satisfied, and set σ as in* (1.11).

Consider a sequence of operators $(\mathcal{V}_h)_{h>0}$ defined on V_h such that for all $h > 0$, $\mathcal{V}_h$ is self-adjoint and positive definite. Also assume that for all $h > 0$, the operators $\mathcal{V}_h$ and P_h (recall that P_h is the orthogonal projection in V_h on $\mathcal{C}_h(\epsilon/h^\sigma)$) commute, and that there exist two positive constants c and C independent of $h > 0$ such that

$$\begin{cases} h^{\sigma/2} \left\| \sqrt{\mathcal{V}_h} z_h \right\|_h \leq C \|z_h\|_h, & \forall z_h \in \mathcal{C}_h(\epsilon/h^\sigma), \\ h^{\sigma/2} \left\| \sqrt{\mathcal{V}_h} z_h \right\|_h \geq c \|z_h\|_h, & \forall z_h \in \mathcal{C}_h(\epsilon/h^\sigma)^\perp, \end{cases} \tag{7.4}$$

where ϵ is the one of Theorem 1.3 (Observability).

Then the space semi-discrete systems

$$i\dot{z}_h = A_{0h}z_h - iB_h^* B_h z_h - ih^\sigma \mathcal{V}_h z_h, \quad t \geq 0, \qquad z_h(0) = z_{0h} \in V_h, \tag{7.5}$$

are exponentially stable, uniformly with respect to the space discretization parameter $h > 0$: there exist two positive constants μ_0 and ν_0 independent of $h > 0$ such that for any $h > 0$, any solution z_h of (7.5) *satisfies*

$$\|z_h(t)\|_h \leq \mu_0 \, \|z_h(0)\|_h \, \exp(-\nu_0 t), \quad t \geq 0. \tag{7.6}$$

Note that, since we assumed B bounded on X, $\kappa = 0$ in Theorem 1.3, and then σ coincides with $2\theta/3$.

The conditions (7.4) on the viscosity operator, roughly speaking, say that the operator $h^\sigma \mathcal{V}_h$ is negligible for frequencies smaller than ϵ/h^σ and is dominant for frequencies higher than ϵ/h^σ. In other words, the viscosity operator $h^\sigma \mathcal{V}_h$ modifies significantly the dynamical properties of system (7.5) only at high frequencies.

In general, the viscosity operator is chosen as a function of A_{0h}, for instance as:

$$\mathcal{V}_{1h} = A_{0h}, \qquad \mathcal{V}_{2h} = \frac{A_{0h}}{I + h^\sigma A_{0h}}, \qquad \mathcal{V}_{3h} = h^\sigma A_{0h}^2.$$

In particular, note that the knowledge of ϵ is not needed to prove (7.4) with these choices. Here, the choice $\mathcal{V}_{2h}$ has the advantage that the operator $h^\sigma \mathcal{V}_{2h}$ is bounded on the whole of V_h rather than only on $\mathcal{C}_h(\epsilon/h^\sigma)$. Remark that this viscosity operator $\mathcal{V}_{2h}$ also coincides with the elliptic regularization operator $\tilde{A}_{0h}$ introduced in (6.20).

Remark 7.2. In [15], several time discrete approximation schemes are proposed to guarantee uniform exponential decay properties for the energy of the time semi-discrete schemes as a consequence of the exponential decay of the energy of the time continuous system. Since the results of [15] also apply to families of uniformly exponentially stable systems, one can apply them to fully discrete approximation schemes of (7.1).

8 Further comments

1. One of the interesting features of our approach is that it works in any dimension and in a very general setting. To our knowledge, this is the first work which proves in such a systematic way admissibility and observability properties for space semi-discrete approximation schemes as a consequence of the ones of the continuous setting.

2. A widely open question consists in finding the sharp filtering scale. We think that the results in [8, 9], which prove the lack of observability for the 1d wave equation in a highly heterogeneous media, might give some insights on the best results we can expect on the filtering scale.

3. Our methods and results require the observation operator B to be continuous on $\mathcal{D}(A_0^\kappa)$, with $\kappa < 1/2$. However, in several relevant applications, as for instance when dealing with the boundary observation of the Schrödinger equation (see for instance [33, 30]), this is not the case. This question deserves further work.

4. An interesting issue for Schrödinger type equations concerns their dispersive properties. To our knowledge, this question, which has been extensively studied in the last decades (see for instance [26] and the references therein), has been successfully addressed for numerical approximation schemes discretized using finite difference (or finite elements) methods on uniform meshes in dimension 1 and 2, see [22, 21, 23]. We think that, similarly as for the observability properties, one could use spectral conditions to derive uniform dispersive properties for space semi-discretizations of Schrödinger equations in a very general setting, for instance by adapting Morawetz's estimates (see [39]).

5. Following the same ideas as the ones presented here, one can derive admissibility and observability results for space semi-discretizations of wave type equations derived from the finite element method. This issue is currently investigated by the author and will be published elsewhere [11].

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