# Admissibility and observability for Wave systems: Applications to finite element approximation schemes

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**Abstract.** In this article, we derive uniform admissibility and observability properties for the finite element space semi-discretizations of $\ddot{u} + A_0 u = 0$, where $A_0$ is an unbounded self-adjoint positive definite operator with compact resolvent. To address this problem, we present a new spectral approach based on several spectral criteria for admissibility and observability of such systems. Our approach provides very general admissibility and observability results for finite element approximation schemes of $\ddot{u} + A_0 u = 0$, which stand in any dimension and for any *regular* mesh (in the sense of finite elements). Our results can be combined with previous works to derive admissibility and observability properties for full discretizations of $\ddot{u}+A_0 u = 0$. We also present applications of our results to controllability and stabilization problems.

## 1. Introduction

Let $X$ be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A_0 : \mathcal{D}(A_0) \subset X \to X$ be a self-adjoint positive definite operator with compact resolvent.

Let us consider the following abstract system:

$$\ddot{u}(t) + A_0 u(t) = 0, \quad t \in \mathbb{R}, \qquad u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{1.1}$$

where solutions are meant in the semigroup sense.

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time $t$. In (1.1), the initial state $(u_0, u_1)$ lies in $\mathfrak{X} = \mathcal{D}(A_0^{1/2}) \times X$.

Such systems are often used as models of vibrating systems (e.g., the wave and beams equations).

Note that the energy

$$E(t) = \frac{1}{2} \left\| A_0^{1/2} u(t) \right\|_X^2 + \frac{1}{2} \|\dot{u}(t)\|_X^2 \tag{1.2}$$

of solutions of (1.1) is constant.

Assume that $Y$ is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X,Y)$ the space of bounded linear operators from $X$ to $Y$, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ be an observation operator and define the output function

$$y(t) = B\dot{u}(t). \tag{1.3}$$

We assume that the operator $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ is admissible for system (1.1) in the following sense (see e.g. [41]):

**Definition 1.1.** System (1.1)-(1.3) is admissible if for every $T > 0$ there exists a constant $K_T > 0$ such that any solution of (1.1) with initial data $(u_0, u_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ satisfies:

$$\int_0^T \|B\dot{u}(t)\|_Y^2 \, dt \leq K_T \Big( \left\|A_0^{1/2} u_0\right\|_X^2 + \|u_1\|_X^2 \Big). \tag{1.4}$$

This condition guarantees that the output function in (1.3) is well-defined as a function of $L^2(0,T;Y)$ for any solution of (1.1) in the energy space.

Note that if $B$ is *bounded* on $X$, i.e. if it can be extended in such a way that $B \in \mathfrak{L}(X,Y)$, then $B$ is obviously an admissible observation operator, and $K_T$ can be chosen as $K_T = T\|B\|^2_{\mathfrak{L}(X,Y)}$. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property of the solutions of the evolution equation (1.1).

The exact observability property for system (1.1)-(1.3) can be formulated as follows:

**Definition 1.2.** System (1.1)-(1.3) is exactly observable in time $T$ if there exists $k_T > 0$ such that any solution of (1.1) with initial data $(u_0, u_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ satisfies:

$$k_T \Big( \left\|A_0^{1/2} u_0\right\|_X^2 + \|u_1\|_X^2 \Big) \leq \int_0^T \|B\dot{u}(t)\|_Y^2 \, dt. \tag{1.5}$$

Moreover, system (1.1)-(1.3) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that observability and admissibility issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [25]). These links will be clarified later in Sections 6 and 7.

There is an extensive literature providing observability results for wave and plate equations, among other models, and by various methods including microlocal analysis [2, 3], multipliers techniques [23, 33] and Carleman estimates [20, 43], etc. Our goal in this paper is to develop a theory allowing to get observability results for space semi-discrete systems as a direct consequence of those corresponding to the continuous ones, thus avoiding technical developments in the discrete setting.

One of the interesting features of the approach we will develop here is that it works in any dimension and in a very general setting. To our knowledge, this is the first work (namely with the companion paper [10]) which proves in a systematic way observability properties for space semi-discrete systems from the ones of the continuous setting.

Let us now introduce the finite element method for (1.1).

Let $(V_h)_{h>0}$ be a sequence of vector spaces of finite dimension $n_h$ which are embedded into $X$ via a linear injective map $\pi_h : V_h \to X$. For each $h \in (0,1)$, the inner product $< \cdot, \cdot >_X$ in $X$ induces a structure of Hilbert space for $V_h$ endowed by the scalar product $< \cdot, \cdot >_h = < \pi_h \cdot, \pi_h \cdot >_X$.

We assume that for each $h > 0$, the vector space $\pi_h(V_h)$ is a subspace of $\mathcal{D}(A_0^{1/2})$. We thus define the linear operator $A_{0h} : V_h \to V_h$ by

$$< A_{0h}\phi_h, \psi_h >_h = < A_0^{1/2}\pi_h\phi_h, A_0^{1/2}\pi_h\psi_h >_X, \quad \forall (\phi_h, \psi_h) \in V_h^2. \tag{1.6}$$

The operator $A_{0h}$ defined in (1.6) obviously is self-adjoint and positive definite. If we introduce the adjoint $\pi_h^*$ of $\pi_h$, definition (1.6) implies that

$$A_{0h} = \pi_h^* A_0 \pi_h. \tag{1.7}$$

This operator $A_{0h}$ corresponds to the finite element discretization of the operator $A_0$ (see [36]). We thus consider the following space semi-discretizations for (1.1):

$$\ddot{u}_h + A_{0h}u_h = 0, \quad t \geq 0, \qquad u_h(0) = u_{0h} \in V_h, \quad \dot{u}_h(0) = u_{1h} \in V_h. \tag{1.8}$$

In this context, for all $h > 0$, the observation operator naturally becomes

$$y_h(t) = B_h\dot{u}_h(t) = B\pi_h\dot{u}_h(t). \tag{1.9}$$

Note that, since $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$, this definition always makes sense since $\pi_h(V_h) \subset \mathcal{D}(A_0^{1/2})$.

We now make precise the assumptions we have, usually, on $\pi_h$, and which will be needed in our analysis. One easily checks that $\pi_h^*\pi_h = Id_{V_h}$. Besides, the embedding $\pi_h$ describes the finite element approximation we have chosen. In particular, the vector space $\pi_h(V_h)$ approximates, in the sense given hereafter, the space $\mathcal{D}(A_0^{1/2})$: There exist $\theta > 0$ and $C_0 > 0$, such that for all $h > 0$,

$$\left\{ \begin{array}{ll} \left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \leq C_0 \left\| A_0^{1/2}\phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0^{1/2}), \\ \left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \leq C_0 h^\theta \left\| A_0\phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0). \end{array} \right. \tag{1.10}$$

Note that in many applications, and in particular for $A_0$ the Laplace operator on a bounded domain with Dirichlet boundary conditions, estimates (1.10) are satisfied for $\theta = 1$ when discretizing on regular meshes (see [36] and Section 4).

We will not discuss convergence results for the numerical approximation schemes presented here, which are classical under assumption (1.10), and which can be found for instance in the textbook [36].

In the sequel, our goal is to obtain uniform admissibility and observability properties for (1.8)-(1.9) similar to (1.4) and (1.5) respectively.

Let us mention that similar questions have already been investigated in [21] for the 1d wave equation observed from the boundary on a 1d mesh. In [21], it has been proved that, for the space semi-discrete schemes derived from a finite element method for the 1d wave equation on uniform meshes (which is a particular instance of (1.1)), observability properties do not hold uniformly with respect to the discretization parameter, because of the presence of spurious high frequency solutions which do not travel. However, if the initial data are filtered in a suitable way, then observability inequalities hold uniformly with respect to the space discretization parameter. Actually, as pointed out by Otared Kavian in [45], it may even happen that unique continuation properties do not hold anymore in the discrete setting due to the existence of localized high-frequency solutions.

Therefore, it is natural to restrict ourselves to classes of suitable filtered initial data. For all $h > 0$, since $A_{0h}$ is a self-adjoint positive definite matrix, the spectrum of $A_{0h}$ is given by a sequence of positive eigenvalues

$$0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{n_h}^h, \tag{1.11}$$

and normalized (in $V_h$) eigenvectors $(\Phi_j^h)_{1 \leq j \leq n_h}$. For any $s > 0$, we can now define, for each $h > 0$, the filtered space

$$\mathcal{C}_h(s) = \operatorname{span}\Big\{\Phi_j^h \text{ such that the corresponding eigenvalue satisfies } |\lambda_j^h| \leq s\Big\}.$$

We are now in position to state the main results of this article:

**Theorem 1.3.** *Let $A_0$ be a self-adjoint positive definite operator with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. Assume that the maps $(\pi_h)_{h>0}$ satisfy property* (1.10)*. Set*

$$\sigma = \theta \min\{2(1 - 2\kappa), 1\}. \tag{1.12}$$

**Admissibility:** *Assume that system* (1.1)-(1.3) *is admissible.*

*Then, for any $\eta > 0$ and $T > 0$, there exists a positive constant $K_{T,\eta}$ such that, for any $h \in (0, 1)$, any solution of* (1.8) *with initial data*

$$(u_{0h}, u_{1h}) \in \mathcal{C}_h(\eta/h^\sigma)^2 \tag{1.13}$$

*satisfies*

$$\int_0^T \|B_h \dot{u}_h(t)\|_Y^2 \, dt \leq K_{T,\eta}\Big(\Big\|A_{0h}^{1/2} u_{0h}\Big\|_h^2 + \|u_{1h}\|_h^2\Big). \tag{1.14}$$

**Observability:** *Assume that system* (1.1)-(1.3) *is admissible and exactly observable.*

*Then there exist $\epsilon > 0$, a time $T^*$ and a positive constant $k_* > 0$ such that, for any $h \in (0, 1)$, any solution of* (1.8) *with initial data*

$$(u_{0h}, u_{1h}) \in \mathcal{C}_h(\epsilon/h^\sigma)^2 \tag{1.15}$$

*satisfies*

$$k_*\Big(\Big\|A_{0h}^{1/2} u_{0h}\Big\|_h^2 + \|u_{1h}\|_h^2\Big) \leq \int_0^{T^*} \|B_h \dot{u}_h(t)\|_Y^2 \, dt. \tag{1.16}$$

These two results are based on new spectral characterizations of admissibility and exact observability for (1.1)-(1.3).

To characterize the admissibility property, we use the results in [12, 10] to obtain a characterization based on a resolvent estimate.

Our characterization of the exact observability property is deduced from the resolvent estimates in [26, 34, 40] and the wave packet characterization obtained in [34] and made more precise in [40]. However, our approach requires explicit estimates, which, to our knowledge, cannot be found in the literature. We thus propose a new proof of the wave packet spectral characterization in [34], which yields quantitative estimates.

The main idea, then, consists in proving uniform (in $h$) resolvent estimates for the operators $A_{0h}$ and $B_h$, in order to recover uniform (in $h$) admissibility and observability estimates. This idea is completely natural since the operators $A_{0h}$ and $B_h$ correspond to discrete versions of $A_0$ and $B$, respectively.

Theorem 1.3 has several important applications. As a straightforward corollary of the results in [12], one can derive observability properties for general fully discrete approximation schemes based on (1.8). Precise statements will be given in Section 5.

Besides, it also has relevant applications in control theory. Indeed, it implies that the Hilbert Uniqueness Method (see [25]) can be adapted in the discrete setting to provide efficient algorithms to compute approximations of exact controls for the continuous systems. This will be clarified in Section 6.

In Section 7, we will also present consequences of Theorem 1.3 to stabilization issues for space semi-discrete damped models. These will be deduced from [15], which addressed this problem in a very general setting which includes our models.

In Section 8, we finally investigate observability properties for space semi-discretizations of the wave equation (1.1) observed through $y(t) = Bu(t)$ instead of (1.3), for which we can adapt the method we will develop to prove Theorem 1.3.

Let us briefly comment some related works. Similar problems have been extensively studied in the last decade for various space semi-discretizations of the 1d wave equation, see for instance the review article [45] and the references therein. The numerical schemes on uniform meshes provided by finite difference and finite element methods do not have uniform observability properties, whatever the time $T$ is ([21]). This is due to high frequency waves which do not propagate, see [39, 27]. In other words, these numerical schemes create some spurious high-frequency wave solutions which are localized.

In this context, filtering techniques have been extensively developed. It has been proved in [21, 44] that filtering the initial data removes these spurious waves, and makes possible uniform observability properties to hold. Other ways to filter these spurious waves exist, for instance using a wavelet filtering approach [31] or bi-grids techniques [16, 32]. However, to the best of our knowledge, these methods have been analyzed only for uniform grids in small dimensions (namely in 1d or

2d). Also note that these results prove uniform observability properties for larger classes of initial data than the ones stated here, but in more particular cases. In particular, Theorem 1.3 depends on neither the dimension nor the uniformity of the meshes.

Let us also mention that observability properties are equivalent to stabilization properties (see [19]), when the observation operator is bounded. Therefore, observability properties can be deduced from the literature in stabilization theory. In particular, we refer to the works [38, 37, 30, 14], which prove uniform stabilization results for damped space semi-discrete wave equations in 1d and 2d, discretized on uniform meshes using finite difference approximation schemes, in which a numerical viscosity term has been added. Again, these results are better than the ones derived here, but apply in the more restrictive context of 1d or 2d wave equations on uniform meshes. Similar results have also been proved in [35], but using a non trivial spectral condition on $A_0$, which reduces the scope of applications mainly to 1d equations.

To the best of our knowledge, there are very few paper dealing with nonuniform meshes. A first step in this direction can be found in the context of the stabilization of the 1d wave equation in [35]: Indeed, stabilization properties are equivalent (see [19]) to observability properties for the corresponding undamped systems. The results in [35] can therefore be applied to 1d wave equations on nonuniform meshes to derive uniform observability results within the class $\mathcal{C}_h(\epsilon/h^\theta)$ for $\epsilon > 0$ small enough. However, they strongly use a spectral gap condition on the eigenvalues of the operator $A_0$, which does not hold for the wave operator in dimension higher than one. In the following, we will get rid of that additional assumption and consider more general observation operator $B$.

Another result in this direction is presented in [9], in the context of the 1d wave equation discretized using a mixed finite element method as in [1, 5]. In [9], it is proved that observability properties for schemes derived from a mixed finite element method hold uniformly within a large class of nonuniform meshes.

Also remark that observability and admissibility properties have been derived recently in [10] for Schrödinger type equations discretized using finite element methods. The results in [10] are strongly based on spectral characterizations of admissibility and observability properties for abstract systems. Actually, the present work follows the investigation in [10]. The main difference is that there is no available spectral characterization of the exact observability property which gives an explicit proof of the observability inequality, with in particular explicit dependence of the constants. This point is needed when considering, as in our setting, family of operators. This then requires to design new spectral characterizations of the admissibility and exact observability properties adapted to deal with systems (1.1)-(1.3).

We shall also mention recent works on spectral characterizations of exact observability for abstract conservative linear systems. In that context, a conservative linear system simply corresponds to a system of the form $\dot{z} = \mathcal{A}z$, $\mathcal{A}$ being a skew-adjoint operator. We refer to [4, 28] for a very general approach of observability properties for such conservative linear systems, which yields a necessary and sufficient resolvent condition for exact observability to hold. Let us also mention the articles [26, 34], which derived several spectral conditions for the exact observability of wave type equations. In [34], a spectral characterization of the exact observability property based on wave packets is also given. Our approach is inspired in all these works.

We also mention the recent article [12], which proved admissibility and observability estimates for general time semi-discrete conservative linear systems. In [12], a very general approach is given, which allows to deal with a large class of time discrete approximation schemes. This approach is based, as here, on a spectral characterization of exact observability for conservative linear systems (namely the one in [4, 28]). Later on in [15] (see also [13]), the stabilization properties of time discrete approximation schemes of damped systems were studied. In particular, [15] introduces time discrete schemes which are guaranteed to enjoy uniform (in the time discretization parameter) stabilization properties.

This article is organized as follows:

In Section 2, we present several spectral conditions for admissibility and exact observability properties of abstract systems (1.1)-(1.3). In Section 3, we prove Theorem 1.3. In Section 4, we give some precise examples of applications. In Section 5, we consider admissibility and exact observability properties for fully discrete approximation schemes of (1.8). In Section 6, we present applications of Theorem 1.3 to controllability issues. In Section 7, we also present applications to stabilization theory. In Section 8, we present similar results for the wave equation (1.1) observed through $y(t) = Bu(t)$ instead of (1.3). We finally present some further comments in Section 9.

## 2. Spectral methods

This section recalls and presents various spectral characterizations of the admissibility and exact observability properties for abstract systems (1.1)-(1.3). Here, we are not dealing with the discrete approximation schemes (1.8).

To state our results properly, we introduce some notations.

When dealing with the abstract system (1.1), it is convenient to introduce the spectrum of the operator $A_0$. Since $A_0$ is self-adjoint, positive definite and with compact resolvent, its spectrum is given by a sequence of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to \infty, \tag{2.1}$$

and normalized (in $X$) eigenvectors $(\Phi_j)_{j\in\mathbb{N}^*}$.

Since some of the results below extend to a larger class of systems than (1.1)-(1.3), we also introduce the following abstract system

$$\left\{ \begin{array}{l} \dot{z} = \mathcal{A}z, \quad t \geq 0, \\ z(0) = z_0 \in \mathfrak{X}, \end{array} \right. \qquad y(t) = Cz(t), \tag{2.2}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathfrak{X} \to \mathfrak{X}$ is an unbounded skew-adjoint operator with compact resolvent and $C \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$. In particular, the spectrum of $\mathcal{A}$ is given by a sequence $(i\mu_j)_j$, where the constants $\mu_j$ are real and $|\mu_j| \to \infty$ when $j \to \infty$, and the corresponding eigenvectors $(\Psi_j)$ (normalized in $\mathfrak{X}$) constitute an orthonormal basis of $\mathfrak{X}$. Note that systems of the form (1.1)-(1.3) indeed are particular instances of (2.2).

This section is organized as follows.

First, we present spectral characterizations for the admissibility properties of systems (2.2) and (1.1)-(1.3), based on the results in [10, 12], which we will recall. Then we present spectral characterizations for the exact observability properties of systems (2.2) and (1.1)-(1.3), based on the articles [34, 26].

### 2.1. Characterizations of admissibility

Note that for (2.2), the admissibility inequality consists in the existence, for all $T > 0$, of a positive constant $K_T$ such that any solution $z$ of (2.2) with initial data $z_0 \in \mathcal{D}(\mathcal{A})$ satisfies

$$\int_0^T \|Cz(t)\|_Y^2 \, dt \le K_T \|z_0\|_{\mathfrak{X}}^2 . \tag{2.3}$$

**2.1.1. Resolvent characterization.** The following result was proved in [10, 12]:

**Theorem 2.1.** *Let $\mathcal{A}$ be a skew-adjoint operator on $\mathfrak{X}$ with compact resolvent and $C$ be in $\mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$. The following statements are equivalent:*

1. *System (2.2) is admissible.*
2. *There exist $r > 0$ and $D > 0$ such that*

$$\forall \mu \in \mathbb{R},\ \forall z = \sum_{l \in J_r(\mu)} c_l \Psi_l, \quad \|Cz\|_Y \le D \|z\|_{\mathfrak{X}} , \tag{2.4}$$

*where*

$$J_r(\mu) = \{l \in \mathbb{N},\ \text{such that } |\mu_l - \mu| \le r\}. \tag{2.5}$$

*Besides, if (2.4) holds, then system (2.2) is admissible, and the constant $K_T$ in (2.3) can be chosen as follows:*

$$K_T = K_{\pi/2r} \left\lceil \frac{2rT}{\pi} \right\rceil, \quad \text{with } K_{\pi/2r} = \frac{3\pi^4 D}{4r}. \tag{2.6}$$

3. *There exist positive constants $m$ and $M$ such that*

$$\|Cz\|_Y^2 \le M^2 \|(\mathcal{A} - i\omega I)z\|_{\mathfrak{X}}^2 + m^2 \|z\|_{\mathfrak{X}}^2 , \quad \forall z \in \mathcal{D}(\mathcal{A}), \forall \omega \in \mathbb{R}. \tag{2.7}$$

The proof of Theorem 2.1 in [10] is based on the previous work [12] which proves a wave packet characterization for the admissibility of systems (2.2).

**2.1.2. Applications to Wave type equations.** We now consider the abstract setting (1.1)-(1.3), which is a particular instance of (2.2) with $\mathfrak{X} = \mathcal{D}(A_0^{1/2}) \times X$, and

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ -A_0 & 0 \end{pmatrix}, \quad C = (\ 0\ ,\ B). \tag{2.8}$$

In particular, the domain of $\mathcal{A}$ simply is $\mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ and the conditions $C \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$ and $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ are equivalent.

**Theorem 2.2.** *Let $A_0$ be a self-adjoint positive definite operator on $X$ with compact resolvent and $B$ be in $\mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$. The following statements are equivalent:*

1. *System* (1.1)-(1.3) *is admissible in the sense of* (1.4).
2. *There exist positive constants $m$ and $M$ such that:*

$$\omega \left\| B\phi \right\|_Y^2 \leq M^2 \left\| (A_0 - \omega I)\phi \right\|_X^2 + m^2 \omega \left\| \phi \right\|_X^2, \quad \forall \phi \in \mathcal{D}(A_0), \forall \omega \in \mathbb{R}_+, \tag{2.9}$$

*or, equivalently,*

$$\omega \left\| B\phi \right\|_Y^2 \leq M^2 \left\| (A_0 - \omega I)\phi \right\|_X^2 + m^2 \omega \left\| \phi \right\|_X^2, \quad \forall \phi \in \mathcal{D}(A_0), \forall \omega \in I(A_0), \tag{2.10}$$

*where $I(A_0)$ denotes the convex hull of the spectrum of $A_0$, denoted by $\Lambda(A_0)$.*

*Besides, if* (2.10) *(or* (2.9)*) holds, then system* (1.1)-(1.3) *is admissible, and the constant $K_T$ in* (1.4) *can be chosen as follows:*

$$K_T = K_{\pi/2} \left\lceil \frac{2T}{\pi} \right\rceil, \quad \text{with } K_{\pi/2} = \frac{3\pi^4}{4\sqrt{2}} \sqrt{9M^2 + m^2}. \tag{2.11}$$

3. *There exist positive constants $\alpha$ and $\beta$ such that*

$$\left\| A_0^{1/2} \phi \right\|_X^2 + \alpha^2 \left\| B\phi \right\|_Y^2 \leq \left\| \phi \right\|_X \left\| A_0 \phi \right\|_X + \beta^2 \left\| \phi \right\|_Y^2, \quad \forall \phi \in \mathcal{D}(A_0). \tag{2.12}$$

*Besides, if* (2.12) *holds, then system* (1.1)-(1.3) *is admissible, and the constant $K_T$ in* (1.4) *can be chosen as follows:*

$$K_T = K_{\pi/2} \left\lceil \frac{2T}{\pi} \right\rceil, \quad \text{with } K_{\pi/2} = \frac{3\pi^4}{8\alpha} \sqrt{9 + 2\beta^2}. \tag{2.13}$$

The remark that estimates (2.10) and (2.9) are equivalent is due to Luc Miller [29].

*Proof.* Let us first prove that statements 1 and 2 are equivalent.

Assume that system (1.1)-(1.3) is admissible. Then, from Theorem 2.1, there exist positive constants $\tilde{m}$ and $\tilde{M}$ such that (2.7) holds:

$$\left\| Bv \right\|_Y^2 \leq \tilde{M}^2 \Big( \left\| A_0^{1/2} \big( v - i\tilde{\omega} u \big) \right\|_X^2 + \left\| A_0 u + i\tilde{\omega} v \right\|_X^2 \Big) + \tilde{m}^2 \Big( \left\| A_0^{1/2} u \right\|_X^2 + \left\| v \right\|_X^2 \Big),$$
$$\forall \tilde{\omega} \in \mathbb{R},\ \forall (u,v) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2}).$$

Taking $\phi \in \mathcal{D}(A_0)$, setting $u = \phi$ and $v = i\tilde{\omega}\phi$ in this last expression, we obtain

$$\tilde{\omega}^2 \left\| B\phi \right\|_Y^2 \leq \tilde{M}^2 \left\| (A_0 - \tilde{\omega}^2 I)\phi \right\|_X^2 + \tilde{m}^2 \Big( \tilde{\omega}^2 \left\| \phi \right\|_X^2 + \left\| A_0^{1/2} \phi \right\|_X^2 \Big). \tag{2.14}$$

Now, we shall get rid of the last term, using interpolation properties:

$$\left\| A_0^{1/2}\phi \right\|_X^2 \leq \|\phi\|_X \|A_0\phi\|_X \leq \frac{1}{4\tilde{\omega}^2} \|A_0\phi\|_X^2 + \tilde{\omega}^2 \|\phi\|_X^2 .$$

But

$$\|A_0\phi\|_X^2 \leq \left( \left\|(A_0 - \tilde{\omega}^2 I)\phi\right\|_X + \left\|\tilde{\omega}^2\phi\right\|_X \right)^2 \leq 2 \left\|(A_0 - \tilde{\omega}^2 I)\phi\right\|_X^2 + 2\tilde{\omega}^4 \|\phi\|_X^2 .$$

This yields

$$\left\| A_0^{1/2}\phi \right\|_X^2 \leq \frac{1}{2\tilde{\omega}^2} \left\|(A_0 - \tilde{\omega}^2 I)\phi\right\|_X^2 + \frac{3}{2}\tilde{\omega}^2 \|\phi\|_X^2 .$$

Setting $\omega = \tilde{\omega}^2$, we obtain (2.10) by setting

$$M^2 = \tilde{M}^2 + \frac{\tilde{m}^2}{2\lambda_1}, \qquad m^2 = \frac{5}{2}\tilde{m}^2.$$

We now show that the *a priori* weaker condition (2.10) actually is equivalent to (2.9). Fix $\phi \in \mathcal{D}(A_0)$ and expand it as $\phi = \sum a_j \Phi_j$. We then study on $\mathbb{R}_+^*$ the function

$$\omega \mapsto \frac{1}{\omega} \|(A_0 - \omega I)\phi\|_X^2 = \frac{1}{\omega} \sum_j |a_j|^2 (\lambda_j - \omega)^2. \tag{2.15}$$

It is easy to check that this function has only one critical point $\omega_\phi \in \mathbb{R}_+^*$, which is a minimum and satisfies:

$$\omega_\phi^2 \sum_j |a_j|^2 = \sum_j |a_j|^2 \lambda_j^2.$$

Since this obviously implies that for all $\phi \in \mathcal{D}(A_0)$, $\omega_\phi \in I(A_0)$, then for all $\phi \in \mathcal{D}(A_0)$ and $\omega \in \mathbb{R}_+^*$,

$$\frac{1}{\omega} \|(A_0 - \omega I)\phi\|_X^2 \geq \inf_{\omega \in I(A_0)} \left\{ \frac{1}{\omega} \|(A_0 - \omega I)\phi\|_X^2 \right\}.$$

Then, (2.10) implies (2.9) for all $\phi \in \mathcal{D}(A_0)$ and $\omega \in \mathbb{R}_+^*$. Since (2.9) obviously holds for $\omega = 0$, the equivalence between (2.10) and (2.9) is proved.

To prove that (2.9) implies the admissibility of (1.1)-(1.3), we use the wave packet criterion (2.4). Before going into the proof, let us recall that the spectrum $(i\mu_j, \Psi_j)_{j\in\mathbb{Z}^*}$ of $\mathcal{A}$ can be deduced from the spectrum $(\lambda_j, \Phi_j)_{j\in\mathbb{N}^*}$ of $A_0$ as follows:

$$\mu_{\pm j} = \pm\sqrt{\lambda_j}, \quad j \in \mathbb{N}^*, \qquad \Psi_{\pm j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{\pm 1}{i\sqrt{\lambda_j}} \Phi_j \\ \Phi_j \end{pmatrix}, \quad j \in \mathbb{N}^*. \tag{2.16}$$

Now, let $\mu$ be a real number, take $r = 1$ and consider a wave packet

$$z = \sum_{l \in J_1(\mu)} c_l \Psi_l = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{2.17}$$

For $|\mu| \geq 1$, applying (2.10) to $z_2$ for $\omega = \mu^2$, we get

$$\|Cz\|_Y^2 = \|Bz_2\|_Y^2 \leq \frac{M^2}{\mu^2} \left\|(A_0 - \mu^2 I)z_2\right\|_X^2 + m^2 \|z_2\|_X^2 .$$

But, using the explicit expansion of $z_2$, one easily checks that

$$\begin{aligned} \left\|(A_0 - \mu^2 I)z_2\right\|_X^2 &= \frac{1}{2} \sum_{|\mu_j - \mu| \leq 1} |\mu_j + \mu|^2 |\mu_j - \mu|^2 c_j^2 \\ &\leq 2(|\mu| + 1)^2 \|z_2\|_X^2 \leq 8|\mu|^2 \|z_2\|_X^2 . \end{aligned}$$

Using $\|z\|_{\mathfrak{X}}^2 = 2\|z_2\|_X^2$, we then obtain

$$\|Cz\|_Y^2 \leq \left(4M^2 + \frac{m^2}{2}\right) \|z\|_{\mathfrak{X}}^2 . \tag{2.18}$$

We now need to prove a similar estimate for $z$ as in (2.17) with $|\mu| < 1$. In this case, we apply (2.10) for $\omega = 1$, and as before, we obtain

$$\|Cz\|_Y^2 \leq M^2 \|(A_0 - I)z_2\|_X^2 + m^2 \|z_2\|_X^2 \leq \left(\frac{9M^2 + m^2}{2}\right) \|z\|_{\mathfrak{X}}^2 , \tag{2.19}$$

where we used that for $z$ as in (2.17), $\|z\|_{\mathfrak{X}}^2 = 2\|z_2\|_X^2$ and, when $|\mu| < 1$,

$$\|(A_0 - I)z_2\|_X^2 \leq 9 \|z_2\|_X^2 .$$

Combining (2.18) and (2.19), we get (2.4) for any wave packet $z$ with $r = 1$ and

$$D = \sqrt{\frac{9M^2 + m^2}{2}}.$$

The estimate (2.11) then follows from (2.6).

We now prove that statements 2 and 3 are equivalent. As in [10], the idea consists in noticing that (2.9) is equivalent to the non-negativity on $\mathbb{R}_+$ of the quadratic form

$$\omega^2 \|\phi\|_X^2 - 2\omega\left( \left\|A_0^{1/2}\phi\right\|_X^2 + \frac{1}{2M^2} \|B\phi\|_Y^2 - \frac{m^2}{2M^2} \|\phi\|_X^2 \right) + \|A_0\phi\|_X^2 ,$$

which is equivalent to (as one can easily check by studying the positivity of the quadratic form $x \mapsto ax^2 - 2bx + c$ on $\mathbb{R}_+$ for $a > 0$ and $c > 0$):

$$\left\|A_0^{1/2}\phi\right\|_X^2 + \frac{1}{2M^2} \|B\phi\|_Y^2 - \frac{m^2}{2M^2} \|\phi\|_X^2 \leq \|\phi\|_X \|A_0\phi\|_X ,$$

or, equivalently, (2.12) with $\alpha = 1/(\sqrt{2}M)$ and $\beta = m/(\sqrt{2}M)$. Conversely, if (2.12) holds, then we can take $M = 1/(\sqrt{2}\alpha)$ and $m = \beta/\alpha$ in (2.9), and this completes the proof of Theorem 2.2. $\square$

### 2.2. Characterizations of observability

We first recall the following criterion for the observability of (1.1)-(1.3):

**Theorem 2.3 ([34], see also [26]).** *Let $A_0$ be a self-adjoint positive definite operator on $X$ with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$. Assume that system (1.1)-(1.3) is admissible in the sense of (1.4).*

*Then system (1.1)-(1.3) is exactly observable if and only if there exist positive constants $m$ and $M$ such that*

$$\omega \left\|u\right\|_X^2 \leq M^2 \left\|(A_0 - \omega I)u\right\|_X^2 + m^2 \omega \left\|Bu\right\|_Y^2, \quad \forall u \in \mathcal{D}(A_0),\ \forall \omega \in \mathbb{R}_+. \quad (2.20)$$

Note that Theorem 2.3 does not any provide precise estimates on the constants in (1.5). This is due to the proof of this theorem, based on Theorem 2.4 below.

Before stating Theorem 2.4, note that for (2.2), the exact observability property consists in the existence of a time $T$ and a positive constant $k_T > 0$ such that any solution of (2.2) with initial data $z_0 \in \mathcal{D}(\mathcal{A})$ satisfies

$$k_T \left\|z_0\right\|_{\mathfrak{X}}^2 \leq \int_0^T \left\|Cz(t)\right\|_Y^2 \, dt. \quad (2.21)$$

**Theorem 2.4 ([34]).** *Let $\mathcal{A}$ be a skew-adjoint operator on $\mathfrak{X}$ with compact resolvent, and $C \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$. Assume that system (2.2) is admissible in the sense of (2.3).*

*Then system (2.2) is exactly observable if and only if there exist $\rho > 0$ and $d > 0$ such that*

$$\forall \mu \in \mathbb{R},\ \forall z = \sum_{l \in J_\rho(\mu)} c_l \Psi_l, \quad d \left\|z\right\|_{\mathfrak{X}} \leq \left\|Cz\right\|_Y, \quad (2.22)$$

*where $J_\rho(\mu)$ is as in (2.5).*

Here again, no estimates on the constants entering in (2.21) are given. Though, a non-explicit constant is given in [40], but which makes the use of Theorems 2.3 and 2.4 delicate for the applications we have in mind, which involve sequences of operators.

Therefore, we present below a new proof of the fact that (2.22) implies the exact observability of system (2.2), which yields explicit estimates in Theorem 2.3 as well. These estimates are crucial in our setting.

**A refined version of Theorem 2.4.**

**Theorem 2.5.** *Let $\mathcal{A}$ be a skew-adjoint operator on $\mathfrak{X}$ with compact resolvent, and $C \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$. Assume that system (2.2) is admissible in the sense of (2.3).*

*If (2.22) holds, then system (2.2) is exactly observable in any time $T > T^*$, for*

$$T^* = \frac{2e}{\rho}\Big(\frac{\pi}{4}\ln(L) + \frac{3\pi}{4}\Big)^{1+1/\ln(L)}, \quad (2.23)$$

*where*

$$L = \frac{2\pi}{3}\frac{K_{1/\rho}\rho}{d^2}. \quad (2.24)$$

*Besides, the constant $k_T$ in* (2.21) *can be chosen as*

$$k_T = \frac{\pi d^2}{\rho}\Big(1 - \Big(\frac{T^*}{T}\Big)^{2n^*-1}\Big), \quad \textit{where } n^* = \Big\lceil \frac{1}{2}\Big(\ln(L)+1\Big)\Big\rceil. \tag{2.25}$$

*Remark* 2.6. Note that the constant $L$ is always greater than $2\pi/3$, and then $\ln(L) > 0$. Indeed, one can consider the solution $z(t) = \exp(i\mu_1 t)\Psi_1$ of (2.2), for which we get

$$\int_0^{1/\rho} \|Cz(t)\|_Y^2 \ dt \le K_{1/\rho},$$

as a consequence of the admissibility of system (2.2), and, from (2.22),

$$\int_0^{1/\rho} \|Cz(t)\|_Y^2 \ dt \ge \int_0^{1/\rho} d^2 \|z(t)\|_{\mathfrak{X}}^2 \ dt \ge \frac{d^2}{\rho}.$$

*Proof.* Set $z_0 \in \mathfrak{X}$, and denote by $z(t)$ the solution of (2.2) with initial data $z_0$. Set

$$g(t) = \chi(t)z(t), \tag{2.26}$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a function whose Fourier transform is smooth and satisfies

$$\operatorname{Supp} \hat{\chi} \subset (-\rho, \rho). \tag{2.27}$$

Note that these conditions imply that $\chi$ is in the Schwartz class $\mathcal{S}(\mathbb{R})$ and therefore $g$ and $\hat{g}$ both are in $L^2(\mathbb{R}, \mathfrak{X})$.

We expand $z_0$ and $z(t)$ on the basis $\Psi_j$:

$$z_0 = \sum_j a_j \Psi_j, \qquad z(t) = \sum_j a_j \exp(i\mu_j t)\Psi_j. \tag{2.28}$$

One then easily check that

$$\hat{g}(\omega) = \sum_j a_j \hat{\chi}(\omega - \mu_j)\Psi_j. \tag{2.29}$$

In particular, due to the property (2.27), for all $\omega$, $\hat{g}(\omega)$ is a wave packet and therefore (2.22) implies

$$d^2 \|\hat{g}(\omega)\|_{\mathfrak{X}}^2 \le \|C\hat{g}(\omega)\|_Y^2. \tag{2.30}$$

Note that, due to the explicit expansion (2.29), we have the identity

$$\|\hat{g}(\omega)\|_{\mathfrak{X}}^2 = \sum_j |a_j|^2 |\hat{\chi}(\omega - \mu_j)|^2.$$

Then, integrating (2.30) in $\omega$, and using Parseval's identity on the right hand-side of (2.30), one easily obtains

$$d^2\Big(\int \hat{\chi}^2(\omega)d\omega\Big)\ \Big(\sum_j |a_j|^2\Big) \le \int_{\mathbb{R}} \|Cg(t)\|_Y^2 \ dt = \int_{\mathbb{R}} \chi^2(t)\, \|Cz(t)\|_Y^2 \ dt, \tag{2.31}$$

where the last equality comes from the definition (2.26) of $g$.

Now, since $\chi \in \mathcal{S}(\mathbb{R})$, we know that for each $n \in \mathbb{N}^*$, there exists a constant $c_n$ such that

$$|\chi(t)| \le c_n \frac{1}{|t|^n}, \qquad \forall t \neq 0. \tag{2.32}$$

Hence, for any time $T > 0$, using the admissibility in time $T$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}} \chi^2(t) \left\| Cz(t) \right\|_Y^2 \, dt &\le \int_{-T}^{T} \chi^2(t) \left\| Cz(t) \right\|_Y^2 \, dt + 2\Big( \sum_{k=1}^{\infty} \frac{1}{(kT)^{2n}} \Big) c_n^2 K_T \left\| z_0 \right\|_{\mathfrak{X}}^2 \\ &\le \int_{-T}^{T} \chi^2(t) \left\| Cz(t) \right\|_Y^2 \, dt + \frac{\pi^2}{3} c_n^2 \frac{1}{T^{2n}} K_T \left\| z_0 \right\|_{\mathfrak{X}}^2 . \end{aligned} \tag{2.33}$$

We therefore need to estimate $c_n$ in (2.32). Of course, one cannot expect it to be uniform in the whole Schwartz class, and it will strongly depend on the choice of $\chi$. By a scaling argument, we assume without loss of generality that

$$\chi(t) = \psi(t\rho), \quad \hat{\chi}(\omega) = \frac{1}{\rho} \hat{\psi}\Big(\frac{t}{\rho}\Big), \tag{2.34}$$

where $\psi$ belongs to the Schwartz class and satisfies Supp $\hat{\psi} \subset (-1, 1)$.

Remark that integrations by parts then yield:

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int \hat{\psi}(\omega) \exp(i\omega t) \, d\omega = \frac{1}{\sqrt{2\pi}(it)^n} \int \hat{\psi}^{(n)} \exp(i\omega t) \, d\omega.$$

Thus we obtain the following decay estimate on $\psi$:

$$|\psi(t)| \le \frac{1}{\sqrt{\pi}} \frac{1}{|t|^n} \Big( \int \left| \hat{\psi}^{(n)} \right|^2 \, d\omega \Big)^{1/2}, \quad t \in \mathbb{R}^*.$$

Therefore $\chi$ satisfies

$$|\chi(t)| \le \frac{1}{\sqrt{\pi}} \Big( \frac{1}{\rho |t|} \Big)^n \Big( \int \left| \hat{\psi}^{(n)} \right|^2 \, d\omega \Big)^{1/2}, \quad t \in \mathbb{R}^*. \tag{2.35}$$

Also note that the $L^\infty$ norm of $\chi$ can be estimated by the $L^2$ norm of $\psi$:

$$|\chi(t)| = |\psi(t\rho)| = \left| \frac{1}{\sqrt{2\pi}} \int \hat{\psi}(\omega) \exp(i\omega t \rho) \, d\omega \right| \le \frac{1}{\sqrt{\pi}} \left( \int \left| \hat{\psi} \right|^2 \, d\omega \right)^{1/2}.$$

Besides, since

$$\int |\hat{\chi}|^2 \, d\omega = \frac{1}{\rho} \int \left| \hat{\psi} \right|^2 d\omega,$$

we obtain from (2.31), (2.33) and (2.35) that

$$\begin{aligned} &\Big( \frac{1}{\rho} d^2 \int \left| \hat{\psi} \right|^2 d\omega - K_T \frac{\pi}{3} \Big( \frac{1}{\rho T} \Big)^{2n} \int \left| \hat{\psi}^{(n)} \right|^2 \, d\omega \Big) \left\| z_0 \right\|_{\mathfrak{X}}^2 \\ &\qquad \le \int_{-T}^{T} \chi^2(t) \left\| Cz(t) \right\|_Y^2 \, dt \le \frac{1}{\pi} \Big( \int \left| \hat{\psi} \right|^2 d\omega \Big) \int_{-T}^{T} \left\| Cz(t) \right\|_Y^2 \, dt. \end{aligned} \tag{2.36}$$

Let us now assume that $T\rho$ is strictly greater than 1. In this case, we can estimate $K_T$ by

$$K_T \le K_{1/\rho}(1+T\rho) \le 2K_{1/\rho}T\rho. \tag{2.37}$$

Therefore, to guarantee that the left hand side of (2.36) is positive, we only need $T\rho > 1$ and

$$T\rho > \inf_n \left\{ \left(\frac{2\pi K_{1/\rho}\rho}{3d^2}\right)^{1/(2n-1)} \inf_{\hat\psi \in \mathcal{D}(-1,1)} \left\{ \frac{\left\|\hat\psi^{(n)}\right\|_{L^2}^2}{\left\|\hat\psi\right\|_{L^2}^2} \right\}^{1/(2n-1)} \right\}. \tag{2.38}$$

We now derive an estimate on the following coefficient:

$$\gamma_n = \left( \inf_{\phi \in \mathcal{D}(-1,1)} \frac{\left\|\phi^{(n)}\right\|_{L^2}^2}{\left\|\phi\right\|_{L^2}^2} \right)^{1/2n}. \tag{2.39}$$

**Lemma 2.7.** *We have the following estimate:*

$$\gamma_n \le \frac{n\pi}{2}, \quad \forall n \in \mathbb{N}^*. \tag{2.40}$$

*Proof of Lemma* 2.7. Set $n \in \mathbb{N}^*$. Let us consider

$$\phi_n(x) = \sin\left(\frac{\pi}{2}(x+1)\right)^n,$$

which belongs to $H_0^n(-1,1)$, and which, by density, is admissible as a test function in the infimum (2.39).

Consider the Fourier development of $\phi_n$, which takes the form

$$\phi_n(x) = \sum_{k=-n}^{n} a_k \exp\left(\frac{ik\pi x}{2}\right).$$

Then we have

$$\left\|\phi_n^{(n)}\right\|_{L^2}^2 = \sum_{k=-n}^{n} |a_k|^2 \left(\frac{k\pi}{2}\right)^{2n} \le \left(\frac{n\pi}{2}\right)^{2n} \sum_{k=-n}^{n} |a_k|^2 \le \left(\frac{n\pi}{2}\right)^{2n} \|\phi_n\|_{L^2}^2.$$

Lemma 2.7 follows. □

Therefore, using the constant $L$ introduced in (2.24), we need to minimize on $\mathbb{N}$

$$f(n) = L^{1/(2n-1)} \left(\frac{n\pi}{2}\right)^{2n/(2n-1)}.$$

In $\mathbb{R}$, the infimum is attained in $\tilde n$ such that

$$2\tilde n - 1 = \ln(L) + \ln\left(\frac{\tilde n \pi}{2}\right).$$

Therefore, when $L$ is large, a good approximation of the minimizer of $f$ on $\mathbb{N}$ is given by $n^*$ as in (2.25), for which we have

$$f(n^*) \le e\left(\frac{\pi}{4}\ln(L) + \frac{3\pi}{4}\right)^{1+1/\ln(L)} = \frac{T^*\rho}{2}.$$

Choosing $n = n^*$ in (2.36) and using (2.37), we obtain that

$$\begin{aligned}\int_{-T}^{T} \|Cz(t)\|_Y^2 \ dt &\geq \frac{\pi d^2}{\rho}\Big(1 - \frac{L}{(T\rho)^{2n^*-1}}\Big(\frac{n^*\pi}{2}\Big)^{2n^*}\Big) \|z_0\|_{\mathfrak{X}}^2 \\ &\geq \frac{\pi d^2}{\rho}\Big(1 - \Big(\frac{T^*}{2T}\Big)^{2n^*-1}\Big) \|z(-T)\|_{\mathfrak{X}}^2 .\end{aligned}$$

Since the semi-group generated by (2.2) is a bijective isometry on $\mathfrak{X}$, this gives, for any $z_0 \in \mathfrak{X}$,

$$\int_0^{2T} \|Cz(t)\|_Y^2 \ dt \geq \frac{\pi d^2}{\rho}\Big(1 - \Big(\frac{T^*}{2T}\Big)^{2n^*-1}\Big) \|z_0\|_{\mathfrak{X}}^2 .$$

This completes the proof of Theorem 2.5 by replacing $2T$ by $T$. □

*Remark* 2.8. The time estimate we obtain with this strategy strongly depends on the estimate (2.40) on $\gamma_n$ defined in (2.39). To our knowledge, though this problem might seem classical, there is no precise bounds on $\gamma_n$. In particular, note that if we were able to prove that $\liminf_{n\to\infty} \gamma_n = \aleph < \infty$, then condition (2.38) would simply become $T\rho > 2\aleph$, which would be very similar to the assumptions of Ingham's Lemma [22] (see also [42] on the completeness of non harmonic Fourier series in $L^2(0, T)$).

**Application to Theorem** 2.3. We can now make precise the estimates in Theorem 2.3.

**Theorem 2.9.** *Under the assumptions of Theorem* 2.3*, assume* (2.20) *or, equivalently, that*

$$\omega \|u\|_X^2 \leq M^2 \|(A_0 - \omega I)u\|_X^2 + m^2 \omega \|Bu\|_Y^2 , \quad \forall u \in \mathcal{D}(A_0), \ \forall \omega \in I(A_0), \quad (2.41)$$

*where $I(A_0)$ is the convex hull of the spectrum $\Lambda(A_0)$ of $A_0$. Also assume that the first eigenvalue of $A_0$ satisfies $\lambda_1 \geq \gamma > 0$.*

*Set*

$$\rho = \min\Big\{\frac{\sqrt{2}}{\sqrt{5}M}, \frac{\sqrt{\gamma}}{2}\Big\}, \quad d = \frac{1}{2m}. \quad (2.42)$$

*Then system* (1.1)-(1.3) *is exactly observable in any time $T > T^*$, for $T^*$ as in* (2.23)*. Besides, the constant $k_T$ in* (1.5) *can be chosen as in* (2.25) *as an explicit expression of $T$, $m$, $M$, $\gamma$, and the admissibilty constant $K_{1/\rho}$.*

*Proof.* The fact that the resolvent conditions (2.20) and (2.41) are equivalent can be done as in the proof of Theorem 2.2 by studying the function (2.15).

The proof of Theorem 2.9 then combines the estimates given in Theorem 2.5 with the following proposition:

**Proposition 2.10.** *Let $\mathcal{A}$, $A_0$, $B$ and $C$ be related as in* (2.8)*. Under the assumptions of Theorem* 2.9*, setting $\rho$ and $d$ as in* (2.42)*, the wave packet estimate* (2.22) *holds.*

*Proof.* First, we remark that, since $\rho \leq \sqrt{\gamma}/2$, when $|\mu| < \sqrt{\gamma}/2$, the set $J_\rho(\mu)$ is empty. Besides, when $\mu \in [\sqrt{\gamma}/2, \sqrt{\lambda_1}]$, $J_\rho(\mu) \subset J_\rho(\sqrt{\lambda_1})$. Therefore we only need to prove (2.22) only for $|\mu| \geq \sqrt{\lambda_1}$. When $\Lambda(A_0)$ is bounded from above, the same argument shows that we can restrict ourselves to $\mu$ such that $\mu^2 \in I(A_0)$.

Fix $\mu$ such that $\mu^2 \in I(A_0)$, let $z$ be a wave packet

$$z = \sum_{l \in J_\rho(\mu)} c_l \Psi_l = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

In particular, we have

$$z_2 = \frac{1}{\sqrt{2}} \sum_{l \in J_\rho(\mu)} c_l \Phi_l, \quad \text{and} \quad \|z_2\|_X^2 = \frac{1}{2} \sum_{l \in J_\rho(\mu)} |c_l|^2 = \frac{1}{2} \|z\|_{\mathfrak{X}}^2 .$$

Applying (2.41) to $z_2$ and $\omega = \mu^2$, we obtain

$$\begin{aligned} \frac{1}{2} \|z\|_{\mathfrak{X}}^2 = \|z_2\|_X^2 &\leq m^2 \|Bz_2\|_Y^2 + \frac{M^2}{\mu^2} \left\|(A_0 - \mu^2) z_2\right\|_X^2 \\ &= m^2 \|Cz\|_Y^2 + \frac{M^2}{\mu^2} \left\|(A_0 - \mu^2) z_2\right\|_X^2 . \end{aligned}$$

But the last term satisfies

$$\begin{aligned} \left\|(A_0 - \mu^2) z_2\right\|_X^2 &= \frac{1}{2} \sum_{l \in J_\rho(\mu)} |c_l|^2 \Big(\mu_l^2 - \mu^2\Big)^2 \\ &\leq 2 \sum_{l \in J_\rho(\mu)} |c_l|^2 \Big(\frac{\mu_l + \mu}{2}\Big)^2 (\mu_l - \mu)^2 \\ &\leq 2\rho^2 \sum_{l \in J_\rho(\mu)} |c_l|^2 \Big(|\mu| + \frac{\rho}{2}\Big)^2 \leq \frac{25}{8} \rho^2 \mu^2 \|z\|_{\mathfrak{X}}^2 , \end{aligned}$$

where we used that, for $l \in J_\rho(\mu)$ and $|\mu| \geq \sqrt{\gamma} \geq 2\rho > 0$, $|\mu_l| \leq |\mu| + \rho \leq 3|\mu|/2$.

With the choice of $\rho$ given in (2.42), we thus obtain

$$\|z\|_{\mathfrak{X}}^2 \leq 4m^2 \|Cz\|_Y^2 ,$$

and Proposition 2.10 follows. □

Theorem 2.9 then directly follows from Theorem 2.5. □

**An interpolation criterion.** We finally deduce another criterion for the observability of wave type equations (1.1)-(1.3).

**Theorem 2.11.** *Let $A_0 : \mathcal{D}(A_0) \subset X \to X$ be a self adjoint positive definite operator with compact resolvent, and let $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ be an admissible observation operator for* (1.1)-(1.3). *Assume that there exists a positive constant $\gamma$ such that the first eigenvalue of $A_0$ is greater than $\gamma$.*

*Then system* (1.1)-(1.3) *is exactly observable if and only if there exist positive constants* $\alpha$ *and* $\beta$ *such that*

$$\left\|A_0^{1/2}u\right\|_X^2 \leq \|u\|_X \|A_0 u\|_X + \alpha^2 \|Bu\|_Y^2 - \beta^2 \|u\|_X^2, \quad \forall u \in \mathcal{D}(A_0). \tag{2.43}$$

*Besides, if* (2.43) *holds, then time* $T$ *and the constant* $k_T$ *in* (1.5) *can be chosen explicitly as functions of* $\alpha$*,* $\beta$*,* $\gamma$ *and the admissibility constants.*

*Proof.* The proof is based on Theorem 2.9. In view of Theorem 2.9, it is sufficient to prove that conditions (2.43) and (2.20) are equivalent. This can be done similarly as in the equivalence of the statements 2 and 3 in Theorem 2.2 by writing (2.20) as the nonnegativity on $\mathbb{R}_+$ of a quadratic form in $\omega$. □

## 3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Below, we assume that the assumptions of Theorem 1.3 are satisfied.

For convenience, since $B$ is assumed to belong to $\mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, we introduce a constant $K_B$ such that

$$\|B\phi\|_Y \leq K_B \|A_0^\kappa \phi\|_X, \quad \forall \phi \in \mathcal{D}(A_0^\kappa). \tag{3.1}$$

### 3.1. Admissibility

*Proof of Theorem* 1.3*: Admissibility.* Assume that system (1.1)-(1.3) is admissible. Then, from Theorem 2.2, there exist positive constants $m$ and $M$ such that (2.9) holds.

Again using Theorem 2.2, but for the operator $A_{0h}|_{\mathcal{C}_h(\eta/h^\sigma)}$, our goal is to prove the existence of positive constants $m_*$ and $M_*$ such that for any $h > 0$,

$$\|B_h u_h\|_Y^2 \leq \frac{M_*^2}{\omega} \|(A_{0h} - \omega I)u_h\|_h^2 + m_*^2 \|u_h\|_h^2, \ \forall u_h \in \mathcal{C}_h(\eta/h^\sigma), \forall \omega \in (0, \eta/h^\sigma]. \tag{3.2}$$

For $h > 0$, we fix $\omega \in (0, \eta/h^\sigma]$ and $u_h \in \mathcal{C}_h(\eta/h^\sigma)$. Similarly as in [10], we introduce $U_h \in \mathcal{D}(A_0)$, defined by

$$A_0 U_h = \pi_h \pi_h^* A_0 \pi_h u_h = \pi_h A_{0h} u_h. \tag{3.3}$$

This defines an element $U_h \in \mathcal{D}(A_0)$, which we expect to be close to $u_h$.

Since $U_h \in \mathcal{D}(A_0)$, inequality (2.10) applies:

$$\|BU_h\|_Y^2 \leq \frac{M^2}{\omega} \|(A_0 - \omega I)U_h\|_X^2 + m^2 \|U_h\|_X^2. \tag{3.4}$$

The computations below are the same as in [10]. For convenience, we recall them.

We first estimate $U_h - \pi_h u_h$. Using (1.7) and (3.3), for all $\phi \in \mathcal{D}(A_0)$, we have:

$$< U_h - \pi_h u_h, A_0\phi >_X =< A_0 U_h - A_0 \pi_h u_h \phi >_X$$
$$=< (\pi_h\pi_h^* - I)A_0\pi_h u_h, \phi >_X =< A_0^{1/2}\pi_h u_h, A_0^{1/2}(\pi_h\pi_h^* - I)\phi >_X . \quad (3.5)$$

But, for any $\delta \in [0,1]$, in view of (1.10), interpolation properties yield

$$\left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \le C_0 h^{\theta(1-\delta)} \left\| A_0^{1-\delta/2}\phi \right\|_X, \quad \forall \phi \in \mathcal{D}(A_0^{1-\delta/2}),$$

and therefore, using (3.5),

$$\begin{aligned}
\left\| A_0^{\delta/2}(U_h - \pi_h u_h) \right\|_X &= \sup_{\substack{\phi \in \mathcal{D}(A_0^{1-\delta/2}),\\ \left\| A_0^{1-\delta/2}\phi \right\|_X = 1}} \left\{ < A_0^{\delta/2}(U_h - \pi_h u_h), A_0^{1-\delta/2}\phi >_X \right\} \\
&= \sup_{\substack{\phi \in \mathcal{D}(A_0^{1-\delta/2}),\\ \left\| A_0^{1-\delta/2}\phi \right\|_X = 1}} \left\{ < (U_h - \pi_h u_h), A_0\phi >_X \right\} \\
&\le \left\| A_0^{1/2}\pi_h u_h \right\|_X \sup_{\substack{\phi \in \mathcal{D}(A_0^{1-\delta/2}),\\ \left\| A_0^{1-\delta/2}\phi \right\|_X = 1}} \left\| A_0^{1/2}(\pi_h\pi_h^* - I)\phi \right\|_X \\
&\le C_0 h^{\theta(1-\delta)} \left\| A_0^{1/2}\pi_h u_h \right\|_X .
\end{aligned}$$

Besides, using the definition (1.6) of $A_{0h}$, one easily gets

$$\left\| A_{0h}^{1/2}\phi_h \right\|_h = \left\| A_0^{1/2}\pi_h\phi_h \right\|_X, \quad \forall \phi_h \in V_h. \quad (3.6)$$

It follows that

$$\left\{ \begin{array}{l} \|U_h - \pi_h u_h\|_X \le C_0 h^\theta \left\| A_{0h}^{1/2} u_h \right\|_h, \\ \|A_0^\kappa (U_h - \pi_h u_h)\|_X \le C_0 h^{\theta(1-2\kappa)} \left\| A_{0h}^{1/2} u_h \right\|_h . \end{array} \right. \quad (3.7)$$

Remark that, from (3.3), $(A_0 - \omega)U_h = \pi_h(A_{0h} - \omega)u_h + \omega(\pi_h u_h - U_h)$. It follows that

$$\Big| \|(A_0 - \omega)U_h\|_X - \|(A_{0h} - \omega)u_h\|_h \Big| \le C_0 h^\theta \omega \left\| A_{0h}^{1/2} u_h \right\|_h . \quad (3.8)$$

Besides, using (3.1) and (3.7), we obtain

$$\Big| \|BU_h\|_Y - \|B_h u_h\|_Y \Big| \le K_B C_0 h^{\theta(1-2\kappa)} \left\| A_{0h}^{1/2} u_h \right\|_h . \quad (3.9)$$

Estimates (3.7)-(3.8)-(3.9) then yield

$$\begin{cases} \|U_h\|_X^2 \leq 2\|u_h\|_h^2 + 2C_0^2 h^{2\theta} \left\| A_{0h}^{1/2} u_h \right\|_h^2, \\ \dfrac{1}{\omega}\|(A_0-\omega)U_h\|_X^2 \leq \dfrac{2}{\omega}\|(A_{0h}-\omega)u_h\|_h^2 + 2C_0^2 h^{2\theta}\omega \left\| A_{0h}^{1/2} u_h \right\|_h^2, \\ \|BU_h\|_Y^2 \geq \dfrac{1}{2}\|B_h u_h\|_Y^2 - K_B^2 C_0^2 h^{2\theta(1-2\kappa)} \left\| A_{0h}^{1/2} u_h \right\|_h^2. \end{cases}$$

From (3.2), since $\omega \in (0, \eta/h^\sigma]$ and $u_h \in \mathcal{C}_h(\eta/h^\sigma)$, we obtain

$$\begin{aligned} \frac{1}{2}\|B_h u_h\|_Y^2 \leq {} & \frac{2M^2}{\omega}\|(A_{0h}-\omega)u_h\|_h^2 \\ & + \|u_h\|_h^2 \Big(2m^2(1+C_0^2 h^{2\theta-\sigma}\eta) + 2M^2 C_0^2 h^{2\theta-2\sigma}\eta^2 + K_B^2 C_0^2 h^{2\theta(1-2\kappa)-\sigma}\eta\Big). \end{aligned}$$

Then, with $\sigma$ as in (1.12), (3.2) holds uniformly with respect to $h \in (0,1)$, $u_h \in \mathcal{C}_h(\eta/h^\sigma)$ and $\omega \in (0, \eta/h^\sigma]$, by setting

$$M_*^2 = 4M^2, \quad m_*^2 = 4m^2(1+C_0^2\eta) + 4M^2 C_0^2 \eta^2 + 2K_B^2 C_0^2 \eta.$$

Theorem 2.2 then applies, and yields the admissibility property stated in Theorem 1.3 uniformly with respect to $h \in (0,1)$. Besides, one can obtain explicit estimates on the constants in (1.14). □

### 3.2. Observability

*Proof of Theorem* 1.3*: Observability.* Assume that system (1.1)-(1.3) is admissible and exactly observable. Then, from Theorem 2.9, there exist positive constants $m$ and $M$ such that (2.20) holds.

Our proof is now based on the spectral criterion given in Theorem 2.9.

We shall first prove that there exist positive constants $m_*$ and $M_*$ such that for any $h > 0$, the following inequality holds:

$$\|u_h\|_h^2 \leq \frac{M_*^2}{\omega}\|(A_{0h}-\omega I)u_h\|_h^2 + m_*^2\|B_h u_h\|_Y^2, \ \forall u_h \in \mathcal{C}_h(\epsilon/h^\sigma), \forall \omega \in (0, \epsilon/h^\sigma]. \tag{3.10}$$

In the sequel, we fix $h > 0$, $u_h \in \mathcal{C}_h(\epsilon/h^\sigma)$ and $\omega \in (0, \epsilon/h^\sigma]$, where $\epsilon$ is a positive parameter independent of $h > 0$ which we will choose later on, and, similarly as in (3.3), we introduce $U_h \in \mathcal{D}(A_0)$ defined by (3.3).

Since $U_h$ belongs to $\mathcal{D}(A_0)$, (2.43) applies:

$$\|U_h\|_X^2 \leq \frac{M^2}{\omega}\|(A_0-\omega I)U_h\|_X^2 + m^2\|BU_h\|_Y^2. \tag{3.11}$$

Similarly as in the admissibility case, using $\omega \in (0, \epsilon/h^\sigma]$ and $u_h \in \mathcal{C}_h(\epsilon/h^\sigma)$, we obtain from (3.7)-(3.8)-(3.9) that

$$\begin{cases} \|U_h\|_X^2 \geq \dfrac{1}{2} \|u_h\|_h^2 - 2C_0^2 h^{2\theta-\sigma}\epsilon \|u_h\|_h^2, \\ \dfrac{1}{\omega} \|(A_0-\omega)U_h\|_X^2 \leq \dfrac{2}{\omega} \|(A_{0h}-\omega)u_h\|_h^2 + 2C_0^2 h^{2\theta-2\sigma}\epsilon^2 \|u_h\|_h^2, \\ \|BU_h\|_Y^2 \leq 2\|B_h u_h\|_Y^2 + 2K_B^2 C_0^2 h^{2\theta(1-2\kappa)-\sigma}\epsilon \|u_h\|_h^2. \end{cases}$$

Now, plugging these estimates into (3.11), we get:

$$\begin{aligned} \|u_h\|_h^2 &\left(\frac{1}{2} - 2C_0 h^{2\theta-\sigma}\epsilon - 2M^2C_0^2 h^{2\theta-2\sigma}\epsilon^2 - 2K_B^2 C_0^2 h^{2\theta(1-2\kappa)-\sigma}\epsilon\right) \\ &\leq \frac{2M^2}{\omega} \|(A_{0h}-\omega I)u_h\|_h^2 + 2m^2 \|B_h u_h\|_Y^2. \end{aligned} \quad (3.12)$$

Therefore, with $\sigma$ as in (1.12), setting $\epsilon$ such that

$$2C_0\epsilon + 2M^2C_0^2\epsilon^2 + 2K_B^2C_0^2\epsilon = \frac{1}{4},$$

we obtain (3.10) uniformly with respect to $h \in (0,1)$, $\omega \in (0, \epsilon/h^\sigma]$ and $u_h \in \mathcal{C}_h(\epsilon/h^\sigma)$ by setting $M_* = 2M$ and $m_* = 2m$.

Now, we need to check that the first eigenvalues $\lambda_1^h$ of the operators $A_{0h}$ are uniformly bounded from below by a positive constant. This can be easily deduced from the Rayleigh characterization of the first eigenvalues of $A_{0h}$ and $A_0$:

$$\lambda_1^h = \inf_{\phi_h \in V_h} \frac{\left\|A_{0h}^{1/2}\phi_h\right\|_h^2}{\|\phi_h\|_h^2}, \quad \lambda_1 = \inf_{\phi \in \mathcal{D}(A_0^{1/2})} \frac{\left\|A_0^{1/2}\phi\right\|_X^2}{\|\phi\|_X^2}. \quad (3.13)$$

Indeed, from (3.6), identities (3.13) imply

$$\lambda_1^h = \inf_{\phi_h \in V_h} \frac{\left\|A_{0h}^{1/2}\phi_h\right\|_h^2}{\|\phi_h\|_h^2} = \inf_{\phi_h \in V_h} \frac{\left\|A_0^{1/2}\pi_h\phi_h\right\|_X^2}{\|\pi_h\phi_h\|_X^2} \geq \lambda_1 > 0. \quad (3.14)$$

The discrete systems (1.8)-(1.9) then satisfy uniformly the assumptions of Theorem 2.9, and then the exact observability property stated in Theorem 1.3 follows. □

*Remark* 3.1. In a first version of this work, instead of using the resolvent estimates (2.10) and (2.41) to prove the admissibility and observability results in Theorem 1.3, we used the interpolation inequalities (2.12) and (2.43). However, this yielded the same result with a smaller filtering parameter $\sigma$, namely $\sigma = \theta\min\{2(1-2\kappa), 2/3\}$ instead of (1.12), the difficulty coming from the comparisons of $\left\|A_{0h}^{1/2}u_h\right\|_h$ and $\left\|A_0^{1/2}U_h\right\|_X$. The remark that this earlier result could be improved is due to Luc Miller [29].

## 4. Examples

In this section, we present several applications of Theorem 1.3, and compare our results with the existing ones.

### 4.1. The 1d wave equation

Let us consider the classical 1d wave equation:

$$\begin{cases} \ddot{u} - \partial_{xx}^2 u = 0, & (t,x) \in \mathbb{R} \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad \dot{u}(0,x) = u_1(x), & x \in (0,1). \end{cases} \tag{4.1}$$

For $(a,b)$ a subset of $(0,1)$, we observe system (4.1) through

$$y(t,x) = \dot{u}(t,x)\chi_{(a,b)}(x), \tag{4.2}$$

where $\chi_{(a,b)}$ is the characteristic function of $(a,b)$.

This model indeed enters in the abstract framework considered in this article, by setting $A_0 = -\partial_{xx}^2$ on $(0,1)$ with Dirichlet boundary conditions, and $B = \chi_{(a,b)}$. Indeed, $A_0$ is self-adjoint, positive definite with compact resolvent in $L^2(0,1)$. The operator $B$ obviously is continuous on $L^2(0,1)$ with values in $L^2(0,1)$. The admissibility of (4.1)-(4.2) is then straightforward.

The observability property for (4.1)-(4.2) is well-known to hold for any time $T > 2\min\{a, 1-b\}$. It can be proved by using, for instance, multiplier techniques [23].

To construct the space $V_h$, we use P1 finite elements. More precisely, for $n_h \in \mathbb{N}$, set $h = 1/(n_h+1) > 0$ and define the points $x_j = jh$ for $j \in \{0, \cdots, n_h+1\}$. We define the basis functions

$$e_j(x) = \Big[1 - \frac{|x - x_j|}{h}\Big]^+, \quad \forall j \in \{1, \cdots, n_h\}.$$

Now, $V_h = \mathbb{R}^{n_h}$, and the embedding $\pi_h$ simply is

$$\begin{array}{rcl} \pi_h : V_h = \mathbb{R}^{n_h} & \rightarrow & L^2(0,1) \\ u_h = \begin{pmatrix} u_1 \\ \vdots \\ u_{n_h} \end{pmatrix} & \mapsto & \pi_h u_h(x) = \displaystyle\sum_{j=1}^{n_h} u_j e_j(x). \end{array}$$

Usually, the resulting schemes are written as

$$\begin{cases} M_h \ddot{u}_h(t) + K_h u_h(t) = 0, \quad t \in \mathbb{R}, \\ u_h(0) = u_{0h}, \quad \dot{u}_h(0) = u_{1h}, \end{cases} \qquad y_h(t) = B\pi_h \dot{u}_h(t), \quad t \in \mathbb{R}, \tag{4.3}$$

where $M_h$ and $K_h$ are $n_h \times n_h$ matrices defined by $(M_h)_{i,j} = \int_0^1 e_i(x)e_j(x)\ dx$ and $(K_h)_{i,j} = \int_0^1 \partial_x e_i(x) \partial_x e_j(x)\ dx$. Note that, since $M_h$ is a Gram matrix corresponding to a linearly independent family, it is invertible, self-adjoint and positive definite, and thus the following defines a scalar product:

$$< \phi_h, \psi_h >_h = \phi_h^* M_h \psi_h, \quad (\phi_h, \psi_h) \in V_h^2. \tag{4.4}$$

Besides, from the definition of $M_h$, one easily checks that

$$< \phi_h, \psi_h >_h = \int_0^1 \pi_h(\phi_h)(x)\pi_h(\psi_h)(x)\ dx, \quad \forall(\phi_h, \psi_h) \in V_h^2,$$

as presented in the introduction.

Similarly, one obtains that, for all $(\phi_h, \psi_h) \in V_h^2$,

$$\phi_h^* K_h \psi_h = \phi_h^* M_h M_h^{-1} K_h \psi_h = < \phi_h, M_h^{-1} K_h \psi_h >_h = \phi_h^* K_h M_h^{-1} M_h \psi_h$$
$$= < M_h^{-1} K_h \phi_h, \psi_h >_h = \int_0^1 \partial_x(\pi_h \phi_h)(x) \partial_x(\pi_h \psi_h)(x)\ dx.$$

In other words, the operator $M_h^{-1} K_h$ coincides with the operator $A_{0h}$ in our framework. Note that this operator indeed is self-adjoint, but with respect to the scalar product (4.4) and not with the usual euclidean norm of $\mathbb{R}^{n_h}$.

It is by now a common feature of finite element techniques (see for instance [36]) that estimates (1.10) hold for $\theta = 1$. We can thus apply Theorem 1.3 (with $\kappa = 0$) to systems (4.3):

**Theorem 4.1.** *There exist $\epsilon > 0$, a time $T^*$ and a constant $k_*$ such that for any $h > 0$, any solution $u_h$ of* (4.3) *with initial data $(u_{0h}, u_{1h}) \in \mathcal{C}_h(\epsilon/h)^2$ satisfies* (1.16).

This result is to be compared with the better ones obtained in [21]: In [21], it is proved that, for finite element approximation schemes of the 1d wave equation, observability properties hold uniformly within the larger class $\mathcal{C}_h(\alpha/h^2)$ for $\alpha < 4$.

Though, as we will see hereafter, we can tackle more general cases, even in 1d, for instance taking sequence of nonuniform meshes. In this case, we obtain the same result as in [35], which was stated from the stabilization point of view, see Section 7.

### 4.2. More general cases

Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^N$, with $N \geq 1$, and consider the following wave equation:

$$\begin{cases} \ddot{u} - \operatorname{div}(M(x)\nabla u) = 0, & (x,t) \in \Omega \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{4.5}$$

where $M(x)$ is a $C^1$ function on $\bar{\Omega}$ with values in the self-adjoint $N \times N$ matrices. We also assume that there exist positive constants $\alpha$ and $\beta$ such that for all $\xi \in \mathbb{R}^N$,

$$\alpha|\xi|^2 \leq (M(x)\xi, \xi) \leq \beta|\xi|^2, \quad \forall x \in \Omega, \tag{4.6}$$

where $(\cdot, \cdot)$ is the canonical scalar product of $\mathbb{R}^N$ and $|\cdot|$ is the corresponding norm.

Under these assumptions, it is well-known that system (4.5) is well-posed for initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

System (4.5) is a particular instance of (1.1) for $A_0 = -\mathrm{div}(M(x)\nabla\cdot)$ on $\Omega$ with Dirichlet boundary condition. This operator is indeed self-adjoint positive definite with compact resolvent, and its domain is $\mathcal{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega)$.

Now, set $\omega \subset \Omega$, and consider the observation

$$y(x,t) = \chi_\omega(x)\dot{u}(x,t), \quad (x,t) \in \Omega \times (0,T). \tag{4.7}$$

This defines a bounded operator $B$ on $L^2(\Omega)$. Therefore, the admissibility condition for (4.5)-(4.7) is obvious.

The observability property for (4.5)-(4.7) is well-known to hold if and only if the *Geometric Control Condition* is satisfied, see [2, 3]. This condition, roughly speaking, asserts the existence of a time $T^*$ such that all the rays of Geometric Optics enters in the observation domain in a time smaller than $T^*$. Note that, in our case, the rays are not necessarily straight lines, but correspond to the bicharacteristic rays of the pseudo-differential operator $\tau^2 - (M(x)\xi, \xi)$. Remark that one can also deduce observability results using multiplier techniques [23, 33]. From now on, we assume that $\omega$ satisfies the *Geometric Control Condition.*

We consider P1 finite elements on meshes $\mathcal{T}_h$. We furthermore assume that the meshes $\mathcal{T}_h$ of the domain $\Omega$ are regular in the sense of finite elements [36, Section 5]. Roughly speaking, this assumption imposes that the polyhedra of $(\mathcal{T}_h)$ are not too flat.

**Definition 4.2.** Let $\mathcal{T} = \cup_{K\in\mathcal{T}} K$ be a mesh of a bounded domain $\Omega$. For each polyhedron $K \in \mathcal{T}$, we define $h_K$ as the diameter of $K$ and $\rho_K$ as the maximum diameter of the spheres $S \subset K$. We then define the regularity of $\mathcal{T}$ as

$$\mathrm{Reg}(\mathcal{T}) = \sup_{K\in\mathcal{T}} \left\{ \frac{h_K}{\rho_K} \right\}.$$

A sequence of mesh $(\mathcal{T}_n)$ is said to be uniformly regular if

$$\sup_n \mathrm{Reg}(\mathcal{T}_n) < \infty.$$

In the following, we will denote sequences of meshes by $(\mathcal{T}_h)_{h\in(0,1)}$ with the implicit assumption that $h = h(\mathcal{T}_h) = \sup_{K\in\mathcal{T}_h} h_K$. In this case, see [36], estimates (1.10) again hold for $\theta = 1$, and Theorem 1.3 implies (here again $\kappa = 0$):

**Theorem 4.3.** *Assume that system* (4.5)-(4.7) *is observable. Given a sequence of uniformly regular meshes* $(\mathcal{T}_h)_{h>0}$*, there exist* $\epsilon > 0$*, a time* $T^*$ *and a constant* $k_*$ *such that for any* $h > 0$*, any solution* $u_h$ *of the P1 finite element approximation scheme of* (4.5)-(4.7) *corresponding to the mesh* $\mathcal{T}_h$ *with initial data* $(u_{0h}, u_{1h}) \in \mathcal{C}_h(\epsilon/h)^2$ *satisfies* (1.16).

To our knowledge, this is the first time that observability properties for space semi-discretizations of (4.5)-(4.7) are derived in such generality for the wave equation. In particular, we emphasize that the only non-trivial assumption we used is (1.10), which is needed anyway to guarantee the convergence of the numerical schemes.

## 5. Fully discrete approximation schemes

This section is based on the article [12], which studied admissibility and exact observability properties of time discrete conservative linear systems. As said in [12], this study can be combined with admissibility and observability results on space semi-discrete systems to deduce admissibility and observability properties for fully discrete systems. Below, we present an application of the results in [12].

Let $\beta \geq 1/4$ and consider the following time discrete approximation scheme - the so-called Newmark method, see for instance [36] - of (1.8):

$$\begin{cases} \dfrac{u_h^{k+1} + u_h^{k-1} - 2u_h^k}{(\triangle t)^2} + A_{0h}\Big(\beta u_h^{k-1} + (1-2\beta)u_h^k + \beta u_h^{k+1}\Big) = 0, \quad k \in \mathbb{N}^*, \\ \Big(\dfrac{u_h^0 + u_h^1}{2}, \dfrac{u_h^1 - u_h^0}{\triangle t}\Big) = (u_{0h}, u_{1h}) \in V_h^2, \end{cases} \tag{5.1}$$

where $u_h^k$ corresponds to an approximation of the solution $u_h$ of (1.8) at time $t_k = k\triangle t$.

The energy of solutions $u_h$ of (5.1), defined by

$$\begin{aligned} E_h^{k+1/2} = \frac{1}{2}\left\| A_{0h}^{1/2}\Big(\frac{u_h^k + u_h^{k+1}}{2}\Big)\right\|_h^2 + \frac{1}{2}\left\|\frac{u_h^{k+1} - u_h^k}{\triangle t}\right\|_h^2 \\ + \frac{(\triangle t)^2}{8}(4\beta - 1)\left\| A_{0h}^{1/2}\Big(\frac{u_h^{k+1} - u_h^k}{\triangle t}\Big)\right\|_h^2, \quad k \in \mathbb{N}, \end{aligned} \tag{5.2}$$

is constant.

Then we get the following admissibility and observability results (see [12]):

**Theorem 5.1.** *Let $A_0$ be a self-adjoint positive definite unbounded operator with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$.*

*Assume that the maps $(\pi_h)_{h>0}$ satisfy property* (1.10)*. Let $\beta \geq 1/4$, and consider the fully discrete approximation scheme* (5.1)*. Set $\sigma$ as in* (1.12)*, and $\delta > 0$.*

**Admissibility:** *Assume that system* (1.1)-(1.3) *is admissible.*

*Then, for any $\eta > 0$ and $T > 0$, there exists a positive constant $K_{T,\eta} > 0$ such that, for any $h > 0$ and $\triangle t > 0$, any solution of* (5.1) *with initial data*

$$(u_{0h}, u_{1h}) \in \Big(\mathcal{C}_h(\eta/h^\sigma) \cap \mathcal{C}_h(\delta^2/(\triangle t)^2)\Big)^2 \tag{5.3}$$

*satisfies*

$$\triangle t \sum_{k\triangle t \in [0,T]} \left\| B_h\Big(\frac{u_h^{k+1} - u_h^k}{\triangle t}\Big)\right\|_Y^2 \leq K_{T,\eta} E_h^{1/2}. \tag{5.4}$$

**Observability:** *Assume that system* (1.1)-(1.3) *is admissible and exactly observable.*

*Then there exist $\epsilon > 0$, a time $T^*$ and a positive constant $k_* > 0$ such that, for any $\triangle t > 0$ small enough, for any $h > 0$, any solution of* (5.1) *with initial data*

$$(u_{0h}, u_{1h}) \in \Big(\mathcal{C}_h(\epsilon/h^\sigma) \cap \mathcal{C}_h(\delta^2/(\triangle t)^2)\Big)^2 \tag{5.5}$$

*satisfies*

$$k_* E_h^{1/2} \leq \triangle t \sum_{k\triangle t \in [0,T^*]} \left\| B_h\Big(\frac{u_h^{k+1} - u_h^k}{\triangle t}\Big)\right\|_Y^2. \tag{5.6}$$

Obviously, inequalities (5.4)-(5.6) are time discrete counterparts of (1.14)-(1.16). Remark that, as in Theorem 1.3, a filtering condition is needed, but which now depends on both time and space discretization parameters.

Also remark that if $(\triangle t)^2 h^{-\sigma} \leq \delta^2/\epsilon$, the filtered space $\mathcal{C}_h(\epsilon/h^\sigma) \cap \mathcal{C}_h(\delta^2/(\triangle t)^2)$ coincides with $\mathcal{C}_h(\epsilon/h^\sigma)$. Roughly speaking, this indicates that under the CFL type condition $(\triangle t)^2 h^{-\sigma}$ small enough, system (5.1) behaves, with respect to the admissibility and observability properties, similarly as the space semi-discrete equations (1.8).

*Remark* 5.2. We restrict our presentation to the Newmark method, but similar results hold for a large range of time discrete approximation schemes of (1.8). We refer to [12, Section 3] for the precise assumptions on the time-discrete approximation schemes under which we can guarantee uniform observability properties to hold.

## 6. Controllability properties

This section aims at discussing applications of Theorem 1.3 to controllability properties for space semi-discretizations of wave type equations such as (1.1). The approach presented below is strongly inspired by previous works [17, 21, 44, 45, 10], and closely follows [10].

In the whole section, we assume that the hypotheses of Theorem 1.3 are satisfied.

### 6.1. The continuous setting

Consider the following control problem: Given $T > 0$, for any $(w_0, w_1) \in \mathcal{D}(A_0^{1/2}) \times X$, find a control $v \in L^2(0, T; Y)$ such that the solution $w$ of

$$\ddot{w} + A_0 w = B^* v(t), \quad t \in [0, T], \qquad w(0) = w_0, \quad \dot{w}(0) = w_1, \tag{6.1}$$

satisfies

$$w(T) = 0, \quad \dot{w}(T) = 0. \tag{6.2}$$

The controllability issue in time $T$ for (6.1) is equivalent to the observability property in time $T$ for (1.1)-(1.3) (see for instance [25]). Indeed, these two properties are dual, and this duality can be made precise using the Hilbert Uniqueness Method (HUM in short), see [25].

More precisely, the control of minimal $L^2(0,T;Y)$ norm for (6.1), that we will denote by $v_{HUM}$, is characterized through the minimizer of the functional $\mathcal{J}$ defined on $\mathcal{D}(A_0^{1/2}) \times X$ by:

$$\mathcal{J}(u_{0T}, u_{1T}) = \frac{1}{2} \int_0^T \|B\dot{u}(t)\|_Y^2 \; dt \\ + < A_0^{1/2} u(0), A_0^{1/2} w_0 >_X + < \dot{u}(0), w_1 >_X, \quad (6.3)$$

where $u$ is the solution of

$$\ddot{u} + A_0 u = 0, \quad t \in [0,T], \qquad u(T) = u_{0T}, \quad \dot{u}(T) = u_{1T}. \qquad (6.4)$$

Indeed, if $(u_{0T}^*, u_{1T}^*)$ is the minimizer of $\mathcal{J}$, then $v_{HUM}(t) = B\dot{u}^*(t)$, where $u^*$ is the solution of (6.4) with initial data $(u_{0T}^*, u_{1T}^*)$.

Besides, the only admissible control for (6.1) which can be written as $B\dot{u}(t)$ for a solution $u$ of (6.4) is the HUM control $v_{HUM}$. This characterization will be used in the following.

Note that the observability property (1.5) for (1.1)-(1.3) implies the strict convexity and coercivity of $\mathcal{J}$ and then guarantees the existence of a unique minimizer for $\mathcal{J}$.

### 6.2. The semi-discrete setting

The natural idea which consists in computing the discrete HUM controls for discrete versions of (6.1) may fail in providing good approximations of the HUM control for (6.1). We refer for instance to the survey article [45] for a detailed presentation of this fact in the context of the 1d wave equation. We thus use filtering techniques developed for instance in [17, 21, 44, 45, 10] to overcome the problems created by the spurious high-frequency components created by the scheme.

Our presentation closely follows the one in [10]. The proofs of the result below will be only sketched, and can be done similarly as in [10].

Since we assumed that the hypotheses of Theorem 1.3 hold, there exists a time $T^*$ such that (1.16) holds for any solution of (1.8) with initial data in the filtered space $\mathcal{C}_h(\epsilon/h^\sigma)^2$.

We now fix $T \geq T^*$.

Following the strategy of HUM, we introduce the adjoint problem

$$\ddot{u}_h + A_{0h} u_h = 0, \quad t \in [0,T], \quad (u_h, \dot{u}_h)(T) = (u_{0Th}, u_{1Th}). \qquad (6.5)$$

**6.2.1. Method I.** For any $h > 0$, we consider the following control problem: For any $(w_{0h}, w_{1h}) \in V_h^2$, find $v_h \in L^2(0,T;Y)$ of minimal $L^2(0,T;Y)$ such that the solution $w_h$ of

$$\ddot{w}_h + A_{0h} w_h = B_h^* v_h(t), \quad t \in [0,T], \qquad w_h(0) = w_{0h}, \quad \dot{w}_h(0) = w_{1h}, \qquad (6.6)$$

satisfies

$$P_h w_h(T) = 0, \quad P_h \dot{w}_h(T) = 0, \qquad (6.7)$$

where $P_h$ is the orthogonal projection in $V_h$ on $\mathcal{C}_h(\epsilon/h^\sigma)$.

To deal with this problem, we introduce the functional $\mathcal{J}_h$ defined for $(u_{0Th}, u_{1Th})$ in $\mathcal{C}_h(\epsilon/h^\sigma)^2$ by

$$\mathcal{J}_h(u_{0Th}, u_{1Th}) = \frac{1}{2}\int_0^T \|B_h \dot{u}_h(t)\|_Y^2 \ dt \\ + < A_{0h}^{1/2} w_{0h}, A_{0h}^{1/2} u_h(0) >_h + < w_{1h}, \dot{u}_h(0) >_h, \quad (6.8)$$

where $u_h$ is the solution of (6.5).

For each $h > 0$, the functional $\mathcal{J}_h$ is strictly convex and coercive (see (1.16)), and thus has a unique minimizer $(u_{0Th}^*, u_{1Th}^*) \in \mathcal{C}_h(\epsilon/h^\sigma)^2$.

Besides, we have:

**Lemma 6.1.** *For all $h > 0$, let $(u_{0Th}^*, u_{1Th}^*) \in \mathcal{C}_h(\epsilon/h^\sigma)^2$ be the unique minimizer of $\mathcal{J}_h$ (on $\mathcal{C}_h(\epsilon/h^\sigma)^2$), and denote by $u_h^*$ the corresponding solution of* (6.5)*. Then the solution of* (6.6) *with $v_h = B_h \dot{u}_h^*$ satisfies* (6.7)*.*

*Sketch of the proof.* We present briefly the proof, which is standard (see for instance [25]).

On one hand, multiplying (6.6) by $\dot{u}_h$ solution of (6.5) with initial data $(u_{0Th}, u_{1Th})$, we get, for all $(u_{0Th}, u_{1Th}) \in V_h^2$,

$$\int_0^T < v_h(t), B_h \dot{u}_h(t) >_Y \ dt + < A_{0h}^{1/2} w_{0h}, A_{0h}^{1/2} u_h(0) >_h + < w_{1h}, \dot{u}_h(0) >_h \\ - < A_{0h}^{1/2} w_h(T), A_{0h}^{1/2} u_{0Th} >_h - < \dot{w}_h(T), u_{1Th} >_h = 0. \quad (6.9)$$

On the other hand, the Fréchet derivative of the functional $\mathcal{J}_h$ at $(u_{0Th}^*, u_{1Th}^*)$ yields:

$$\int_0^T < B_h \dot{u}_h^*(t), B_h \dot{u}_h(t) >_Y \ dt + < A_{0h}^{1/2} w_{0h}, A_{0h}^{1/2} u_h(0) >_h \\ + < w_{1h}, \dot{u}_h(0) >_h = 0, \quad \forall (u_{0Th}, u_{1Th}) \in \mathcal{C}_h(\epsilon/h^\sigma)^2. \quad (6.10)$$

Therefore, setting $v_h = B_h \dot{u}_h^*$, subtracting (6.9) to (6.10), we obtain

$$< A_{0h}^{1/2} w_h(T), A_{0h}^{1/2} u_{0Th} >_h + < \dot{w}_h(T), u_{1Th} >_h = 0, \quad \forall (u_{0Th}, u_{1Th}) \in \mathcal{C}_h(\epsilon/h^\sigma)^2$$

or, equivalently, (6.7). □

As in [10], we then investigate the convergence of the discrete controls $v_h$ obtained in Lemma 6.1.

**Theorem 6.2.** *Assume that the hypotheses of Theorem* 1.3 *are satisfied. Also assume that*

$$Y_X = \Big\{ y \in Y, \ \textit{such that } B^* y \in X \Big\} \quad (6.11)$$

*is dense in $Y$.*

*Let* $(w_0, w_1) \in \mathcal{D}(A_0^{1/2}) \times X$*, and consider a sequence* $(w_{0h}, w_{1h})_{h>0}$ *such that* $(w_{0h}, w_{1h})$ *belongs to* $V_h^2$ *for any* $h > 0$ *and*

$$(\pi_h w_{0h}, \pi_h w_{1h}) \to (w_0, w_1) \quad \text{in } \mathcal{D}(A_0^{1/2}) \times X. \tag{6.12}$$

*Then the sequence* $(v_h)_{h>0}$ *of discrete controls given by Lemma* 6.1 *strongly converges in* $L^2(0,T;Y)$ *to the HUM control* $v_{HUM}$ *of* (6.1) *associated to the initial data* $(w_0, w_1)$*.*

Remark that, for $w \in \mathcal{D}(A_0)$, in view of (1.10), the sequence $(w_h)_h = (\pi_h^* w)$ converges to $w$ in $\mathcal{D}(A_0^{1/2})$ in the sense that the sequence $(\pi_h w_h)$ converges to $w$ in $\mathcal{D}(A_0^{1/2})$. For $(w_0, w_1) \in \mathcal{D}(A_0^{1/2}) \times X$, one can then find a sequence $(w_{0h}, w_{1h})_{h>0}$ satisfying (6.12) and $(w_{0h}, w_{1h}) \in V_h^2$ for any $h > 0$ by using the density of $\mathcal{D}(A_0)^2$ into $\mathcal{D}(A_0^{1/2}) \times X$.

The technical assumption $\bar{Y}_X = Y$ on $B$ is usually satisfied, and thus does not limit the range of applications of Theorem 6.2. In particular, when $B$ is bounded from $X$ to $Y$, the space $Y_X$ coincides with $Y$ and then this condition is automatically satisfied.

The proof of Theorem 6.2 uses precisely the same ingredients as the one in [10], and is briefly sketched for the convenience of the reader.

*Sketch of the proof. Step 1.* The discrete controls $v_h$ are bounded in $L^2(0,T;Y)$. This follows from the inequality

$$\mathcal{J}_h(u_{0Th}^*, u_{1Th}^*) \le \mathcal{J}_h(0,0) = 0,$$

and the observability inequality (1.16). Hence the controls are bounded, and, up to an extraction, the sequence $(v_h)$ weakly converges to some function $v$ in $L^2(0,T;Y)$. Besides, the sequence $(u_{0Th}^*, u_{1Th}^*)$ is also bounded in $\mathcal{D}(A_0^{1/2}) \times X$, and therefore weakly converges in $\mathcal{D}(A_0^{1/2}) \times X$ to some couple of functions $(\tilde{u}_{0T}, \tilde{u}_{1T})$.

*Step 2.* The weak limit $v$ is an admissible control for (6.1) associated to the data $(w_0, w_1)$. This can be deduced, as in [10], from the convergence properties of the approximation schemes (1.8) (or equivalently (6.5)), which can be found for instance in [36, Chapter 8].

*Step 3.* The weak limit $v$ is the HUM control for (6.1) associated to the data $(w_0, w_1)$. This is also based on a convergence result which can be found in [36, Chapter 8], and which guarantees that $v = B\dot{\tilde{u}}$, where $\tilde{u}$ is the solution of (6.4) with initial data $(\tilde{u}_{0T}, \tilde{u}_{1T})$. This also proves that $(\tilde{u}_{0T}, \tilde{u}_{1T})$ coincides with the minimizer $(u_{0T}^*, u_{1T}^*)$ of the continuous functional $\mathcal{J}$ in (6.3). Assumption (6.11) is needed in this step to identify the limit of $(B\dot{u}_h^*)$ with $B\dot{\tilde{u}}$.

*Step 4.* Finally, the strong convergence of the controls is proved using the convergence of the $L^2(0,T;Y)$ norms. Compute first the Fréchet derivative of $\mathcal{J}$

at $(u_{0T}^*, u_{1T}^*)$: for $(u_{0T}, u_{1T}) \in \mathcal{D}(A_0^{1/2}) \times X$, we obtain

$$\int_0^T < B\dot{u}^*(t), B\dot{u}(t) >_Y \ dt + < A_0^{1/2} u(0), A_0^{1/2} w_0 >_X + < \dot{u}(0), w_1 >_X = 0. \tag{6.13}$$

Now, applying (6.10) to $(u_{0Th}^*, u_{1Th}^*)$ and (6.13) to $(u_{0T}^*, u_{1T}^*)$, the assumptions on the convergence of $(w_{0h}, w_{1h})$ imply the convergence of the $L^2(0,T;Y)$ norms of $v_h$ to the $L^2(0,T;Y)$ norm of $v$. □

**6.2.2. Method II.** As in [10], one can prefer a method which does not involve a filtering process in the discrete setting. We thus recall the works [17, 45, 10], which propose an alternate process based on a Tychonoff regularization of $\mathcal{J}_h$.

**Theorem 6.3.** *Assume that the hypotheses of Theorem* 1.3 *are satisfied. Also assume that $B \in \mathfrak{L}(X,Y)$, which, in particular, implies that $\sigma = \theta$.*
*Let $(w_0, w_1) \in \mathcal{D}(A_0^{1/2}) \times X$, and consider a sequence $(w_{0h}, w_{1h})_{h>0}$ such that $(w_{0h}, w_{1h})$ belongs to $V_h^2$ for any $h > 0$ and* (6.12) *holds.*

*For any $h > 0$, consider the functionals $\mathcal{J}_h^*$, defined for $(u_{0Th}, u_{1Th}) \in V_h^2$ by*

$$\begin{aligned} \mathcal{J}_h^*(u_{0Th}, u_{1Th}) = \frac{1}{2} \int_0^T \|B_h \dot{u}_h(t)\|_Y^2 \ dt + \frac{h^\sigma}{2} \Big( \left\| \tilde{A}_{0h}^{1/2} A_{0h}^{1/2} u_{0Th} \right\|_h^2 \\ + \left\| \tilde{A}_{0h}^{1/2} u_{1Th} \right\|_h^2 \Big) + < A_{0h}^{1/2} w_{0h}, A_{0h}^{1/2} u_h(0) >_h + < w_{1h}, \dot{u}_h(0) >_h, \end{aligned} \tag{6.14}$$

*where*

$$\tilde{A}_{0h} = A_{0h}(Id_{V_h} + h^\sigma A_{0h})^{-1}, \tag{6.15}$$

*and $u_h$ is the solution of* (6.5) *with initial data $(u_{0Th}, u_{1Th})$.*

*Then, for any $h > 0$, the functional $\mathcal{J}_h^*$ admits a unique minimizer $(U_{0Th}, U_{1Th})$ in $V_h^2$. Besides, setting $v_h(t) = B_h \dot{U}_h(t)$, where $U_h$ is the solution of* (6.5) *with initial data $(U_{0Th}, U_{1Th})$, one gets the following convergence results:*

$$v_h \longrightarrow v_{HUM} \quad \text{in } L^2(0,T;Y), \tag{6.16}$$

*where $v_{HUM}$ denotes the HUM control for* (6.1).

Theorem 6.3 proposes a numerical process based on the minimization of the functional $\mathcal{J}_h^*$ defined for *any* element of $V_h^2$. Though, the functional $\mathcal{J}_h^*$ involves the regularizing term

$$h^\sigma \left\| \tilde{A}_{0h}^{1/2} u_{1Th} \right\|_h^2 + h^\sigma \left\| \tilde{A}_{0h}^{1/2} A_{0h}^{1/2} u_{0Th} \right\|_h^2.$$

This term is small for data in $\mathcal{C}_h(\epsilon/h^\sigma)$ and of unit order for frequencies higher than $1/h^\sigma$. Also note that this term can be computed easily since

$$h^\sigma \left\| \tilde{A}_{0h}^{1/2} \phi_h \right\|_h^2 = h^\sigma < \tilde{A}_{0h} \phi_h, \phi_h >_h = h^\sigma < A_{0h} \tilde{\phi}_h, \phi_h >_h,$$

where $\tilde{\phi}_h$ is the solution of

$$\Big( Id_{V_h} + h^\sigma A_{0h} \Big) \tilde{\phi}_h = \phi_h. \tag{6.17}$$

In other words, the operator $\tilde{A}_{0h}$ simply introduces an elliptic regularization of the data, and the regularizing terms can be computed explicitly by solving the elliptic equation (6.17).

The keynote of the proof of Theorem 6.3 is to remark that the functionals $\mathcal{J}_h^*$ are uniformly coercive. This can be done as in [10] by decomposing the solutions of (6.5) into low- and high-frequency components.

The proof of Theorem 6.3 can then be done similarly as the one of Theorem 6.2 (see also [10] for technical details), and thus is left to the reader.

*Remark* 6.4. Similar results can be obtained for fully discrete approximation schemes derived from Newmark time discretizations of (1.8) (or more general time discrete approximation scheme, see Remark 5.2). The proof can then be done similarly as in the time continuous setting, using the observability inequality (5.6) and convergence properties for the fully discrete approximation schemes, which can be found for instance in [36, Chapter 8].

## 7. Stabilization properties

This section is mainly based on the articles [15, 13], in which stabilization properties are derived for abstract linear damped systems.

Below, we assume that $A_0$ is self-adjoint, positive definite and with compact resolvent, and that $B \in \mathfrak{L}(X, Y)$.

### 7.1. The continuous setting

Consider the following damped wave type equations:

$$\ddot{u} + A_0 u + B^* B \dot{u} = 0, \quad t \geq 0, \qquad (u(0), \dot{u}(0)) = (u_0, u_1) \in \mathcal{D}(A_0^{1/2}) \times X. \quad (7.1)$$

The energy of solutions of (7.1), defined by (1.2), satisfies the dissipation law

$$\frac{dE}{dt}(t) = - \|B\dot{u}(t)\|_Y^2, \quad t \geq 0. \quad (7.2)$$

System (7.1) is said to be exponentially stable if there exists positive constants $\mu$ and $\nu$ such that any solution of (7.1) with initial data $(u_0, u_1) \in \mathcal{D}(A_0^{1/2}) \times X$ satisfies

$$E(t) \leq \mu E(0) \exp(-\nu t). \quad (7.3)$$

It is by now well-known (see [19]) that this property holds if and only if the observability inequality (1.5) holds for solutions of (1.1).

### 7.2. The space semi-discrete setting

We now assume that system (1.1)-(1.3) is observable in the sense of (1.5), or, equivalently (see [19]), that system (7.1) is exponentially stable.

Then, combining Theorem 1.3 and the results in [15], we get:

**Theorem 7.1.** *Let $B$ be a bounded operator in $\mathfrak{L}(X,Y)$, and assume that system* (7.1) *is exponentially stable in the sense of* (7.3)*. Also assume that the hypotheses of Theorem* 1.3 *are satisfied.*

*Then the space semi-discrete systems*

$$\begin{cases} \ddot{u}_h + A_{0h}u_h + B_h^* B_h \dot{u}_h + h^\theta A_{0h}\dot{u}_h = 0, \quad t \geq 0, \\ (u_h(0), \dot{u}_h(0)) = (u_{0h}, u_{1h}) \in V_h^2, \end{cases} \tag{7.4}$$

*are exponentially stable, uniformly with respect to the space discretization parameter $h > 0$: there exist two positive constants $\mu_0$ and $\nu_0$ independent of $h > 0$ such that for any $h > 0$, any solution $u_h$ of* (7.4) *satisfies, for $t \geq 0$,*

$$\left\| A_{0h}^{1/2} u_h(t) \right\|_h^2 + \|\dot{u}_h(t)\|_h^2 \leq \mu_0 \left( \left\| A_{0h}^{1/2} u_h(0) \right\|_h^2 + \|\dot{u}_h(0)\|_h^2 \right) \exp(-\nu_0 t). \tag{7.5}$$

Here, several other viscosity operators could have been chosen: We refer to [15] for the precise assumptions required on the viscosity operator introduced in (7.4) for which we can guarantee uniform stabilization results.

Note that systems (7.4) are similar to the numerical approximation schemes of the 1d and 2d wave equations studied in [38, 37, 30, 14], which were dealt with using multiplier techniques. In [38, 37, 30, 14], the viscosity term $h^2 A_{0h}$, instead of what would correspond to $hA_{0h}$ in our setting, has been proved to be sufficient to guarantee the uniform exponential decay of the energy. However, the range of applications of [38, 37, 30, 14] is limited to the case of uniform meshes and of wave equations with constant velocity.

Thus, in many situations, our results are not sharp. However, they apply for a wide range of applications: in particular, no condition is required on the dimension or on the uniformity of the meshes.

Besides, our results generalize the ones in [35], where uniform stabilization results are derived for general damped wave equations (7.1) using a non-trivial spectral conditions on $A_0$. Indeed, in [35], a non-trivial spectral gap condition on the eigenvalues of $A_0$ is needed, which restricts the range of direct applications to the 1d case only.

*Remark* 7.2. One can use the results in [15] to design fully discrete approximation schemes of (7.1) for which one can guarantee uniform (in both time and space discretization parameters) stabilization properties.

## 8. A wave equation observed through $y(t) = Bu(t)$

In this section, rather than studying an observation operator which involves the time derivative of solutions of (1.1) as in (1.3), we focus on the case of an observation of the form

$$y(t) = Bu(t). \tag{8.1}$$

The operator $B$ is now assumed to belong to $\mathfrak{L}(\mathcal{D}(A_0), Y)$, where $Y$ is an Hilbert space.

Now, the admissibility property for (1.1)-(8.1) consists in the existence, for every $T > 0$, of a constant $K_T$ such that any solution of (1.1) with initial data $(u_0, u_1) \in \mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ satisfies

$$\int_0^T \|Bu(t)\|_Y^2 \, dt \le K_T \Big( \left\| A_0^{1/2} u_0 \right\|_X^2 + \|u_1\|_X^2 \Big). \tag{8.2}$$

In particular, when $B$ belongs to $\mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$, system (1.1)-(8.1) is obviously admissible because of the conservation of the energy (1.2).

The observability property for (1.1)-(8.1) now reads as follows: There exist a time $T$ and a positive constant $k_T > 0$ such that

$$k_T \Big( \left\| A_0^{1/2} u_0 \right\|_X^2 + \|u_1\|_X^2 \Big) \le \int_0^T \|Bu(t)\|_Y^2 \, dt. \tag{8.3}$$

Similarly as before, assuming that system (1.1)-(8.1) is admissible and exactly observable, one can ask if the discrete systems (1.8) observed through

$$y_h(t) = B\pi_h u_h(t), \tag{8.4}$$

are uniformly admissible and exactly observable in a convenient filtered class.

Below, we provide a partial answer to that question. As before, we can only consider operators $B$ which belong to $\mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$ for $\kappa < 1/2$. This makes the admissibility properties obvious since the observation operators $B_h = B\pi_h$ are then uniformly bounded as operators from $V_h$, endowed with the norm $\left\| A_{0h}^{1/2} \cdot \right\|_h = \left\| A_0^{1/2} \pi_h \cdot \right\|_X$ (see (3.6)), to $Y$.

We therefore focus on the observability properties of (1.8)-(8.4), for which we obtain the following:

**Theorem 8.1.** *Let $A_0$ be a self-adjoint positive definite operator with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$ with $\kappa < 1/2$. Assume that the maps $(\pi_h)$ satisfy property* (1.10)*.*

*Assume that system* (1.1)-(8.1) *is exactly observable. Then there exist $\epsilon > 0$, a time $T^*$ and a positive constant $k_* > 0$ such that, for any $h \in (0,1)$, any solution of* (1.8) *with initial data $(u_{0h}, u_{1h}) \in \mathcal{C}_h(\epsilon/h^\theta)^2$ satisfies*

$$k_* \Big( \left\| A_{0h}^{1/2} u_{0h} \right\|_h^2 + \|u_{1h}\|_h^2 \Big) \le \int_0^{T^*} \|B\pi_h u_h(t)\|_Y^2 \ dt. \tag{8.5}$$

The proof of Theorem 8.1 is based on the following spectral characterization:

**Theorem 8.2.** *Let $A_0$ be a self-adjoint positive definite operator on $X$ with compact resolvent and $B \in \mathfrak{L}(\mathcal{D}(A_0), Y)$. Assume that system* (1.1)-(8.1) *is admissible in the sense of* (8.2)*. Then system* (1.1)-(8.1) *is exactly observable if and only if there exist positive constants $m$ and $M$ such that*

$$\left\| A_0^{1/2} u \right\|_X^2 \le M^2 \|(A_0 - \omega I)u\|_X^2 + m^2 \|Bu\|_Y^2, \ \forall u \in \mathcal{D}(A_0), \ \forall \omega \in I(A_0), \tag{8.6}$$

*where $I(A_0)$ is the convex hull of the spectrum of $A_0$.*

*Besides, if* (8.6) *holds, then the time* $T$ *and the constants* $k_T$ *in* (8.3) *can be chosen explicitly as functions of the admissibility constants, the first eigenvalue* $\lambda_1$ *of* $A_0$ *and* $(m, M)$.

The proof of Theorem 8.2 is left to the reader. We only briefly indicate the method one can use to show Theorem 8.2. The fact that the exact observability of (1.1)-(8.1) implies (8.6) can be derived easily using the resolvent estimate in [28]. The reverse implication, similarly as in Theorem 2.9, can be proved using Theorem 2.5.

Once Theorem 8.2 is proved, one only needs to prove that there exists $\epsilon > 0$ small enough and positive constants $m_*$ and $M_*$ such that for any $h \in (0,1)$,

$$\left\|A_{0h}^{1/2}u_h\right\|_h^2 \leq M_*^2 \left\|(A_{0h} - \omega I)u_h\right\|_h^2 + m_*^2 \left\|B_h u_h\right\|_Y^2, \ \forall u \in \mathcal{C}_h(\epsilon/h^\theta), \forall \omega \in [0, \epsilon/h^\theta]. \tag{8.7}$$

The proof of (8.7) can be done similarly as in Subsection 3.2. The only new estimate required is the following one: For $u_h \in \mathcal{C}_h(\epsilon/h^\sigma)$,

$$\begin{aligned} \left| \left\|A_0^{1/2}U_h\right\|_X^2 - \left\|A_{0h}^{1/2}u_h\right\|_h^2 \right| &= \left|< A_{0h}u_h, \pi_h^*(U_h - \pi_h u_h) >_h\right| \\ &\leq C_0 h^\theta \left\|A_{0h}^{1/2}u_h\right\|_h \left\|A_{0h}u_h\right\|_h \leq C_0 \epsilon h^{\theta - \sigma/2} \left\|A_{0h}^{1/2}u_h\right\|_h^2. \end{aligned}$$

The rest of the proof is left to the reader.

## 9. Further comments

1. A widely open question consists in finding the sharp filtering scale. We think that the works [6, 7], which present a study of the observability properties of the 1d wave equation in highly heterogeneous media, might give some insights to address this issue.

2. In this article, we assumed that the continuous systems are exactly observable. However, there are several important models of vibrations where the energy is only weakly observable. That is the case for instance for networks of vibrating strings [8] or when the *Geometric Control Condition* is not fulfilled (see [2, 24]). It would be interesting to address the observability issues for the space semi-discretizations of such systems. To our knowledge, this issue is widely open.

3. When looking at Theorem 8.1, it is surprising that $B$ has to be assumed in $\mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$ with $\kappa < 1/2$. There are several cases of interests, as for instance the wave equation observed from the normal derivative of the boundary, in which $B$ is not in this class. This question thus deserves further work.

4. Theorem 1.3 can also be applied to Schrödinger systems observed from an open subset which satisfies the Geometric Control Condition. In this case, it is indeed well-known that exact observability properties for the continuous Schrödinger system hold in arbitrary small time (see [28]). As a by-product of our analysis of the discrete waves, one can obtain uniform admissibility and exact observability

results for the discrete Schrödinger systems, still in the same classes of data filtered at the scale $1/h^{\sigma}$ with $\sigma$ as in (1.12). This improves the results in [10] where the same results were stated but at a scale $1/h^{\varsigma}$ with $\varsigma = \theta \min\{2(1-2\kappa), 2/3\}$. However, to prove that, under the Geometric Control Condition, the time $T$ of exact observability for the discrete Schrödinger equations can be made arbitrary small is not straightforward. A proof is presented in [11], adapting [18] in the discrete cases.

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## References

[1] H. T. Banks, K. Ito, and C. Wang. Exponentially stable approximations of weakly damped wave equations. In *Estimation and control of distributed parameter systems (Vorau, 1990)*, volume 100 of *Internat. Ser. Numer. Math.*, pages 1–33. Birkhäuser, Basel, 1991.

[2] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control and Optim.*, 30(5):1024–1065, 1992.

[3] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.

[4] N. Burq and M. Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2):443–471 (electronic), 2004.

[5] C. Castro and S. Micu. Boundary controllability of a linear semi-discrete 1-d wave equation derived from a mixed finite element method. *Numer. Math.*, 102(3):413–462, 2006.

[6] C. Castro and E. Zuazua. Low frequency asymptotic analysis of a string with rapidly oscillating density. *SIAM J. Appl. Math.*, 60(4):1205–1233 (electronic), 2000.

[7] C. Castro and E. Zuazua. Concentration and lack of observability of waves in highly heterogeneous media. *Arch. Ration. Mech. Anal.*, 164(1):39–72, 2002.

[8] R. Dáger and E. Zuazua. *Wave propagation, observation and control in* 1*-d flexible multi-structures*, volume 50 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006.

[9] S. Ervedoza. Observability of the mixed finite element method for the 1d wave equation on nonuniform meshes. *To appear in ESAIM Control Optim. Calc. Var.*

[10] S. Ervedoza. Admissibility and observability for Schrödinger systems: Applications to finite element approximation schemes. *Preprint*, 2008.

[11] S. Ervedoza. Observability in arbitrary small time for discrete approximations of conservative systems. *Preprint*, 2009.

[12] S. Ervedoza, C. Zheng, and E. Zuazua. On the observability of time-discrete conservative linear systems. *J. Funct. Anal.*, 254(12):3037–3078, June 2008.

[13] S. Ervedoza and E. Zuazua. Uniform exponential decay for viscous damped systems. *To appear in Proc. of Siena "Phase Space Analysis of PDEs 2007", Special issue in honor of Ferrucio Colombini.*

[14] S. Ervedoza and E. Zuazua. Perfectly matched layers in 1-d: Energy decay for continuous and semi-discrete waves. *Numer. Math.*, 109(4):597–634, 2008.

[15] S. Ervedoza and E. Zuazua. Uniformly exponentially stable approximations for a class of damped systems. *J. Math. Pures Appl.*, 91:20–48, 2009.

[16] R. Glowinski. Ensuring well-posedness by analogy: Stokes problem and boundary control for the wave equation. *J. Comput. Phys.*, 103(2):189–221, 1992.

[17] R. Glowinski, C. H. Li, and J.-L. Lions. A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods. *Japan J. Appl. Math.*, 7(1):1–76, 1990.

[18] A. Haraux. Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire. *J. Math. Pures Appl. (9)*, 68(4):457–465 (1990), 1989.

[19] A. Haraux. Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. *Port. Math.*, 46(3):245–258, 1989.

[20] O. Y. Imanuvilov. On Carleman estimates for hyperbolic equations. *Asymptot. Anal.*, 32(3-4):185–220, 2002.

[21] J.A. Infante and E. Zuazua. Boundary observability for the space semi discretizations of the 1-d wave equation. *Math. Model. Num. Ann.*, 33:407–438, 1999.

[22] A. E. Ingham. Some trigonometrical inequalities with applications to the theory of series. *Math. Z.*, 41(1):367–379, 1936.

[23] V. Komornik. *Exact controllability and stabilization.* RAM: Research in Applied Mathematics. Masson, Paris, 1994. The multiplier method.

[24] G. Lebeau. Équations des ondes amorties. *Séminaire sur les Équations aux Dérivées Partielles, 1993–1994,École Polytech.*, 1994.

[25] J.-L. Lions. *Contrôlabilité exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1. Contrôlabilité exacte*, volume RMA 8. Masson, 1988.

[26] K. Liu. Locally distributed control and damping for the conservative systems. *SIAM J. Control Optim.*, 35(5):1574–1590, 1997.

[27] F. Macià. The effect of group velocity in the numerical analysis of control problems for the wave equation. In *Mathematical and numerical aspects of wave propagation—WAVES 2003*, pages 195–200. Springer, Berlin, 2003.

[28] L. Miller. Controllability cost of conservative systems: resolvent condition and transmutation. *J. Funct. Anal.*, 218(2):425–444, 2005.

[29] L. Miller. Private communication, 2009.

[30] A. Münch and A. F. Pazoto. Uniform stabilization of a viscous numerical approximation for a locally damped wave equation. *ESAIM Control Optim. Calc. Var.*, 13(2):265–293 (electronic), 2007.

[31] M. Negreanu, A.-M. Matache, and C. Schwab. Wavelet filtering for exact controllability of the wave equation. *SIAM J. Sci. Comput.*, 28(5):1851–1885 (electronic), 2006.

[32] M. Negreanu and E. Zuazua. Convergence of a multigrid method for the controllability of a 1-d wave equation. *C. R. Math. Acad. Sci. Paris*, 338(5):413–418, 2004.

[33] A. Osses. A rotated multiplier applied to the controllability of waves, elasticity, and tangential Stokes control. *SIAM J. Control Optim.*, 40(3):777–800 (electronic), 2001.

[34] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. *J. Funct. Anal.*, 226(1):193–229, 2005.

[35] K. Ramdani, T. Takahashi, and M. Tucsnak. Uniformly exponentially stable approximations for a class of second order evolution equations—application to LQR problems. *ESAIM Control Optim. Calc. Var.*, 13(3):503–527, 2007.

[36] P.-A. Raviart and J.-M. Thomas. *Introduction à l'analyse numérique des équations aux dérivées partielles.* Collection Mathématiques Appliquées pour la Maitrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983.

[37] L. R. Tcheugoué Tebou and E. Zuazua. Uniform boundary stabilization of the finite difference space discretization of the $1d$ wave equation. *Adv. Comput. Math.*, 26(1-3):337–365, 2007.

[38] L.R. Tcheugoué Tébou and E. Zuazua. Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity. *Numer. Math.*, 95(3):563–598, 2003.

[39] L. N. Trefethen. Group velocity in finite difference schemes. *SIAM Rev.*, 24(2):113–136, 1982.

[40] M. Tucsnak and G. Weiss. Observation and control for operator semigroups, 2008.

[41] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27(3):527–545, 1989.

[42] R. M. Young. *An introduction to nonharmonic Fourier series.* Academic Press Inc., San Diego, CA, first edition, 2001.

[43] X. Zhang. Explicit observability estimate for the wave equation with potential and its application. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 456(1997):1101–1115, 2000.

[44] E. Zuazua. Boundary observability for the finite-difference space semi-discretizations of the 2-D wave equation in the square. *J. Math. Pures Appl. (9)*, 78(5):523–563, 1999.

[45] E. Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2):197–243 (electronic), 2005.

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