LONG-TIME BEHAVIOR FOR THE TWO-DIMENSIONAL MOTION OF A DISK IN A VISCOUS FLUID

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Abstract. In this article, we study the long-time behavior of solutions of the two-dimensional fluid-rigid disk problem. The motion of the fluid is modeled by the two-dimensional Navier-Stokes equations, and the disk moves under the influence of the forces exerted by the viscous fluid. We first derive L^p-L^q decay estimates for the linearized equations and compute the first term in the asymptotic expansion of the solutions of the linearized equations. We then apply these computations to derive time-decay estimates for the solutions to the full Navier-Stokes fluid-rigid disk system.

Contents

1. Introduction

We consider the system formed by a rigid disk and a viscous fluid filling the whole plane $\mathbb{R}^2$. We assume that the body initially occupies the disk B_0 and rigidly moves so that at time t it occupies a domain denoted by $B(t)$ that is isometric to B_0. We denote $\mathcal{F}(t) := \mathbb{R}^2 \setminus B(t)$ the domain occupied by the fluid at time t starting from the initial domain $\mathcal{F}_0 := \mathbb{R}^2 \setminus B_0$.

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When the fluid has constant viscosity $\nu > 0$, the equations modeling the dynamics of the system fluid-rigid disk read

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0 \qquad \text{for } t\in(0,\infty),\ x\in\mathcal{F}(t), \tag{1.1}$$

$$\operatorname{div} u = 0 \qquad \text{for } t\in(0,\infty),\ x\in\mathcal{F}(t), \tag{1.2}$$

$$u(t,x) = h'(t) + \omega(t)(x-h(t))^{\perp} \qquad \text{for } t\in(0,\infty),\ x\in\partial B(t), \tag{1.3}$$

$$\lim_{|x|\to\infty} |u(t,x)| = 0 \qquad \text{for } t\in(0,\infty), \tag{1.4}$$

$$mh''(t) = -\int_{\partial B(t)} \Sigma n \,\mathrm{d}s \qquad \text{for } t\in(0,\infty), \tag{1.5}$$

$$\mathcal{J}\omega'(t) = -\int_{\partial B(t)} (x-h(t))^{\perp}\cdot\Sigma n \,\mathrm{d}s \qquad \text{for } t\in(0,\infty), \tag{1.6}$$

$$u|_{t=0} = u_0 \qquad \text{for } x\in\mathcal{F}_0, \tag{1.7}$$

$$h(0)=h_0,\ h'(0)=\ell_0,\ \omega(0)=\omega_0. \tag{1.8}$$

Here, $u=(u_1,u_2)$ denotes the velocity-field, p the pressure and Σ is the Cauchy stress tensor of the fluid:

$$\Sigma = -p\mathrm{Id} + 2\nu D(u), \tag{1.9}$$

where Id is the 2×2 identity matrix and:

$$D(u)_{k,\ell} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\Big) \qquad 1\le k,\ell\le 2\,.$$

The constants m and $\mathcal{J}$ denote respectively the mass and the inertia of the body while the fluid is supposed to be homogeneous, of density 1 to simplify notations. In this work, we assume that the solid is homogeneous of density m/π, implying in particular $\mathcal{J} = m/2$ (we discuss in Section 5 about a generalization). When $x=(x_1,x_2)\in\mathbb{R}^2$, the vector $x^{\perp}$ stands for $x^{\perp} = (-x_2,x_1)$, n denotes the unit normal vector to $\partial B(t)$ pointing outside the fluid domain, $h'(t)$ is the velocity of the center of mass $h(t)$ of the body and $\omega(t)$ denotes the angular velocity of the rigid body. Indeed, since $B(t)$ is isometric to B_0 there exists a rotation matrix

$$S_{\theta(t)} := \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

such that the lagrangian coordinates $\eta(t,x)$ associated to the body read:

$$\eta(t,x) := h(t) + S_{\theta(t)}(x-h_0).$$

Furthermore, the angle θ satisfies $\theta'(t)=\omega(t)$, and is chosen such that $\theta(0)=0$. Without loss of generality, we assume that B_0 is the unit disk centered at the origin: $h(0)=0$.

Given $(u_0,\ell_0,\omega_0)\in H^1(\mathcal{F}_0)\times\mathbb{R}^2\times\mathbb{R}$, satisfying the compatibility condition:

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}_0, \quad u_0 = \ell_0 + \omega_0 x^{\perp} \text{ on } \partial B_0,$$

T. Takahashi and M. Tucsnak prove in [22] that there exists a unique global strong solution (u,p,h,ω) of (1.1)-(1.8). The construction is based on the change of variable:

$$v(t,x) = u(t,x-h(t))\,, \qquad \tilde{p}(t,x) = p(t,x-h(t))\,, \qquad \ell(t) = h'(t)\,. \tag{1.10}$$

The new unknowns $(v,\tilde{p})$ are then defined in the fixed domain $[0,\infty)\times(\mathbb{R}^2\setminus B_0)$ and system (1.1)-(1.8) reads, in terms of $(v,\tilde{p},\ell,\omega)$:

$$\frac{\partial v}{\partial t}+\Big((v-\ell)\cdot\nabla\Big)v-\nu\Delta v+\nabla\tilde{p}=0 \qquad \text{for } (t,x)\in(0,\infty)\times\mathcal{F}_0, \tag{1.11}$$

$$\operatorname{div} v=0 \qquad \text{for } (t,x)\in(0,\infty)\times\mathcal{F}_0, \tag{1.12}$$

$$v(t,x)=\ell(t)+\omega(t)x^{\perp} \qquad \text{for } (t,x)\in(0,\infty)\times\partial B_0, \tag{1.13}$$

$$\lim_{|x|\to\infty}|v(t,x)|=0, \qquad \text{for } t\in(0,\infty), \tag{1.14}$$

$$m\ell'(t)=-\int_{\partial B_0}\Sigma n\,\mathrm{d}s, \qquad \text{for } t\in(0,\infty), \tag{1.15}$$

$$\mathcal{J}\omega'(t)=-\int_{\partial B_0}x^{\perp}\cdot\Sigma n\,\mathrm{d}s, \qquad \text{for } t\in(0,\infty), \tag{1.16}$$

$$v|_{t=0}=v_0 \qquad \text{for } x\in\mathcal{F}_0, \tag{1.17}$$

$$\ell(0)=\ell_0\,,\ \omega(0)=\omega_0, \tag{1.18}$$

with

$$\Sigma=-\tilde{p}\,\mathrm{Id}+2\nu D(v).$$

These solutions verify the following energy decay estimate:

$$\frac{1}{2}\left[\int_{\mathcal{F}_0}|v(t,x)|^2\,\mathrm{d}x+\big(m|\ell(t)|^2+\mathcal{J}|\omega(t)|^2\big)\right]+2\nu\int_0^t\int_{\mathcal{F}_0}|D(v)(\tau,x)|^2\,\mathrm{d}\tau\mathrm{d}x$$
$$\leq\frac{1}{2}\left[\int_{\mathcal{F}_0}|v_0(x)|^2\,\mathrm{d}x+\big(m|\ell_0|^2+\mathcal{J}|\omega_0|^2\big)\right]\,,\quad\forall\,t>0\,. \tag{1.19}$$

Relying on this estimate, T. Takahashi and M. Tucsnak prove the existence and uniqueness of a global weak solution to (1.11)–(1.18) for initial data (v_0,ℓ_0,ω_0) such that $v_0\in L^2(\mathcal{F}_0)$ and

$$\operatorname{div} v_0=0,\ \text{in } \mathcal{F}_0,\qquad v_0\cdot n=(\ell_0+\omega_0x^{\perp})\cdot n,\ \text{on } \partial B_0. \tag{1.20}$$

In this article, we aim at studying the long-time behavior of these weak solutions.

The long-time behavior of solutions to fluid-structure interaction systems has already been tackled in different ways. In a series of papers, several authors study the asymptotics of systems without pressure, *i.e.* where the Navier Stokes equations are replaced by a heat equation [18, 19, 24, 25]. In this simplified case, the force applied by the fluid on a solid is modeled by the circulation of the normal derivative of the velocity-field u on the solid boundaries. In the one-dimensional case in [24, 25], and then in several dimensions in [19, 18], the authors show that the multiplier method introduced in [7] to study the asymptotic behavior of solutions to convection-diffusion equations (also applied to the porous medium equation in [23]) enables to compute sharp decay estimates and asymptotic expansions of solutions up to the second order. Even if the divergence-free condition (1.12) significantly modifies the equations, we will strongly use the results in [19, 18].

The long-time behavior of solutions for the full Navier Stokes equations in the whole space is also a long-standing question that has motivated numerous studies. Applying a Fourier decomposition, M. E. Schonbek and M. Wiegner show in [21, 27] that the L^2 norm of the Navier-Stokes solution decreases to zero, which was a question raised by J. Leray [16]. In [3], A. Carpio obtains a sharp description of the pressure, which is given by $p=\Delta^{-1}(\operatorname{div}(u\cdot\nabla u))$. Representing then the velocity-field by a Duhamel formula and using a scaling argument, she computes the development of the solution for long times up to the second order.

Another approach consists in removing the pressure by taking the curl of the momentum equation:

$$\partial_t\operatorname{curl} u+u\cdot\nabla\operatorname{curl} u-\nu\Delta\operatorname{curl} u=0, \tag{1.21}$$

where $\operatorname{curl} u$ is the vorticity of the fluid. Without boundaries, such an equation yields the decay of the L^p norms of the vorticity $\operatorname{curl} u$. For $\operatorname{curl} u_0\in L^1(\mathbb{R}^2)$, such that $\int\operatorname{curl} u_0\neq0$, T. Gallay and C.E.

Wayne prove in [11] that the vorticity behaves as $t \to \infty$ like the heat kernel

$$\left(\int \operatorname{curl} u_0 \right) \frac{e^{-\frac{|x|^2}{4t}}}{4\pi t}.$$

Note that if $\operatorname{curl} u_0$ is compactly supported and integrable, then thanks to the Biot-Savart law, we have, for large x,

$$u_0(x) = \frac{\int \operatorname{curl} u_0}{2\pi} \frac{x^\perp}{|x|^2} + \mathcal{O}_{|x|\to\infty}\Big(\frac{1}{|x|^2}\Big).$$

Of course, this implies that u_0 does not belong to $L^1(\mathbb{R}^2) \cup L^2(\mathbb{R}^2)$ if $\int \operatorname{curl} u_0 \neq 0$. Consequently, this theory corresponds to solutions with infinite energy. For instance, in [11], T. Gallay and C.E. Wayne deduce that the velocity behaves asymptotically as $t \to \infty$ like the Lamb-Oseen vector field:

$$\frac{\int \operatorname{curl} u_0}{2\pi} \frac{x^\perp}{|x|^2}\Big(1 - e^{-\frac{|x|^2}{4t}}\Big)$$

which has infinite energy.

In a domain with boundaries, system (1.21) has to be completed with boundary conditions. When Dirichlet boundary conditions are imposed for the velocity-field, one might compute Robin boundary conditions for the vorticity but with non-dissipative coefficients. Therefore, working on the vorticity seems difficult. In the case of one obstacle surrounded by a viscous fluid (*i.e.*, when B_0 is fixed and the system reduces to the Navier-Stokes equations in the exterior domain $\mathcal{F}_0$ completed with homogeneous Dirichlet boundary condition on $\partial\mathcal{F}_0$), the recent works [10, 12] prove that the first term in the long-time behavior of the velocity-field is given by the Lamb-Oseen vector field. Their proofs consist in a perturbative argument showing that the decay estimates for the solutions of the Stokes problem, which were established in [5, 6, 17], implies that the nonlinear terms tend faster to zero than the Stokes solution. To our knowledge, such decay estimates on the Stokes semigroup are only known for fixed domains with homogeneous Dirichlet boundary conditions for the velocity-field.

The only result considering the long-time behavior of a moving particle inside a Navier Stokes fluid is due to E. Feireisl and S. Nečasovà [8]. However, they assume the whole system to be confined in a bounded container and they take into account the influence of gravity. Hence, they obtain completely different results with completely different methods. Broadly speaking, they prove that, if the container has no vertical wall and contains only one particle, the particle reaches the bottom of the container asymptotically in time.

One of the main steps in [10, 12] is to establish $L^p - L^q$ decay estimates for solutions to the linear Stokes equations underlying the Navier Stokes system. Such results are known for fixed domains with homogeneous Dirichlet boundary condition (see [5, 17]), but in our case, the linearized Stokes fluid-solid system reads:

$$\frac{\partial v}{\partial t} - \nu\Delta v + \nabla p = 0 \qquad \text{for } (t,x) \in (0,\infty)\times\mathcal{F}_0, \tag{1.22}$$

$$\operatorname{div} v = 0 \qquad \text{for } (t,x) \in (0,\infty)\times\mathcal{F}_0, \tag{1.23}$$

$$v = \ell(t) + \omega(t)x^\perp \qquad \text{for } (t,x) \in (0,\infty)\times\partial B_0, \tag{1.24}$$

$$\lim_{|x|\to\infty} |v(t,x)| = 0 \qquad \text{for } t \in (0,\infty), \tag{1.25}$$

$$m\ell'(t) = -\int_{\partial B_0} \Sigma n \,\mathrm{d}s \qquad \text{for } t \in (0,\infty), \tag{1.26}$$

$$\mathcal{J}\omega'(t) = -\int_{\partial B_0} x^\perp \cdot \Sigma n \,\mathrm{d}s \qquad \text{for } t \in (0,\infty), \tag{1.27}$$

$$v|_{t=0} = v_0 \qquad \text{for } x \in \mathcal{F}_0, \tag{1.28}$$

$$\ell(0) = \ell_0,\ \omega(0) = \omega_0. \tag{1.29}$$

To our knowledge, $L^p - L^q$ (p, $q \neq 2$) estimates are not available in the literature for solutions of (1.22)–(1.29). To be more precise, in [22] , T. Takahashi and M. Tucsnak construct solutions of (1.22)–(1.29) *via* a semigroup approach. They show that the semigroup is analytic on $\mathcal{L}^2$ (to be defined in (1.30) below) in dimension 2. In [28], the semigroup is also proved to be analytic in the counterpart of the spaces $\mathcal{L}^{6/5} \cap \mathcal{L}^p$ in 3D. However, in both papers the subsequent decay estimates are not sufficient for our purpose. In the first case, the L^2 framework only is considered. In the second case, the authors do not obtain sharp decay estimates.

To state our results precisely we introduce shortly some notations. From a triplet (v_0, ℓ_0, ω_0) verifying (1.20), we define a divergence-free vector field denoted V_0 on $\mathbb{R}^2$ obtained by extending v_0 by $\ell_0 + \omega_0 x^\perp$ in B_0. Adapted to such V_0, we introduce the functional spaces $\mathcal{L}^p$ defined as follows:

$$\mathcal{L}^p = \{V \in L^p(\mathbb{R}^2),\ \operatorname{div} V = 0 \text{ in } \mathbb{R}^2,\ D(V) = 0 \text{ in } B_0\}, \quad (p \in [1,\infty]). \tag{1.30}$$

When $p \in [1,\infty)$, we endow these spaces with the norms

$$\|V\|^p_{\mathcal{L}^p} = \int_{\mathcal{F}_0} |V|^p + \frac{m}{\pi}\int_{B_0} |V|^p .$$

It is easy to check that, if $V \in \mathcal{L}^p$, then $V = \ell_V + \omega_V x^\perp$ on B_0, where

$$\ell_V = \frac{1}{\pi}\int_{B_0} V(x)\,\mathrm{d}x, \quad \omega_V = \frac{2}{\pi}\int_{B_0} V(x)\cdot x^\perp\,\mathrm{d}x, \tag{1.31}$$

and the normal component of V is continuous across ∂B_0 (as in (1.20)). In particular, we remark that, setting $v = V|_{\mathcal{F}_0}$, there holds:

$$\|V\|^p_{\mathcal{L}^p} \sim \|v\|^p_{L^p(\mathcal{F}_0)} + |\ell_V|^p + |\omega_V|^p.$$

Such a space is obviously a Banach space as a closed subspace of $L^p(\mathbb{R}^2)$. A straightforward extension of [9, Theorem III.2.3] yields that $\mathcal{L}^p \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$ is dense in $\mathcal{L}^p$ for arbitrary $p \in (1,\infty)$. For $p = 2$, the space $\mathcal{L}^p$ is a Hilbert space as the norm is associated with the scalar product:

$$\langle V, W\rangle_{\mathcal{L}^2} = \int_{\mathcal{F}_0} V\cdot W + \frac{m}{\pi}\int_{B_0} V\cdot W\,. \tag{1.32}$$

For $p \in (1,\infty)\setminus\{2\}$, the same bilinear form enables to identify the dual of $\mathcal{L}^p$ with $\mathcal{L}^{p'}$ where p' is the conjugate exponent of p.

Naturally, we endow $\mathcal{L}^\infty$ with the norm:

$$\|V\|_{\mathcal{L}^\infty} = \|V\|_{L^\infty(\mathbb{R}^2)}\,.$$

For any $V \in \mathcal{L}^\infty$, we still have $V(x) = \ell_V + \omega_V x^\perp$ in B_0 with ℓ_V and ω_V defined by (1.31). Hence, there holds again:

$$\|V\|_{\mathcal{L}^\infty} \sim \max\left\{\|v\|_{L^\infty(\mathcal{F}_0)}, |\ell_V|, |\omega_V|\right\}\,.$$

Our first results concern the Cauchy problem for (1.22)-(1.29) in $\mathcal{L}^p$ and the decay rates of the constructed solutions. As in [22, 28], we use a semigroup approach:

Theorem 1.1. *For each $q \in (1,\infty)$, the Stokes operator of the linear problem* (1.22)-(1.29) *generates a semigroup $S(t)$ on $\mathcal{L}^q$ which satisfies the following decay estimates:*

- *For $p \in [q,\infty]$, there exists $K_1 = K_1(p,q) > 0$ such that for every $V_0 \in \mathcal{L}^q$:*

$$\|S(t)V_0\|_{\mathcal{L}^p} \le K_1(\nu t)^{\frac{1}{p}-\frac{1}{q}}\|V_0\|_{\mathcal{L}^q} \qquad \text{for all} \quad t > 0. \tag{1.33}$$

- *If $q \le 2$, for $p \in [q,2]$, there exists $K_2 = K_2(p,q) > 0$ such that for every $V_0 \in \mathcal{L}^q$,*

$$\|\nabla S(t)V_0\|_{L^p(\mathcal{F}_0)} \le K_2(\nu t)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\|V_0\|_{\mathcal{L}^q} \qquad \text{for all} \quad t > 0. \tag{1.34}$$

- *For $p \in [\max\{2,q\},\infty)$, there exists $K_3 = K_3(p,q) > 0$ such that for every $V_0 \in \mathcal{L}^q$,*

$$\|\nabla S(t)V_0\|_{L^p(\mathcal{F}_0)} \le \begin{cases} K_3(\nu t)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\|V_0\|_{\mathcal{L}^q} & \text{for all} \quad 0 < t < \frac{1}{\nu}, \\ K_3(\nu t)^{-\frac{1}{q}}\|V_0\|_{\mathcal{L}^q} & \text{for all} \quad t \ge \frac{1}{\nu}. \end{cases} \tag{1.35}$$

Our approach is based on the decomposition of the velocity in spherical harmonics. We show that the 0-spherical harmonic verifies a heat equation (without pressure) with dynamical boundary conditions. This enables us to compute decay estimates using the multiplier method of Escobedo-Zuazua [7] in the same way as in [18, 19]. The 1-spherical harmonic is the hardest part. *A priori*, it verifies an equation with pressure and non-standard boundary conditions. However, we show that there exists an underlying algebra which enables to reduce this equation to a heat equation (without pressure) with dynamical boundary conditions. So, we can again reproduce the method of [7] in the spirit of [18, 19]. We do not expand the remainder (*i.e.*, the k-spherical harmonic for $k \geq 2$) in this part, as we show it satisfies the Stokes equations with Dirichlet boundary condition on B_0 which has been studied formerly in several papers [5, 17].

Going further in the spherical-harmonic decomposition, we are also able to compute an asymptotic expansion of the solution to the Stokes system (1.22)-(1.29) for well-localized initial data:

Theorem 1.2. *For all $p \in [2, \infty]$, and for any $V_0 \in \mathcal{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$, setting $\ell_0 = \ell_{V_0}$ and*

$$\vec{\mathcal{M}} = (m - \pi)\ell_0,$$

we have

$$\lim_{t\to\infty} t^{1-1/p} \left\| S(t)V_0 - U_{\vec{\mathcal{M}}}(t, \cdot)\right\|_{L^p(\mathcal{F}_0)} = 0, \tag{1.36}$$

$$\lim_{t\to\infty} t \left| \ell_{S(t)V_0} - \frac{\vec{\mathcal{M}}}{8\pi\nu t} \right| = 0, \tag{1.37}$$

$$\limsup_{t\to\infty} t^2 \left|\omega_{S(t)V_0}\right| < +\infty, \tag{1.38}$$

where

$$U_{\vec{\mathcal{M}}}(t, x) = \nabla^\perp \left[\frac{1 - e^{-\frac{|x|^2}{4\nu t}}}{2\pi |x|^2} \vec{\mathcal{M}} \cdot x^\perp \right].$$

Before going further, let us emphasize the following important remark, which can be easily deduced from explicit computations: provided $\vec{\mathcal{M}} \neq 0$, for all $p > 1$,

$$0 < \liminf_{t\to\infty} t^{1-1/p} \left\| U_{\vec{\mathcal{M}}}(t, \cdot)\right\|_{L^p(\mathcal{F}_0)} = \limsup_{t\to\infty} t^{1-1/p} \left\| U_{\vec{\mathcal{M}}}(t, \cdot)\right\|_{L^p(\mathcal{F}_0)} = \left\| U_{\vec{\mathcal{M}}}(1, \cdot)\right\|_{L^p(\mathbb{R}^2)} < \infty.$$

The quantity $\vec{\mathcal{M}}$ represents the total momentum of the system. Indeed, since any $V_0 \in L^1(\mathbb{R}^2)$ satisfying the divergence free condition has 0 mean value,

$$\int_{\mathcal{F}_0} V_0 \,\mathrm{d}x + m\ell_0 = -\int_{B_0} V_0 \,\mathrm{d}x + m\ell_0 = (m - \pi)\ell_0 = \vec{\mathcal{M}}.$$

Therefore, if $\vec{\mathcal{M}} \neq 0$, (1.36) shows that $U_{\vec{\mathcal{M}}}$ is the first term in the asymptotic expansion of $S(t)V_0$. We deduce also from (1.37)–(1.38) that, provided $\vec{\mathcal{M}} \neq 0$, the solid, whose center of mass corresponds to $h_{S(t)V_0} = \int_0^t \ell_{S(s)V_0}\,\mathrm{d}s$, goes logarithmically to infinity and stops turning.

If $\vec{\mathcal{M}} = 0$, then a careful reading of the proof of Theorem 1.2 yields

$$\limsup_{t\to\infty} \left(\frac{t^{5/4}}{|\log(t)|^{1/2}} \left|\ell_{S(t)V_0}\right| \right) < +\infty \tag{1.39}$$

which implies that the disk converges to a fixed state when considering the linearized equations (1.22)–(1.29). Note that the condition $\vec{\mathcal{M}} = 0$ is satisfied in the following two cases:

- $m = \pi$, that is the case of a solid having exactly the same density as the fluid.
- $\ell_0 = 0$, that is the case of a solid whose center of mass has zero initial velocity.

Thus, when $\vec{\mathcal{M}} = 0$, we expect a different behavior as $t \to \infty$ of the solutions of the Stokes system (1.22)–(1.29).

With the decay estimates of Theorem 1.1 at hand, we study the long-time behavior of solutions to the Navier Stokes system (1.11)–(1.18). We prove that, for small initial data, such solutions satisfy decay estimates similar to the one of the solutions to the Stokes equations:

Theorem 1.3. *Let $q \in (1,2]$. Then there exists $\lambda_0(q) > 0$ such that, for all initial data $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ satisfying the smallness assumption*

$$\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(q), \tag{1.40}$$

the unique weak solution V of (1.11)–(1.18) with initial data V_0 satisfies the following decay estimates:

- *for all $p \in [2,\infty)$, there exists $H(p,q,V_0)$ such that*

$$\sup_{t>0}\{t^{\frac{1}{q}-\frac{1}{p}}\|V(t)\|_{\mathcal{L}^p}\} \leq H(p,q,V_0). \tag{1.41}$$

- *there exists $H_\ell(q,V_0)$ such that*

$$\sup_{t>0}\{t^{\frac{1}{q}}|\ell_V(t)|\} \leq H_\ell(q,V_0). \tag{1.42}$$

Besides, the function $q \mapsto \lambda_0(q)$ can be chosen as an increasing function of $q \in (1,2]$ which goes to zero as $q \to 1$.

The proof of Theorem 1.3 consists of two steps. First, we consider the case $q = 2$. Following the idea developed by Kato in [14], we construct successive approximations Y_n which verify the decay estimates (1.41) uniformly in n for $p = 2$, $p = 8$ and (1.42). Next, we pass to the limit to get a solution to the Navier-Stokes equations with such a time-decay. To reach $p \in [2,\infty)$ we use a bootstrap argument based on the Duhamel formula as in [3]. We then develop the case $q \in (1,2)$ by showing that estimates (1.41) are satisfied uniformly by the sequence Y_n.

Using then a bootstrap argument allows us to quantify the proximity of the solution of the non-linear system (1.11)–(1.18) and of the linear system (1.22)–(1.29):

Theorem 1.4. *Let $q \in (1,2]$. Taking $\lambda_0(q) > 0$ as in Theorem 1.3, for any $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ verifying (1.40), the unique global solution V of (1.11)-(1.18) with initial data V_0 verifies: for all $p \in [2,\infty)$ there exist constants $C(p,q,V_0) > 0$ for which:*

$$\sup_{t>2}\{t^{1-\frac{1}{p}}\|V(t) - S(t)V_0\|_{\mathcal{L}^p}\} \leq C(p,q,V_0), \qquad \text{if } q \in (1,4/3), \tag{1.43}$$

$$\sup_{t>2}\left\{\frac{t^{1-\frac{1}{p}}}{\log(t)}\|V(t) - S(t)V_0\|_{\mathcal{L}^p}\right\} \leq C(p,q,V_0), \qquad \text{if } q = 4/3, \tag{1.44}$$

$$\sup_{t>2}\{t^{\frac{2}{q}-\frac{1}{2}-\frac{1}{p}}\|V(t) - S(t)V_0\|_{\mathcal{L}^p}\} \leq C(p,q,V_0) \qquad \text{if } q \in (4/3,2]. \tag{1.45}$$

Similarly, there exist constants $C_\ell(q,V_0) > 0$ such that

$$\sup_{t>2}\{t|\ell_V(t) - \ell_{S(t)V_0}|\} \leq C_\ell(q,V_0), \qquad \text{if } q \in (1,4/3), \tag{1.46}$$

$$\sup_{t>2}\left\{\frac{t}{\log(t)}|\ell_V(t) - \ell_{S(t)V_0}|\right\} \leq C_\ell(q,V_0), \qquad \text{if } q = 4/3, \tag{1.47}$$

$$\sup_{t>2}\{t^{\frac{2}{q}-\frac{1}{2}}|\ell_V(t) - \ell_{S(t)V_0}|\} \leq C_\ell(q,V_0). \qquad \text{if } q \in (4/3,2]. \tag{1.48}$$

Let us comment the fact that if the initial data V_0 belongs to $\mathcal{L}^q \cap \mathcal{L}^2$ for some $q \in (1,2)$ and satisfies the smallness condition (1.40), the $\mathcal{L}^p$-norm of the difference between the solution of the complete non-linear system (1.11)–(1.18) and the linear one, given by $S(t)V_0$, decays faster than the a priori decay estimates predicted by Theorem 1.1. Indeed, we check easily that $1 - \frac{1}{p} > \frac{1}{q} - \frac{1}{p}$ for any $q > 1$ and that $\frac{2}{q} - \frac{1}{2} - \frac{1}{p} > \frac{1}{q} - \frac{1}{p}$ for any $q < 2$.

Combining Theorem 1.4 and Theorem 1.1 and taking $\lambda_0 = \lambda_0(5/4)$, we can guarantee that, for all $q \in (1,2]$ and all $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ satisfying $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0$, we have

$$\sup_{t>2}\{t^{\frac{1}{q}-\frac{1}{p}}\|V(t)\|_{\mathcal{L}^p}\} < \infty.$$

Indeed, for $q > 5/4$, it relies on Theorem 1.3 and the fact that $q \mapsto \lambda_0(q)$ is increasing. For $q \in (1, 5/4]$, it is a simple combination of Theorem 1.4 (for $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(5/4)$ because $V_0 \in \mathcal{L}^{5/4}$) with the decay estimates of Theorem 1.1 (because $V_0 \in \mathcal{L}^q$). If V_0 satisfies the further assumption $V_0 \in \mathcal{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\,\mathrm{d}x)$ with $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(5/4)$, we can also combine Theorem 1.4 with Theorem 1.2 yielding that, for all $p \geq 2$:

$$\sup_{t>2} t^{1-\frac{1}{p}} \|V(t)\|_{\mathcal{L}^p} < \infty.$$

In all these cases, we obtain that the solution to the Navier Stokes system thus decays with time (at least) as fast as the solution to the Stokes system.

The paper is organized as follows. In next section, we collect some preliminary results. We explain the decomposition of the velocity-field in spherical harmonics. We then compute the different equations satisfied by the different modes of the velocity-field and we end up the section by several elliptic lemmas that will be used further. Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2. Section 4 contains the proof of Theorem 1.3 and Theorem 1.4. The article ends by some comments and open problems, in particular the lack of the first asymptotic term of the Navier-Stokes solution.

Notations. In the whole article, we use classical notations for function spaces. The symbol $L^p(\Omega, \mathrm{d}\mu)$ stands for the Lebesgue space with respect to measure $\mathrm{d}\mu$ defined on an open set $\Omega \subset \mathbb{R}^n$. If $\mathrm{d}\mu$ is the Lebesgue measure, we drop $\mathrm{d}\mu$. Sobolev spaces are denoted by $H^m(\Omega)$, $m \in \mathbb{Z}$. Further notations for function spaces are introduced along the paper. We shall use extensively symbol L in different fonts (such as $\mathcal{L}^p$, $\mathfrak{L}^p$). This will correspond to variants of Lebesgue spaces. The only exception concerns $\mathscr{L}_c(X)$ (resp. $\mathscr{L}_c(X \to Y)$), which represent the Banach space of continuous linear operators from a Banach space X to itself (resp. a Banach space X to another Banach space Y).

In what follows, we will use capital letters to denote functions defined on $\mathbb{R}^2$, as we did for the velocity V above, and denote by the corresponding small characters the restriction on $\mathcal{F}_0$. To be more precise, for V, W, Z $(\cdots)$ defined on $\mathbb{R}^2$, functions v, w, z denote the corresponding restrictions of V, W, Z on $\mathcal{F}_0$ and ℓ_V, ℓ_W, ℓ_Z the mean value of V, W, Z on B_0. In the sequel, when considering functions W, Z $(\cdots)$ which are constant on B_0, we will identify them with the couple (w, ℓ_W), (z, ℓ_Z) $(\cdots)$ and write $W \doteq (w, \ell_W)$, $Z \doteq (z, \ell_Z)$. In the case of the velocity V in $\mathcal{L}^p$, the restriction of V is $\ell_V + \omega_V x^\perp$ and thus we also identify V with the triplet (v, ℓ_V, ω_V) and note $V \doteq (v, \ell_V, \omega_V)$.

2. Preliminary results

We first recall how the Cauchy problem for (1.22)–(1.29) has been tackled in [22]. Formal energy estimates imply that, for a sufficiently smooth and localized initial data, $V(t) \in \mathcal{L}^2$ for all t. In this framework, system (1.22)–(1.29) reduces to the abstract ODE: $\partial_t V + AV = 0$, where A is the unbounded operator with domain:

$$\mathcal{D}(A) = \{V \in H^1(\mathbb{R}^2),\, V_{|\mathcal{F}_0} \in H^2(\mathcal{F}_0),\ \operatorname{div} V = 0 \text{ in } \mathbb{R}^2,\ D(V) = 0 \text{ in } B_0\} \tag{2.1}$$

such that for $V \in \mathcal{D}(A)$, $AV := \mathbb{P}_2 \mathcal{A} V$ with

$$\mathcal{A}V = \begin{cases} -\nu \Delta V & \text{in } \mathcal{F}_0\,, \\ \dfrac{2\nu}{m} \displaystyle\int_{\partial B_0} D(V) n \,\mathrm{d}s + \dfrac{2\nu}{\mathcal{J}} x^\perp \int_{\partial B_0} y^\perp \cdot D(V) n \,\mathrm{d}s(y) & \text{in } B_0\,, \end{cases}$$

and where $\mathbb{P}_2$ is the orthogonal projector from $L^2(\mathbb{R}^2)$ onto $\mathcal{L}^2$.

For arbitrary $p \in [1, \infty)$, $\mathcal{L}^p$ is a closed subspace of $L^p(\mathbb{R}^2)$. Hence we can define $\mathbb{P}_p$ the projector operators from $L^p(\mathbb{R}^2)$ onto $\mathcal{L}^p$ which coincide with $\mathbb{P}_2$ on $L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ (see e.g. [28]). These projectors are obviously continuous and satisfy $\mathbb{P}_p V = \mathbb{P}_q V$ for all $V \in L^p \cap L^q$. In what follows, we omit the index p. We emphasize that the pressure does not appear in the abstract ODE. But, once a solution V is constructed, one shows the existence of a pressure such that (1.22)–(1.29) holds true (see the proof of Corollary 4.3 in [22]). According to this, for sake of simplicity we omit to mention the pressure when considering solutions of (1.22)–(1.29).

Proposition 4.2 in [22] shows that A is a self-adjoint maximal monotone operator. Therefore, applying Hille-Yosida theorem (see e.g. [2, Theorem 7.7]) yields global solutions to (1.22)–(1.29) for arbitrary initial data $V_0 \in \mathcal{L}^2$. Furthermore, there holds:

$$\|V(t)\|_{\mathcal{L}^2} \leq \|V_0\|_{\mathcal{L}^2}\,, \qquad \forall\, t \in \mathbb{R}^+\,,$$

and there exists a constant C such that

$$\|\partial_t V(t)\|_{\mathcal{L}^2} = \|AV(t)\|_{\mathcal{L}^2} \leq \frac{C}{t}\,\|V_0\|_{\mathcal{L}^2}\,.$$

Using the identity

$$\nu\,\|D(V)\|^2_{L^2(\mathcal{F}_0)} = \langle AV, V\rangle_{\mathcal{L}^2}$$

for $V \in \mathcal{D}(A)$, see [22, p.61], Lemma 4.1 in [22] implies

$$\|\nabla V(t)\|_{L^2(\mathcal{F}_0)} \leq \frac{C}{\sqrt{t}}\,\|V_0\|_{\mathcal{L}^2}\,.$$

Hence, previous results in [22] imply Theorem 1.1 when $q = p = 2$.

To generalize this result to arbitrary values for p and q, we provide here an original decomposition of $V(t)$.

2.1. **Spherical-harmonic decomposition of $\mathcal{L}^p$ spaces.** To motivate the spherical-harmonic decomposition of $\mathcal{L}^p$, assume for instance that $V \in \mathcal{L}^p$ is smooth and denote $(\ell_V, \omega_V) \in \mathbb{R}^2 \times \mathbb{R}$ the only pair such that $V(x) = \ell_V + \omega_V x^\perp$ in B_0. As V is divergence-free, there exists $\tilde{\Psi} \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $V = \nabla^\perp \tilde{\Psi}$. Fixing $\tilde{\Psi}(0) = 0$ yields:

$$\tilde{\Psi}(x) = \frac{1}{2}\omega_V |x|^2 + \ell_V \cdot x^\perp\,, \quad \text{in } \overline{B_0}\,.$$

Consequently, introducing radial coordinates (r,θ) and expanding $\tilde{\Psi}$ in Fourier series:

$$\tilde{\Psi}(r,\theta) = \sum_{k=0}^{\infty} \Psi_k(r)\cos(k\theta) + \sum_{k=1}^{\infty} \Phi_k(r)\sin(k\theta), \quad \forall\,(r,\theta) \in (0,\infty)\times(-\pi,\pi)\,,$$

we observe that, setting $\ell_{V,1} = \ell_V \cdot e_1$, $\ell_{V,2} = \ell_V \cdot e_2$: where e_1 and e_2 is the canonical orthonormal basis of $\mathbb{R}^2$:

$$\begin{aligned}
&\Psi_0(r) = \frac{\omega_V}{2} r^2\,, \quad \Psi_1(r) = \ell_{V,2} r\,, \quad \Phi_1(r) = -\ell_{V,1} r\,, \qquad \forall\, r \in (0,1)\,,\\
&\Psi_k(r) = 0\,, \quad \Phi_k(r) = 0\,, \quad \forall\, k \geq 2\,, \qquad \forall\, r \in (0,1)\,.
\end{aligned}$$

Then, informations on ω_V and ℓ_V are contained in the zero and first modes of Ψ respectively, so that these modes are handled separately from the others. In particular, we focus on $\partial_r \Psi_0, \Phi_1, \Psi_1$, that we denote by W, Φ, Ψ respectively and regroup the other terms into a remainder. In what follows, we still denote (r,θ) radial coordinates and introduce (e_r, e_θ) the associated local basis. Accordingly, we denote by V_r and V_θ the radial and tangential components of a vector V. To state our result precisely, we also introduce, for $p \in [1,\infty]$, the set

$$L^p_\sigma(\mathcal{F}_0) = \{V \in \mathcal{L}^p,\ V = 0 \text{ on } B_0\}.$$

Though this space contains functions defined on $\mathbb{R}^2$, we will often identify the elements of $L^p_\sigma(\mathcal{F}_0)$ with their restrictions on $\mathcal{F}_0$.

Proposition 2.1. *Let $p \in [1,\infty]$ and $V \in \mathcal{L}^p$, then there exists a unique 4-uplet (W, Ψ, Φ, V_R) such that:*

(i) $V(r,\theta) = W(r)\min(r,1)e_\theta(\theta) + \nabla^\perp[\Psi(r)\cos(\theta)] + \nabla^\perp[\Phi(r)\sin(\theta)] + V_R(r,\theta)$,
(ii) $W = W(r) \in L^p((0,\infty), r\mathrm{d}r)$, *and W is constant on $(0,1)$: $W(r) = \ell_W = \omega_V$ for $r \in (0,1)$.*

(iii) $(\Psi,\Phi) = (\Psi(r),\Phi(r)) \in W^{1,p}_{loc}(0,\infty)$ *with*

$$\int_0^\infty \left[\left|\frac{\Psi(r)}{r}\right|^p + \left|\frac{\Phi(r)}{r}\right|^p + |\partial_r \Psi(r)|^2 + |\partial_r\Phi(r)|^p \right] r\mathrm{d}r < \infty \, ,$$

and the functions $\Psi/r, \Phi/r, \partial_r\Psi, \partial_r\Phi$ *are constant on* $(0,1)$*:* $\Psi(r)/r = \partial_r\Psi(r) = \ell_2 = \ell_{V,2}$ *and* $\Phi(r)/r = \partial_r\Phi(r) = -\ell_1 = -\ell_{V,1}$ *for* $r \in (0,1)$*.*

(iv) $V_R = V_R(x) \in L^p_\sigma(\mathcal{F}_0)$ *and the following identities hold true:*

$$\int_0^{2\pi} V_R(r,\theta)\cdot e_r \cos(\theta)\, d\theta = \int_0^{2\pi} V_R(r,\theta)\cdot e_r \sin(\theta)\, d\theta = \int_0^{2\pi} V_R(r,\theta)\cdot e_\theta \, d\theta = 0 \, , \quad \forall\, r \in (1,\infty) \, . \tag{2.2}$$

Furthermore, there exists a constant $C(p)$ *depending only on* p *such that*

$$\begin{aligned} \|W\|_{L^p((0,\infty),r\mathrm{d}r)} &+ \|V_R\|_{L^p(\mathbb{R}^2)} \\ &+ \|\partial_r\Psi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Psi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \\ &+ \|\partial_r\Phi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Phi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \le C(p)\|V\|_{\mathcal{L}^p} \, . \end{aligned} \tag{2.3}$$

There exists also a constant $C(p)$ *depending only on* p *so that conversely:*

$$\begin{aligned} \|V\|_{\mathcal{L}^p} \le\; & C(p)\Big(\|W\|_{L^p((0,\infty),r\mathrm{d}r)} + \|V_R\|_{L^p(\mathbb{R}^2)} + \|\partial_r\Psi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Psi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \\ & + \|\partial_r\Phi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Phi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \Big) \, , \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \|\nabla V\|_{L^p(\mathcal{F}_0)} \le\; & C(p)\Big(\|\partial_r W\|_{L^p((1,\infty),r\mathrm{d}r)} + \|W/r\|_{L^p((1,\infty),r\mathrm{d}r)} + \|\nabla V_R\|_{L^p(\mathcal{F}_0)} \\ & + \|\partial_{rr}\Psi\|_{L^p((1,\infty),r\mathrm{d}r)} + \|\partial_r\Psi/r\|_{L^p((1,\infty),r\mathrm{d}r)} + \|\Psi/r^2\|_{L^p((1,\infty),r\mathrm{d}r)} \\ & + \|\partial_{rr}\Phi\|_{L^p((1,\infty),r\mathrm{d}r)} + \|\partial_r\Phi/r\|_{L^p((1,\infty),r\mathrm{d}r)} + \|\Phi/r^2\|_{L^p((1,\infty),r\mathrm{d}r)} \Big) \, . \end{aligned} \tag{2.5}$$

Proof. Let $p \in [1,\infty]$. We first note that, given $V_R \in L^p_\sigma(\mathcal{F}_0)$, it is possible to define by duality the functions:

$$r \mapsto \int_0^{2\pi} V_R(r,\theta)\cdot e_r\cos(\theta)\,\mathrm{d}\theta \, , \quad r \mapsto \int_0^{2\pi} V_R(r,\theta)\cdot e_r\sin(\theta)\,\mathrm{d}\theta \, , \quad r \mapsto \int_0^{2\pi} V_R(r,\theta)\cdot e_\theta\,\mathrm{d}\theta \, ,$$

on $(0,\infty)$. This yields $L^1_{\mathrm{loc}}(1,\infty)$ functions which might satisfy (2.2). Also, once Φ,Ψ and W, V_R are constructed with the regularity of (ii)–(iv), then (i) yields:

$$V_r(r,\theta) = \frac{\Psi(r)}{r}\sin(\theta) - \frac{\Phi(r)}{r}\cos(\theta) + V_R(r,\theta)\cdot e_r \, , \tag{2.6}$$

$$V_\theta(r,\theta) = W(r)\min(1,r) + \partial_r\Psi(r)\cos(\theta) + \partial_r\Phi(r)\sin(\theta) + V_R(r,\theta)\cdot e_\theta \, . \tag{2.7}$$

This implies (2.4) and (2.5).

To prove existence and uniqueness of W,Φ,Ψ, we assume $V \in \mathcal{L}^p$. With this further assumption, identities (2.6) and (2.7) together with (2.2) imply that the only possible candidates W,Φ,Ψ are the following functions:

$$W(r) := \frac{1}{2\pi\min(1,r)}\int_0^{2\pi} V_\theta(r,\theta)\,\mathrm{d}\theta \, , \tag{2.8}$$

$$\Phi(r) := -\frac{r}{\pi}\int_0^{2\pi} V_r(r,\theta)\cos(\theta)\,\mathrm{d}\theta \, , \tag{2.9}$$

$$\Psi(r) := \frac{r}{\pi}\int_0^{2\pi} V_r(r,\theta)\sin(\theta)\,\mathrm{d}\theta \, . \tag{2.10}$$

Differentiating the formulas (2.9)–(2.10) and recalling that V is divergence-free then yields:

$$\partial_r\Phi(r) = \frac{1}{\pi}\int_0^{2\pi} V_\theta(r,\theta)\sin(\theta)\,\mathrm{d}\theta, \quad \partial_r\Psi(r) = \frac{1}{\pi}\int_0^{2\pi} V_\theta(r,\theta)\cos(\theta)\,\mathrm{d}\theta \, ,$$

(where these identities have to be understood in the sense of $\mathcal{D}'(0,\infty)$) and we have then

$$V_R := V - (W \min(1,r) e_\theta + \nabla^\perp[\Psi\cos(\theta)] + \nabla^\perp[\Phi\sin(\theta)]). \tag{2.11}$$

In the ball, we deduce from these definitions and from $V = \ell_V + \omega_V x^\perp$ that for all $r \in (0,1)$:

$$W(r) = \omega_V, \quad \Psi(r)/r = \partial_r\Psi(r) = \ell_{V,2}, \quad \Phi(r)/r = \partial_r\Phi(r) = -\ell_{V,1}, \quad V_R = 0.$$

From the definition (2.8) of W, Jensen inequality implies that:

$$|W(r)|^p \le \frac{1}{2\pi}\int_0^{2\pi} |V_\theta|^p(r,\theta)\,\mathrm{d}\theta \qquad \forall\, r > 1.$$

Combining with the remark that $W(r) = \omega_V$ for $r < 1$, we obtain there exists a constant C for which $\|W\|_{L^p((0,\infty),r\mathrm{d}r)} \le C(p)\|V\|_{\mathcal{L}^p}$. Similarly, we prove that

$$\|\partial_r\Psi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Psi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \le C(p)\|V\|_{\mathcal{L}^p}\,,$$
$$\|\partial_r\Phi\|_{L^p((0,\infty),r\mathrm{d}r)} + \|\Phi/r\|_{L^p((0,\infty),r\mathrm{d}r)} \le C(p)\|V\|_{\mathcal{L}^p}\,.$$

Finally, straightforward computations yield that V_R is divergence-free, vanishes in B_0 and satisfies (2.2). As, combining previous estimates and (2.11) also yields that $V_R \in \mathcal{L}^p$, we conclude that $V_R \in L^p_\sigma(\mathcal{F}_0)$ and that (2.3) holds true. This ends the proof of Proposition 2.1. □

Of course, Proposition 2.1 and Theorem 1.1 hold true if we replace the norms $\|\cdot\|_{\mathcal{L}^p}$ by $\|\cdot\|_{L^p(\mathbb{R}^2)}$. We have chosen to keep the notations of Takahashi and Tucsnak [22], where they prove that A is a self-adjoint maximal monotone operator for the scalar product (1.32).

Let us also emphasize that in Proposition 2.1, all the functions $(W, \partial_r\Psi, \Psi/r, \partial_r\Phi, \Phi/r, v_R)$ defined on $\mathbb{R}^2$ are constant on B_0, so that we can identify these extensions with the pairs given by their restriction to $\mathcal{F}_0$, denoted will small caps, and their mean value on the ball B_0, denoted ℓ. For instance, we will write $W \doteq (w, \ell_W)$. Moreover, in all the text, we will identify $\ell_W(t) = \ell_{W(t)}$.

2.2. **Decomposition in spherical harmonics of the Stokes semigroup.** In the rest of this section and in Section 3, we only consider smooth initial data, namely $V_0 \in \mathcal{L}^2 \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$. Indeed, it is sufficient to show Theorem 1.1 for smooth initial data, because $\mathcal{L}^2 \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$ is dense in $\mathcal{L}^q$, for the $\mathcal{L}^q$ norm with $q \in (1,\infty)$. So the estimates (1.33)-(1.35) could be extended, thanks to the linearity of the Stokes system.

In this paragraph, we prove that the spherical-harmonic decomposition of $\mathcal{L}^p$ is well-adapted to compute solutions of (1.22)–(1.29). We prove:

Proposition 2.2. *Given $V_0 \in \mathcal{L}^2 \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$, the spherical-harmonic decomposition provided by Proposition 2.1 of the unique solution $V \in C([0,\infty);\mathcal{L}^2)$ of* (1.22)–(1.29) *satisfies:*

- $W \doteq (w, \ell_W)$, *where $w \in \mathcal{C}([0,\infty), L^2((1,\infty), r\mathrm{d}r)) \cap \mathcal{C}^\infty((0,\infty)\times[1,\infty))$ verifies:*

$$\partial_t w + \nu\left(-\frac{1}{r}\partial_r(r\partial_r w) + \frac{1}{r^2}w\right) = 0 \qquad \text{for } (t,r) \in (0,\infty)\times(1,\infty); \tag{2.12}$$
$$w(t,1) = \ell_W(t) \qquad \text{for } t \in (0,\infty); \tag{2.13}$$
$$\ell'_W(t) = \frac{2\nu\pi}{\mathcal{J}}(\partial_r w(t,1) - w(t,1)) \qquad \text{for } t \in (0,\infty); \tag{2.14}$$

- $\partial_r\Psi \doteq (\partial_r\psi, \ell_2)$ *and* $\Psi/r \doteq (\psi/r, \ell_2)$, *where* $\partial_r\psi, \psi/r \in \mathcal{C}([0,\infty); L^2((1,\infty), r\mathrm{d}r))$, $\psi \in \mathcal{C}^\infty((0,\infty)\times[1,\infty))$ *and there exists a pressure* $q_1 \in \mathcal{C}^\infty((0,\infty)\times[1,\infty))$ *satisfying* $\partial_r q_1 \in$

$\mathcal{C}((0,\infty);L^2((1,\infty),r\mathrm{d}r))$ *such that:*

$$\partial_t\psi+\nu\left(-\frac{1}{r}\partial_r(r\partial_r\psi)+\frac{1}{r^2}\psi\right)=-r\partial_r q_1 \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{2.15}$$

$$\partial_t\partial_r\psi+\nu\partial_r\left(-\frac{1}{r}\partial_r(r\partial_r\psi)+\frac{1}{r^2}\psi\right)=-\frac{q_1}{r} \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{2.16}$$

$$\psi(t,1)=\partial_r\psi(t,1)=\ell_2(t) \qquad \text{for } t\in(0,\infty); \tag{2.17}$$

$$\frac{m}{\pi}\ell_2'(t)=-q_1(t,1)-\nu\left(-\frac{1}{r}\partial_r(r\partial_r\psi)+\frac{1}{r^2}\psi\right)(t,1) \qquad \text{for } t\in(0,\infty); \tag{2.18}$$

- $\partial_r\Phi \doteq (\partial_r\varphi,-\ell_1)$ *and* $\Phi/r\doteq(\varphi/r,-\ell_1)$*, where* $(\partial_r\varphi,\varphi/r)\in\mathcal{C}([0,\infty);L^2((1,\infty),r\mathrm{d}r))$, $\varphi\in\mathcal{C}^\infty((0,\infty)\times[1,\infty))$ *and there exists a pressure* $p_1\in\mathcal{C}^\infty((0,\infty)\times[1,\infty))$ *satisfying* $\partial_r p_1\in\mathcal{C}((0,\infty);L^2((1,\infty),r\mathrm{d}r))$ *such that:*

$$\partial_t\varphi+\nu\left(-\frac{1}{r}\partial_r(r\partial_r\varphi)+\frac{1}{r^2}\varphi\right)=r\partial_r p_1 \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{2.19}$$

$$\partial_t\partial_r\varphi+\nu\partial_r\left(-\frac{1}{r}\partial_r(r\partial_r\varphi)+\frac{1}{r^2}\varphi\right)=\frac{p_1}{r} \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{2.20}$$

$$\varphi(t,1)=\partial_r\varphi(t,1)=-\ell_1(t) \qquad \text{for } t\in(0,\infty); \tag{2.21}$$

$$\frac{m}{\pi}\ell_1'(t)=-p_1(t,1)+\nu\left(-\frac{1}{r}\partial_r(r\partial_r\varphi)+\frac{1}{r^2}\varphi\right)(t,1) \qquad \text{for } t\in(0,\infty); \tag{2.22}$$

- $V_R\doteq(v_R,0)$*, where* $v_R\in\mathcal{C}([0,\infty),L^2(\mathcal{F}_0))\cap\mathcal{C}^\infty((0,\infty)\times\bar{\mathcal{F}}_0)$ *and there exists* $p_R\in\mathcal{C}^\infty((0,\infty)\times\mathcal{F}_0)$ *such that:*

$$\partial_t v_R-\nu\Delta v_R+\nabla p_R=0 \qquad \text{for } (t,x)\in(0,\infty)\times\mathcal{F}_0; \tag{2.23}$$

$$\operatorname{div} v_R=0 \qquad \text{for } (t,x)\in(0,\infty)\times\mathcal{F}_0; \tag{2.24}$$

$$v_R(t,x)=0 \qquad \text{for } (t,x)\in(0,\infty)\times\partial B_0. \tag{2.25}$$

We postpone the proof of Proposition 2.2 to Appendix A. This mainly consists of tedious computations.

The main interest of Proposition 2.2 is that it reduces the study of the Stokes semigroup to the study of scalar equations for the modes involving non-trivial boundary conditions and one Stokes equation with homogeneous boundary conditions on the obstacle. Indeed:

- System (2.12)–(2.14) is a scalar heat equation with dynamic boundary condition.
- System (2.23)–(2.25) is a Stokes equation with a fixed obstacle and Dirichlet boundary condition.
- Systems (2.15)–(2.18) and (2.19)–(2.22) are similar one to each other. Actually, (φ,p_1,ℓ_1) solves (2.19)–(2.22) if and only if $(-\varphi,p_1,\ell_1)$ solves (2.15)–(2.18). System (2.15)–(2.18) involves two scalar heat equations (2.15)–(2.16) which contain the term q_1 reminiscent from the pressure. It also involves intricate boundary conditions (2.17)–(2.18) which couples Dirichlet $(\psi(t,1))$, Neumann $(\partial_r\psi(t,1))$ and dynamic (see (2.18)) boundary conditions.

Whereas systems (2.12)-(2.14) and (2.23)-(2.25) are classical and widely studied in the literature, systems (2.15)-(2.18) and (2.19)-(2.22) do not seem known and are the main challenge of our study. Actually, we show that systems (2.15)-(2.18) and (2.19)-(2.22) reduce to a heat equation with dynamic boundary conditions. Concerning ψ for instance, our strategy consists of removing the pressure term and reduce (2.15)-(2.18) to a scalar equation for the new unknown:

$$Z(r):=\partial_r\Psi(r)+\frac{\Psi(r)}{r}=\frac{1}{r}\partial_r[r\Psi(r)]\,,\quad \forall\, r\in(0,\infty), \tag{2.26}$$

which, in particular, is a constant function on the ball B_0, denoted by ℓ_Z, and for which we have

$$Z(r)=\ell_Z \quad \forall\, r\in(0,1), \qquad \ell_Z=2\ell_2. \tag{2.27}$$

Note that, using the definition (2.26) of Z and the fact the $\Psi(r)/r = \ell_2$ on the unit ball, identity (2.27) immediately implies $\partial_r \Psi = \ell_2$ on the unit ball, thus being completely compatible with the boundary conditions (2.17).

Indeed, using this new unknown, we get:

Proposition 2.3. *Given $V_0 \in \mathcal{L}^2 \cap \mathcal{C}_c^\infty(\mathbb{R}^2)$, let (W, Ψ, Φ, V_R) be the spherical-harmonic decomposition given by Proposition 2.1 of the solution $V \in \mathcal{C}([0,\infty); \mathcal{L}^2)$ of (1.22)–(1.29). Then, setting $Z \doteq (z, \ell_Z)$ as in (2.26), (or $Z = -\partial_r(r\Phi)/r$),*

- *$z \in \mathcal{C}([0,\infty); L^2((1,\infty), r\mathrm{d}r)) \cap \mathcal{C}^\infty((0,\infty) \times [1,\infty))$*
- *(z, ℓ_Z) is a solution to:*

$$\partial_t z - \nu\Big(\partial_{rr} + \frac{1}{r}\partial_r\Big) z = 0 \qquad \text{for } (t,r) \in (0,\infty) \times (1,\infty); \tag{2.28}$$

$$z(t,1) = \ell_Z(t) \qquad \text{for } t \in (0,\infty); \tag{2.29}$$

$$\ell_Z'(t) = \alpha_0 \nu \partial_r z(t,1) \qquad \text{for } t \in (0,\infty); \tag{2.30}$$

with

$$\alpha_0 = \frac{4\pi}{\pi + m}. \tag{2.31}$$

Proof. Up to a change of sign, we focus on $Z = \partial_r(r\Psi)/r$. Thanks to the regularity proved in Proposition 2.2, we have $(\partial_r\psi, \psi/r) \in \mathcal{C}([0,\infty); L^2((1,\infty), r\mathrm{d}r))$. Consequently $z = \partial_r\psi + \psi/r$ enjoys the same regularity. The smoothness of z is straightforward.

Differentiating (2.15) with respect to r and subtracting (2.16), the pressure $q_1(t)$ satisfies, for each time $t > 0$:

$$-\frac{1}{r}\partial_r(r\partial_r q_1) + \frac{1}{r^2} q_1 = 0, \qquad \text{for } r \in (1,\infty).$$

Hence $q_1(t,r) = \alpha_1(t) r + \frac{\beta_1(t)}{r}$. Of course, the condition $\partial_r q_1 \in L^2((1,\infty), r\,\mathrm{d}r)$ implies that:

$$q_1(t,r) = \frac{\beta_1(t)}{r}, \tag{2.32}$$

and therefore, for all $t > 0$ and $r \geq 1$,

$$-\partial_r q_1 = \frac{q_1}{r}.$$

With this identity, the pressure can be removed simply by adding (2.16) to $1/r$ times (2.15):

$$\partial_t \left[\left(\partial_r + \frac{Id}{r} \right) \psi \right] + \nu \left(\partial_r + \frac{Id}{r} \right) \left(-\frac{1}{r}\partial_r(r\partial_r\psi) + \frac{1}{r^2}\psi \right) = 0.$$

Using (2.26),

$$\partial_r z = -\left(-\frac{1}{r}\partial_r(r\partial_r\psi) + \frac{1}{r^2}\psi \right), \tag{2.33}$$

and the new variable z in (2.26) solves (2.28).

Concerning the boundary conditions, (2.17) reads as

$$z(t,1) = 2\ell_2(t) = \ell_Z(t)$$

and, using (2.33) and (2.32), (2.18) yields

$$\frac{m}{\pi}\ell_2'(t) = -\beta_1(t) + \nu\partial_r z(t,1).$$

Moreover, still using (2.33) and (2.32), (2.15) for $r = 1$ and (2.17) gives

$$\ell_2'(t) - \nu\partial_r z(t,1) = \beta_1(t).$$

Combining the previous equations, (z, ℓ_Z) solves (2.28)-(2.30). □

Remark 2.4. In what follows, given $V \doteq (v, \ell, \omega)$ a solution to (1.22)–(1.29) on $(0,\infty)$, we keep the convention:

$$Z_\Psi(t,r) := \partial_r \Psi(t,r) + \frac{\Psi(t,r)}{r}\,,\ Z_\Phi(t,r) := -\left(\partial_r \Phi(t,r) + \frac{\Phi(t,r)}{r}\right)\,,\ \forall (t,r) \in [0,\infty) \times (0,\infty)\,.$$

We emphasize that, for $t > 0$, $V(t,\cdot)$ has continuous normal and tangential traces through ∂B_0, and thus then $Z_\Phi(t,\cdot)$ and $Z_\Psi(t,\cdot)$ have continuous traces through the interface $r = 1$.

Remark 2.5. As we recalled in the introduction, the classical approach would rather consist in the elimination of the pressure in the Navier Stokes system by taking the curl of the Navier Stokes equation, yielding that way an equation for the vorticity of the velocity-field. *But this is not the method we choose here.* Indeed, in an exterior domain, one should complete the vorticity equation, and this would yield non-dissipative boundary conditions of Robin type.

2.3. **Some elliptic problems.** To conclude this section, we prove some technical lemmas that will be useful later on. Indeed, in order to compute the decay of the Stokes semigroup, we study the decay of solutions to the heat equation (2.28)-(2.30). This gives the decay of the new unknown z whether it is computed with respect to φ or ψ. However, to our purpose, we need then to invert the definition of z in order to get also the decay of φ and ψ in suitable spaces. This is the content of the following proposition:

Proposition 2.6. *Given $p \in (1,\infty]$ and $(z,\ell) \in L^p((1,\infty), r\mathrm{d}r) \times \mathbb{R}$, there exists a unique $\psi \in W^{1,p}_{loc}(1,\infty)$ solution to the following boundary value problem:*

$$\begin{aligned} \partial_r \psi(r) + \frac{\psi(r)}{r} &= z(r)\,, \qquad \textit{for } r \in (1,\infty)\,, && (2.34)\\ \psi(1) &= \ell\,, && (2.35) \end{aligned}$$

and there exists a constant $C(p)$ depending only on p for which:

$$\|\partial_r \psi\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{\psi}{r}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \le C(p)\left(\|z\|_{L^p((1,\infty),r\mathrm{d}r)} + |\ell|\right)\,. \qquad (2.36)$$

Proof. Let $p \in (1,\infty]$ and (z,ℓ) satisfy the assumptions of Proposition 2.6. It is straightforward that the unique solution to (2.34)-(2.35) reads:

$$\psi(r) = \frac{\ell}{r} + \frac{1}{r}\int_1^r s z(s)\,\mathrm{d}s\,, \quad \forall\, r \ge 1\,.$$

If $p = \infty$, we establish easily (2.36) from this formula. If $p \in (1,\infty)$, up to a regularizing argument we skip for conciseness, we multiply (2.34) by $|\psi|^{p-2}\psi/r^{p-1}$ on $[1,R]$, for arbitrary $R > 1$:

$$\begin{aligned} \int_1^R z|\psi|^{p-2}\frac{\psi}{r^{p-1}} r\,dr &= \left[\frac{|\psi(r)|^p}{r^{p-2}}\right]_1^R - (p-1)\left(\int_1^R \frac{\partial_r|\psi|^p}{p r^{p-2}}\,dr - \int_1^R \frac{|\psi|^p}{r^p} r\,dr\right)\\ &= \frac{1}{p}\left[\frac{|\psi(r)|^p}{r^{p-2}}\right]_1^R + 2\left(1 - \frac{1}{p}\right)\int_1^R \frac{|\psi|^p}{r^{p-1}}\,dr\,,\\ &\ge -\frac{|\ell|^p}{p} + 2\left(1 - \frac{1}{p}\right)\int_1^R \frac{|\psi|^p}{r^{p-1}}\,dr\,. \end{aligned}$$

Hence, for all $p \in (1,\infty)$,

$$\begin{aligned} \left\|\frac{\psi}{r}\right\|^p_{L^p((1,R),r\mathrm{d}r)} &\le C(p)\,\|z\|_{L^p((1,\infty),r\mathrm{d}r)} \left\|\left(\frac{\psi}{r}\right)^{p-1}\right\|_{L^{p'}((1,R),r\mathrm{d}r)} + C(p)|\ell|^p\\ &\le C(p)\,\|z\|_{L^p((1,\infty),r\mathrm{d}r)} \left\|\frac{\psi}{r}\right\|^{p-1}_{L^p((1,R),r\mathrm{d}r)} + C(p)|\ell|^p. \end{aligned}$$

This yields

$$\left\|\frac{\psi}{r}\right\|_{L^p((1,R),r\mathrm{d}r)} \leq C(p)\left(\|z\|_{L^p((1,\infty),r\mathrm{d}r)} + |\ell|\right).$$

Letting then $R \to \infty$ we obtain (2.36). □

Let us now state another elliptic estimate that will be useful in the following:

Proposition 2.7. *Let $p \in (1,\infty)\setminus\{2\}$ and assume that $z \in L^p((1,\infty), r\mathrm{d}r)$ and $\partial_r z \in L^p((1,\infty), r\mathrm{d}r)$. There exists a constant $C(p)$ depending only on p such that:*

$$\left\|\frac{z}{r}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \leq C(p)\left(\|\partial_r z\|_{L^p((1,\infty),r\mathrm{d}r)} + \varepsilon_p|z(1)|\right) \tag{2.37}$$

where $\varepsilon_p = 1$ if $p > 2$ and $\varepsilon_p = 0$ if $p < 2$.

Proof. As z belongs to $W^{1,p}_{loc}(1,\infty)$, we infer that it is continuous and we integrate by parts on $[1, R]$:

$$\begin{aligned}\left\|\frac{z}{r}\right\|^p_{L^p((1,R),r\mathrm{d}r)} &= -\frac{1}{p-2}\int_1^R |z|^p\partial_r\Big(\frac{1}{r^{p-2}}\Big)\,\mathrm{d}r = -\frac{1}{p-2}\Big[\frac{|z|^p}{r^{p-2}}\Big]_1^R + \frac{p}{p-2}\int_1^R \frac{|z|^{p-2}z\partial_r z}{r^{p-1}}\,r\mathrm{d}r\\ &\leq \frac{1}{p-2}\Big(|z(1)|^p - \frac{|z(R)|^p}{R^{p-2}}\Big) + \frac{p}{|p-2|}\|\partial_r z\|_{L^p((1,\infty),r\mathrm{d}r)}\left\|\frac{z}{r}\right\|^{p-1}_{L^p((1,R),r\mathrm{d}r)}.\end{aligned}$$

Then, the following depends on the sign of $p-2$:

- if $p > 2$, we directly have that
$$\left\|\frac{z}{r}\right\|^p_{L^p((1,R),r\mathrm{d}r)} \leq \frac{1}{p-2}|z(1)|^p + \frac{p}{p-2}\|\partial_r z\|_{L^p((1,\infty),r\mathrm{d}r)}\left\|\frac{z}{r}\right\|^{p-1}_{L^p((1,R),r\mathrm{d}r)},$$
which gives (2.37) with $\varepsilon_p = 1$.
- if $p < 2$, we get
$$\left\|\frac{z}{r}\right\|^p_{L^p((1,R),r\mathrm{d}r)} \leq \frac{1}{2-p}\frac{|z(R)|^p}{R^{p-2}} + \frac{p}{2-p}\|\partial_r z\|_{L^p((1,\infty),r\mathrm{d}r)}\left\|\frac{z}{r}\right\|^{p-1}_{L^p((1,R),r\mathrm{d}r)}.$$
To establish (2.37) with $\varepsilon_p = 0$, it is sufficient to find a sequence $R_n \to \infty$ such that $(|z(R_n)|^p R_n)$ tends to zero. This can obviously be done since $r \mapsto r|z(r)|^p$ is assumed to belong to $L^1(1,\infty)$.

□

We finally provide elliptic estimates that will be useful when getting estimates on the 0-mode:

Proposition 2.8. *Let $p \in (1,\infty)$ and assume that $w \in L^p((1,\infty), r\mathrm{d}r)$ satisfies*

$$\begin{aligned}&\partial_{rr}w(r) + \frac{\partial_r w(r)}{r} - \frac{w(r)}{r^2} = f(r)\,, \qquad \textit{for } r \in (1,\infty)\,;\\ &\partial_r w(1) - w(1) = a, \qquad w(1) = b,\end{aligned}$$

for some $f \in L^p((1,\infty), r\mathrm{d}r)$, a, b in $\mathbb{R}$. Then, there exists a constant $C(p)$ depending only on p for which:

$$\|\partial_{rr}w\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{\partial_r w}{r} - \frac{w(r)}{r^2}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \leq C(p)\left(\|f\|_{L^p((1,\infty),r\mathrm{d}r)} + |a|\right). \tag{2.38}$$

Furthermore, if $p \neq 2$,

$$\left\|\frac{\partial_r w}{r}\right\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{w}{r^2}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \leq C(p)\left(\|f\|_{L^p((1,\infty),r\mathrm{d}r)} + |a| + \varepsilon_p|b|\right), \tag{2.39}$$

with $\varepsilon_p = 1$ if $p > 2$ and $\varepsilon_p = 0$ if $p < 2$.

Proof. We define $\tilde{w} = w(r)/r$ for $r \geq 1$. Then, $\tilde{w}$ satisfies

$$r\partial_{rr}\tilde{w}(r) + 3\partial_r\tilde{w}(r) = f(r), \qquad \text{for } r \in (1,\infty); \tag{2.40}$$
$$\partial_r\tilde{w}(1) = a, \tag{2.41}$$

Following the method of the proof of Proposition 2.6, we multiply (2.40) by $|\partial_r\tilde{w}|^{p-2}\partial_r\tilde{w}$ on $[1,R]$. After integration by parts, this yields:

$$\begin{aligned}\int_1^R f|\partial_r\tilde{w}|^{p-2}\partial_r\tilde{w}\, r\mathrm{d}r &= \frac{1}{p}\left[r^2|\partial_r\tilde{w}|^p\right]_1^R + \left(3-\frac{2}{p}\right)\int_1^R |\partial_r\tilde{w}|^p\, r\mathrm{d}r\,,\\ &\geq -\frac{|a|^p}{p} + \left(3-\frac{2}{p}\right)\int_1^R |\partial_r\tilde{w}|^p\, r\mathrm{d}r\,.\end{aligned}$$

We conclude that:

$$\|\partial_r\tilde{w}\|_{L^p((1,\infty),r\mathrm{d}r)} \leq C(p)\left(\|f\|_{L^p((1,\infty),r\mathrm{d}r)} + |a|\right). \tag{2.42}$$

Expanding $\partial_r\tilde{w}$, we remark that $\partial_{rr}w = f - \partial_r\tilde{w}$ so that (2.42) implies (2.38).

If $p \neq 2$, we then apply Proposition 2.7 to $\partial_r\tilde{w}$. This yields

$$\left\|\frac{\tilde{w}}{r}\right\|_{L^p((1,\infty),r\mathrm{d}r)} = \left\|\frac{w}{r^2}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \leq C(p)\left(\|f\|_{L^p((1,\infty),r\mathrm{d}r)} + |a| + \varepsilon_p|b|\right). \tag{2.43}$$

Since $\partial_r\tilde{w} = \partial_r w/r - w/r^2$, estimates (2.42) and (2.43) immediately yield (2.39). ☐

3. Study of solutions to (1.22)–(1.29)

The ultimate goal of this section is to prove Theorem 1.1 and Theorem 1.2. In all this section, we assume that $\nu = 1$ for simplicity. This can be done without loss of generality by setting $(V_\nu(t,x), P_\nu(t,x)) := (V(t/\nu,x), P(t/\nu,x)/\nu)$. Because of the computations we presented in the previous section, we first analyze separately the decay of solutions to the Stokes equation with a fixed obstacle and then, we compute the long-time behavior of solutions to both heat equations with dynamic boundary conditions. We conclude by combining all these computations.

3.1. Decay of solutions to (2.23)–(2.25).

System (2.23)–(2.25) has already been studied in the frame of $L^p_\sigma(\mathcal{F}_0)$ spaces [5, Theorem 1.2], [6]:

Theorem 3.1. *For each $q \in (1,\infty)$, the Stokes operator of the linear problem (2.23)-(2.25) generates a semigroup $S_R(t)$ on $L^q_\sigma(\mathcal{F}_0)$. Moreover, this semigroup satisfies the following decay estimates for $v_R(t,\cdot) = S_R(t)v_R(0,\cdot)$:*

- *For $p \in [q,\infty]$, there exists $K_{1,R} = K_{1,R}(p,q) > 0$ such that for every $v_R(0,\cdot) \in L^q_\sigma(\mathcal{F}_0)$,*

$$\|v_R(t,\cdot)\|_{L^p_\sigma(\mathcal{F}_0)} \leq K_{1,R}\, t^{\frac{1}{p}-\frac{1}{q}}\|v_R(0,\cdot)\|_{L^q_\sigma(\mathcal{F}_0)}, \qquad \text{for all} \quad t > 0. \tag{3.1}$$

- *If $q \leq 2$, for $p \in [q,2]$, there exists $K_{2,R} = K_{2,R}(p,q) > 0$ such that for every $v_R(0,\cdot) \in L^q_\sigma(\mathcal{F}_0)$,*

$$\|\nabla v_R(t,\cdot)\|_{L^p(\mathcal{F}_0)} \leq K_{2,R}\, t^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\|v_R(0,\cdot)\|_{L^q_\sigma(\mathcal{F}_0)}, \qquad \text{for all} \quad t > 0. \tag{3.2}$$

- *For $p \in [\max\{2,q\},\infty)$, there exists $K_{3,R} = K_{3,R}(p,q) > 0$ such that for every $v_R(0,\cdot) \in L^q_\sigma(\mathcal{F}_0)$,*

$$\|\nabla v_R(t,\cdot)\|_{L^p(\mathcal{F}_0)} \leq \begin{cases} K_{3,R}\, t^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}\|v_R(0,\cdot)\|_{L^q_\sigma(\mathcal{F}_0)}, & \text{for all} \quad 0 < t < 1,\\ K_{3,R}\, t^{-\frac{1}{q}}\|v_R(0,\cdot)\|_{L^q_\sigma(\mathcal{F}_0)}, & \text{for all} \quad t \geq 1.\end{cases} \tag{3.3}$$

For localized initial data it is possible to obtain a much sharper description of the long-time behavior of v_R by following the spirit of our spherical-harmonic decomposition. To this end, we need a general result on the decay of solutions to heat equations with dynamic boundary conditions. This result is detailed in the following subsection. So, we postpone the more precise computation of the long-time behavior of v_R to the end of this section (see Theorem 3.13).

3.2. **Semigroup estimates.** We proceed with the computation of the long-time behavior of solutions to (2.12)–(2.14) and (2.28)–(2.30). We note that both equations are examples of the family of systems:

$$\partial_t y + \left(-\frac{1}{r}\partial_r (r\partial_r y) + \frac{k^2}{r^2} y \right) = 0\,, \qquad \text{for } (t,r) \in (0,\infty)\times(1,r)\,; \tag{3.4}$$

$$y(t,1) = \ell_Y(t)\,, \qquad \text{for } (t,r) \in (0,\infty)\times(1,r)\,; \tag{3.5}$$

$$\ell_Y'(t) = \tilde{\alpha}(\partial_r y(t,1) - ky(t,1))\,, \qquad \text{for } t \in (0,\infty)\,; \tag{3.6}$$

with parameters $\tilde{\alpha} > 0$ and $k \in \mathbb{N}\cup\{0\}$. Indeed (z,ℓ_Z) solution of (2.28)–(2.30) is a solution to (3.4)–(3.6) in the case

$$k = 0 \qquad \tilde{\alpha} = \frac{4\pi}{\pi + m}$$

whereas (w,ℓ_W) solution of (2.12)–(2.14) is a solution to (3.4)–(3.6) in the case

$$k = 1 \qquad \tilde{\alpha} = \frac{2\pi}{\mathcal{J}}\,.$$

To compute the decay of solutions to (3.4)–(3.6), we use classical methods for parabolic equations (see [7, 23, 24, 19]). In our context, due to the presence of the solid, we shall refer extensively to the works [19, 18] of A. Munnier and E. Zuazua which study thoroughly the equation

$$\begin{cases} \partial_t \mathrm{v} - \Delta_{\mathbb{R}^n}\mathrm{v} = 0, & \text{for } (t,x)\in(0,\infty)\times\mathbb{R}^n\backslash B(0,1),\\ \mathrm{v}(t,x) = \ell_{\mathrm{v}}(t), & \text{for } (t,x)\in(0,\infty)\times\mathcal{S}^{n-1},\\ \ell_{\mathrm{v}}'(t) = \alpha\displaystyle\int_{\mathcal{S}^{n-1}}\partial_r\mathrm{v}(t,x)\,d\sigma, & \text{for } t\in(0,\infty), \end{cases} \tag{3.7}$$

where $\alpha > 0$ is a fixed real number. Formally, for arbitrary $k \in \mathbb{N}$, (y,ℓ_Y) is a solution to (3.4)–(3.6) if and only if the pair $(\mathrm{v},\ell_{\mathrm{v}})$ defined by

$$\ell_{\mathrm{v}}(t) = \ell_Y(t)\,, \qquad \mathrm{v}(t,r,\omega) := \frac{y(t,r)}{r^k}, \qquad \forall\, r>1\,, \quad \forall\,\omega\in\mathcal{S}^{n-1}, \tag{3.8}$$

is a solution of equation (3.7) for

$$n = 2k+2\,, \quad \alpha = \frac{\tilde{\alpha}}{|\mathcal{S}^{n-1}|}\,. \tag{3.9}$$

In this subsection, we fix $k \in \mathbb{N}\cup\{0\}$ and $\tilde{\alpha} > 0$ and study the long-time behavior of the solution of system (3.4)–(3.6). By (3.9), this fixes also values for n and α.

In order to study system (3.7), A. Munnier and E. Zuazua introduce the functional spaces

$$\mathfrak{L}^p(\mathbb{R}^n) = \{Y \in L^p(\mathbb{R}^n),\ \nabla Y = 0 \text{ in } B(0,1)\}, \qquad (p\in[1,\infty]),$$

endowed with the norm:

$$\|Y\|^p_{\mathfrak{L}^p(\mathbb{R}^n)} = \|y\|^p_{L^p(\mathbb{R}^n\backslash B(0,1))} + \frac{1}{\alpha}|\ell_Y|^p, \qquad \text{when } p<\infty\,,$$

$$\|Y\|_{\mathfrak{L}^\infty(\mathbb{R}^n)} = \max(\|y\|_{L^\infty(\mathbb{R}^n\backslash B(0,1))}, |\ell_Y|), \qquad \text{corresponding to } p=\infty\,,$$

where ℓ_Y is the mean value of Y in the ball:

$$\ell_Y = \frac{1}{|B(0,1)|}\int_{B(0,1)} Y(x)\,\mathrm{d}x.$$

As before, in what follows, we identify $(\mathrm{v},\ell_{\mathrm{v}}) \in L^p(\mathbb{R}^n\setminus B(0,1))\times\mathbb{R}$ with the extension $\mathrm{V}\in\mathfrak{L}^p(\mathbb{R}^n)$ given by $\mathrm{V} = \mathbf{1}_{B(0,1)}\ell_{\mathrm{v}} + \mathbf{1}_{\mathbb{R}^n\backslash B(0,1)}\mathrm{v}$, and we shall write $\mathrm{V} \doteq (\mathrm{v},\ell_{\mathrm{v}})$ to denote this extension.

We also introduce a radial variant of $\mathfrak{L}^p(\mathbb{R}^2)$-spaces:

$$\mathfrak{L}^p := \{Y \doteq (y,\ell_Y) \text{ radial function, such that } Y\in\mathfrak{L}^p(\mathbb{R}^2)\}\,.$$

This space is endowed with the norm:

$$\|Y\|^p_{\mathfrak{L}^p} = \|y\|^p_{L^p(\mathcal{F}_0)} + \frac{2\pi}{\tilde{\alpha}}|\ell_Y|^p, \qquad \text{when } p < \infty\,,$$
$$\|Y\|_{\mathfrak{L}^\infty} = \max(\|y\|_{L^\infty(\mathcal{F}_0)}, |\ell_Y|), \qquad \text{corresponding to } p = \infty\,.$$

In the case $p = 2$, this space is a Hilbert space associated with the scalar product:

$$(Y, \tilde{Y}) = \int_{\mathcal{F}_0} y\tilde{y} + \frac{2\pi}{\tilde{\alpha}}\ell_Y \ell_{\tilde{Y}}.$$

For $p \neq 2$, extending this scalar product by a density argument enables to identify the dual of $\mathfrak{L}^p$ with $\mathfrak{L}^{p'}$ where p' is the conjugate exponent of p.

With these notations, A. Munnier and E. Zuazua prove in [19, 18]:

Theorem 3.2 (Decay estimates for (3.7), [19, 18])**.** *Given* $(\mathrm{v}_0, \ell_{\mathrm{v}_0}) \in \mathfrak{L}^2(\mathbb{R}^n)$*, there exists a unique solution* $(\mathrm{v}, \ell_\mathrm{v}) \in \mathcal{C}([0,\infty); \mathfrak{L}^2(\mathbb{R}^n))$ *of* (3.7) *such that* $(\mathrm{v}(0,\cdot), \ell_\mathrm{v}(0)) = (\mathrm{v}_0, \ell_{\mathrm{v}_0})$*. This solution satisfies:*

$$\|(\mathrm{v}(t,\cdot), \ell_\mathrm{v}(t))\|_{\mathfrak{L}^2(\mathbb{R}^n)} \leq \|(\mathrm{v}_0, \ell_{\mathrm{v}_0})\|_{\mathfrak{L}^2(\mathbb{R}^n)}\,, \quad \forall\, t \geq 0\,. \tag{3.10}$$

Moreover, if $(\mathrm{v}_0, \ell_{\mathrm{v}_0}) \in \mathfrak{L}^q(\mathbb{R}^n)$*, for some* $q \in [1,\infty]$*, for all* $p \in [q,\infty]$*, there exists a constant* $C(p,q)$ *such that*

$$t^{\frac{n}{2}(1/q-1/p)}\|(\mathrm{v}(t,\cdot), \ell_\mathrm{v}(t))\|_{\mathfrak{L}^p(\mathbb{R}^n)} \leq \|(\mathrm{v}_0, \ell_{\mathrm{v}_0})\|_{\mathfrak{L}^q(\mathbb{R}^n)}\,, \quad \forall\, t \geq 1\,. \tag{3.11}$$

Theorem 3.3 (First term in the asymptotic expansion of solutions of (3.7), [19, 18])**.** *Given* $(\mathrm{v}_0, \ell_{\mathrm{v}_0}) \in \mathfrak{L}^2(\mathbb{R}^n)$ *such that* $\mathrm{v}_0 \in L^2(\mathbb{R}^n \setminus B(0,1); \exp(|x|^2/4)\mathrm{d}x)$*, setting*

$$M = \int_{\mathbb{R}^n \setminus B(0,1)} \mathrm{v}_0(x)\,\mathrm{d}x + \frac{1}{\alpha}\ell_{\mathrm{v}_0}\,,$$

we get

- *for all* $t > 0$ *and* $p \in [1,\infty]$*,* $(\mathrm{v}(t,\cdot), \ell_\mathrm{v}(t)) \in \mathfrak{L}^p(\mathbb{R}^n)$
- *for all* $p \in [1,\infty]$*, there exists a constant* C_p *such that for all* $t > 0$*,*

$$\begin{aligned} t^{\frac{n}{2}(1-1/p)}\,\|\mathrm{v}(t,\cdot) - MG(t)\|_{L^p(\mathbb{R}^n\setminus B(0,1))} &\leq C_p R_{1,p}(t), \\ t^{\frac{n}{2}}\left|\ell_\mathrm{v}(t) - \frac{M}{(4\pi t)^{\frac{n}{2}}}\right| &\leq CR_2(t), \end{aligned}$$

where

$$G(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(-\frac{|x|^2}{4t}\right),$$

and, denoting by $\delta_{n,2}$ *the Kronecker symbol:*

$$R_{1,p}(t) = \begin{cases} (\delta_{n,2}|\log(t)| + 1)t^{-1/2} & \text{if } p \in [1,2], \\ (\delta_{n,2}|\log(t)| + 1)t^{-1/2+\theta_{n,p}} & \text{if } p \geq 2, \end{cases} \qquad \text{with } \theta_{n,p} = \frac{n}{2}\frac{(p-1)(p-n)}{p(2p+n(p-1))},$$

$$R_2(t) = (\delta_{n,2}|\log(t)|^{1/2} + 1)t^{-1/(n+2)}.$$

We do not give a comprehensive proof of Theorems 3.2–3.3 and let the reader refer to [18, 19] for further details. Let us only recall that the proof of Theorem 3.2 is based on the remark that (3.7) reduces to the abstract ODE: $\partial_t \mathrm{V} + A_{mz}\mathrm{V} = 0$, where A_{mz} is the unbounded operator with domain

$$\mathcal{D}(A_{mz}) = \{\mathrm{V} \doteq (\mathrm{v}, \ell_\mathrm{v}) \in H^2(\mathbb{R}^n \setminus B(0,1)) \times \mathbb{R} \text{ with } \mathrm{v}_{||x|=1} = \ell_\mathrm{v}\}, \tag{3.12}$$

such that:

$$A_{mz}(\mathrm{v}, \ell_\mathrm{v}) = \begin{pmatrix} -\Delta\cdot & 0 \\ -\alpha\displaystyle\int_{\mathcal{S}^{n-1}} \partial_r \cdot d\sigma & 0 \end{pmatrix}\begin{pmatrix} \mathrm{v} \\ \ell_\mathrm{v} \end{pmatrix} = \begin{pmatrix} -\Delta \mathrm{v} \\ -\alpha\displaystyle\int_{\mathcal{S}^{n-1}} \partial_r \mathrm{v} d\sigma \end{pmatrix}. \tag{3.13}$$

A. Munnier and E. Zuazua show that this operator is maximal monotone which implies the existence of a contraction semigroup on $\mathfrak{L}^2(\mathbb{R}^n)$ representing the unique solution to (3.7). Further classical smoothing properties of this semigroup also yield that, for $(\mathrm{v}_0, \ell_{\mathrm{v}_0}) \in \mathcal{D}(A_{mz})$, the unique solution to (3.7) satisfies:

$$\mathrm{V} \in \mathcal{C}^1([0,\infty); \mathfrak{L}^2(\mathbb{R}^n)) \cap \mathcal{C}([0,\infty); \mathcal{D}(A_{mz}))\,. \tag{3.14}$$

We remark that (3.7) is rotational invariant. Hence, considering radial data and noting that transformation (3.8) is a bi-continuous one-to-one and onto mapping from $\mathfrak{L}^2$ to radial functions in $\mathfrak{L}^2(\mathbb{R}^n)$ (with $n = 2k+2$), Theorem 3.2 implies:

Theorem 3.4. *Given $Y_0 \in \mathfrak{L}^2$ there exists a unique solution $Y \in \mathcal{C}([0,\infty); \mathfrak{L}^2)$ of* (3.4)–(3.6) *such that $Y(0,\cdot) = Y_0$. This solution satisfies:*

$$\|Y(t,\cdot)\|_{\mathfrak{L}^2} \le \|Y_0\|_{\mathfrak{L}^2}\,, \quad \forall\, t \ge 0\,.$$

This theorem implies again that the solution to (3.4)–(3.6) is given by a contraction semigroup on $\mathfrak{L}^2$ denoted by S_y in what follows. The results in [19, 18] are not sufficient for our purposes. Indeed, we also have to compute decay rates in $\mathfrak{L}^p - \mathfrak{L}^q$ spaces, similar to the ones in (3.11). But, when $n \neq 2$ (equivalently $k \neq 0$) and $p \neq 2$, the transformation (3.8) is not an isometry between $\mathfrak{L}^p(\mathbb{R}^n)$ and $\mathfrak{L}^p$, so that the "change of dimension" argument does not yield the expected result. Besides, we will also derive estimates on the $\partial_r y$, y/r, $\partial_{rr} y$, $\partial_r y/r$, and y/r^2 in $L^p(\mathcal{F}_0)$, for which no precise estimates were given in [19, 18], except in the case $p = 2$.

In the following subsection, we adapt the arguments of [19, 18] to system (3.4)–(3.6) to estimate the decay of S_y in $\mathfrak{L}^p$. We then explain how to derive estimates on the derivatives of solutions of (3.4)–(3.6) in $\mathfrak{L}^p$.

3.2.1. *$\mathfrak{L}^p - \mathfrak{L}^q$ estimates on y.* Inspired in [19, 18] , we prove the following $\mathfrak{L}^p - \mathfrak{L}^q$ decay estimates for solutions of (3.4)–(3.6):

Theorem 3.5. *For all $q \in [1,\infty)$, system* (3.4)–(3.6) *is well-posed in $\mathfrak{L}^q$: given $Y_0 \in \mathfrak{L}^q$ there is one unique solution Y of* (3.4)–(3.6) *in $\mathcal{C}([0,\infty); \mathfrak{L}^q)$. This solution satisfies:*

$$\|Y(t)\|_{\mathfrak{L}^q} \le \|Y_0\|_{\mathfrak{L}^q}\,. \tag{3.15}$$

We furthermore have the following $\mathfrak{L}^p$ estimates: for all $p \in [q,\infty]$, Y belongs to $\mathcal{C}((0,\infty); \mathfrak{L}^p)$ and there exists a constant C such that

$$t^{1/q-1/p}\, \|Y(t)\|_{\mathfrak{L}^p} \le C\, \|Y_0\|_{\mathfrak{L}^q}\,, \quad t > 0. \tag{3.16}$$

Furthermore, if Y_0 also belongs to $\mathfrak{L}^\infty$, we also have $\|Y(t)\|_{\mathfrak{L}^\infty} \le \|Y_0\|_{\mathfrak{L}^\infty}$.

Before going into the proof of Theorem 3.5, let us emphasize that estimates (3.15)–(3.16) are different from the ones in (3.11) when $k = 1$, *i.e.* $n = 4$, that corresponds to (w, ℓ_W) solutions of (2.12)–(2.14). To be more precise, in that case, using the transformation (3.8) for $r > 1$, (3.11) would then read: for all $q \in [1,\infty]$, $p \in [q,\infty]$, there exists a constant C such that for all $W_0 \doteq (w_0, \ell_{W_0})$ satisfying $w_0/r \in L^q(\mathbb{R}^4 \setminus B(0,1))$, the solution $W \doteq (w, \ell_W)$ of (2.12)–(2.14) satisfies, for all $t > 0$,

$$t^{2(1/q-1/p)} \Big\| \frac{w(t)}{r} \Big\|_{L^p(\mathbb{R}^4 \setminus B(0,1))} \le C \Big\| \Big(\frac{w_0}{r}, \ell_{W_0}\Big) \Big\|_{\mathfrak{L}^q(\mathbb{R}^4)}. \tag{3.17}$$

Hence, the solution W of (2.12)–(2.14) will simultaneously satisfy the decay estimates (3.16) and (3.17). Actually, as we explain below, both results can be proved following the same strategy based on suitable multipliers, the only difference being Sobolev's embeddings.

Proof. Let $Y_0 \in \mathfrak{L}^2$ and $Y \doteq (y, \ell_Y) \in \mathcal{C}([0,\infty); \mathfrak{L}^2)$ be the unique solution to (3.4)–(3.6) given by Theorem 3.4. Up to assume that Y_0 is sufficiently smooth and vanish sufficiently rapidly at infinity we can apply the regularizing effect of the semigroup in $\mathbb{R}^n$ (see (3.14)) so that, going back in $\mathbb{R}^2$ we have $Y \in \mathcal{C}([0,\infty); H^1(\mathbb{R}^2))$ and $y \in \mathcal{C}([0,\infty); H^2(\mathcal{F}_0))$. Then, the idea is to multiply equation (3.4) by $j'(y)$ for smooth non-decreasing convex functional $j = j(y)$ with at most linear growth at infinity. After integration by parts, this yields

$$\frac{d}{dt}\left(\int_{\mathcal{F}_0} j(y) + \frac{2\pi}{\tilde{\alpha}} j(\ell_Y)\right) + \int_{\mathcal{F}_0} j''(y)|\nabla y|^2 + \frac{2k\pi}{\tilde{\alpha}} j'(\ell_Y)\ell_Y + k^2 \int_{\mathcal{F}_0} \frac{j'(y)y}{|x|^2} = 0. \tag{3.18}$$

After a classical regularization argument, one can show that such estimate can be extended to the convex functionals $j(y) = |y|^q$, for $q \in [1, \infty)$, and this yields:

$$\frac{d}{dt}\left(\|Y\|^q_{\mathfrak{L}^q}\right) \leq 0. \tag{3.19}$$

Similarly, using functionals of the form $j(y) = (y - K)_+$, after a suitable regularization argument, one derives

$$\frac{d}{dt}\left(\|Y\|_{\mathfrak{L}^\infty}\right) \leq 0.$$

Based on the contraction property (3.19), the semigroup $S_y(t)$ can be uniquely extended by density to initial data in $\mathfrak{L}^q$ as an operator from $\mathfrak{L}^q$ to itself. We thus have the well-posedness of (3.4)–(3.6) in *any* $\mathfrak{L}^q$, $q \in [1, \infty)$ ($\mathfrak{L}^2 \cap H^2(\mathbb{R}^2 \setminus B(0,1))$ is not densely embedded in $\mathfrak{L}^\infty$, thus our argument does not apply in that case). This yields also the decay estimates (3.15) for all $q \in [1, \infty]$ and $y_0 \in \mathfrak{L}^q$. Note that the decay estimates (3.15) also coincide with the decay estimates (3.16) for any $p = q < \infty$.

Actually, one can go even further. Taking $j(y) = |y|^p$ for $p \geq 2$, estimate (3.18) implies (forgetting the two last terms which are non-negative):

$$\frac{d}{dt}\left(\|Y\|^p_{\mathfrak{L}^p}\right) + \frac{4(p-1)}{p}\int_{\mathcal{F}_0} |\nabla(|y|^{p/2})|^2 \leq 0. \tag{3.20}$$

Using then suitable Sobolev embeddings and interpolation estimate (actually, this is the only step where the dimension plays a role), one gets Lemma 2.2 in [19] (the proof is done in [18]), and in particular [19, (2.17)]: there exists a constant C such that for all functions $Y \doteq (y, \ell_Y) \in \mathfrak{L}^1$ with $y \in H^1(\mathcal{F}_0)$:

$$\|Y\|^4_{\mathfrak{L}^2} \leq C\|Y\|^2_{\mathfrak{L}^1}\|\nabla y\|^2_{L^2(\mathcal{F}_0)}.$$

Applying it to $|Y|^q$, we get the existence of a constant C such that for all $q \geq 1$,

$$\left(\|Y\|^{2q}_{\mathfrak{L}^{2q}}\right)^2 \leq C\left(\|Y\|^q_{\mathfrak{L}^q}\right)^2\|\nabla(|y|^q)\|^2_{L^2(\mathcal{F}_0)}.$$

Plugging this estimate in (3.20) for $p = 2q$ and using the fact that the $\mathfrak{L}^q$-norm of y decays according to (3.19),

$$\frac{d}{dt}\left(\|Y(t)\|^{2q}_{\mathfrak{L}^{2q}}\right) + \frac{2q-1}{8Cq\|Y_0\|^{2q}_{\mathfrak{L}^q}}\left(\|Y(t)\|^{2q}_{\mathfrak{L}^{2q}}\right)^2 \leq 0.$$

Of course, this implies that there exists a constant C independent of $q \in [1, \infty)$ such that

$$\frac{d}{dt}\left(\|Y(t)\|^{2q}_{\mathfrak{L}^{2q}}\right) + \frac{1}{C\|Y_0\|^{2q}_{\mathfrak{L}^q}}\left(\|Y(t)\|^{2q}_{\mathfrak{L}^{2q}}\right)^2 \leq 0.$$

This yields the following decay property: there exists a constant $C > 0$ independent of $q > 0$ such that for all $q \in [1, \infty)$,

$$t\|Y(t)\|^{2q}_{\mathfrak{L}^{2q}} \leq C\left(\|Y_0\|^q_{\mathfrak{L}^q}\right)^2. \tag{3.21}$$

Then, the iteration argument of [26] based on (3.21) applies and yields

$$t^{1/q}\|Y(t)\|_{\mathfrak{L}^\infty} \leq C\|Y_0\|_{\mathfrak{L}^q}, \quad t > 0.$$

Other estimates in (3.16) are deduced for arbitrary $p \in [q, \infty)$ by interpolating the cases $p = q$ and $p = \infty$. □

As we mentioned in the above proof, the semigroup S_y associated with system (3.4)–(3.6) extends to a semigroup on $\mathfrak{L}^q$ for all $q \in [1, \infty)$ that we still denote the same for simplicity. Consequently, Corollary [20, Corollary 2.5, p.5] implies that it is associated to a closed linear operator. In this case the operator reads A_q where

$$\mathcal{D}(A_q) = \{Y \doteq (y, \ell_Y) \in \mathfrak{L}^p \text{ with } A_qY \in \mathfrak{L}^q\}.$$

and

$$A_qY = A_q(y,\ell_Y) = \begin{pmatrix} -\Delta + \dfrac{k^2}{r^2}\cdot & 0 \\ -\dfrac{\tilde{\alpha}}{2\pi}\displaystyle\int_{\mathcal{S}^1} \partial_r \cdot d\sigma & \tilde{\alpha}k \end{pmatrix} \begin{pmatrix} y \\ \ell_Y \end{pmatrix} = \begin{pmatrix} -\Delta y + \dfrac{k^2 y}{r^2} \\ -\dfrac{\tilde{\alpha}}{2\pi}\displaystyle\int_{\mathcal{S}^1} \partial_r y d\sigma + k\tilde{\alpha}\ell_Y \end{pmatrix}.$$

3.2.2. $\mathfrak{L}^p - \mathfrak{L}^q$ *estimates on* $\partial_t Y$. In the case $p = 2$, as A_2 is self-adjoint (see [18, App. A]), Theorem 7.7 in [2] states that, if $Y_0 \in \mathfrak{L}^2$, the solution Y of (3.4)–(3.6) belongs to $C^\infty((0,\infty);\cap_{\ell\in\mathbb{N}}\mathcal{D}(A_2^\ell))$ and

$$\|\partial_t Y(t)\|_{\mathfrak{L}^2} = \|A_2 Y(t)\|_{\mathfrak{L}^2} \le \frac{C}{t}\|Y_0\|_{\mathfrak{L}^2}.$$

Extending this result to the $\mathfrak{L}^q$ case, for $q \in (1,\infty)$ turns out to be slightly more intricate.

Theorem 3.6. *For all $q \in (1,\infty)$, there exists a constant $C = C(q)$ such that for all $Y_0 \in \mathfrak{L}^q$, the solution y of* (3.4)–(3.6) *satisfies, for all $t > 0$,*

$$\|\partial_t Y(t)\|_{\mathfrak{L}^q} \le \frac{C}{t}\|Y_0\|_{\mathfrak{L}^q}. \tag{3.22}$$

Proof. The proof of such result is rather classical, but we did not find precise reference in our precise setting. We follow the proof of Theorem 3.6 in [20, Chapter 7]. First, we recall that $\mathfrak{L}^q$ is a Banach space whose dual is identified with $\mathfrak{L}^{q'}$, for $q' = q/(q-1)$, when taking the duality pairing

$$\langle Y_1, Y_2\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}} = \int_{\mathcal{F}_0} y_1\overline{y_2} + \frac{2\pi}{\tilde{\alpha}}\ell_{Y_1}\overline{\ell_{Y_2}},$$

for $Y_1 \doteq (y_1,\ell_{Y_1})$, $Y_2 \doteq (y_2,\ell_{Y_2})$. Note that, in this proof only, we extend $\mathfrak{L}^p$ to functions having complex values. We focus on the case $q \ge 2$.

For $Y \in \mathcal{D}(A_q)$, $Y^* = |Y|^{q-2}\overline{Y}$ belongs to $\mathfrak{L}^{q'}$ and satisfies

$$\langle Y, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}} = \|Y\|^q_{\mathfrak{L}^q} \text{ and } \|Y^*\|_{\mathfrak{L}^{q'}} = \|Y\|^{q-1}_{\mathfrak{L}^q}.$$

Besides, easy computations yield

$$\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}} = \frac{q}{2}\int_{\mathcal{F}_0}|y|^{q-2}|\nabla y|^2 + \left(\frac{q}{2}-1\right)\int_{\mathcal{F}_0}|y|^{q-4}(\overline{y}\nabla y)^2 + k^2\int_{\mathcal{F}_0}\frac{|y|^q}{r^2} + 2\pi k|\ell_Y|^q.$$

In particular, both first terms can be expressed easily in terms of $|y|^{\frac{q}{2}-2}\overline{y}\nabla y$. So, we introduce the vectors $\vec{a} = \vec{a}(x)$, and $\vec{b} = \vec{b}(x)$ of $\mathbb{R}^2$ defined by $|y|^{\frac{q}{2}-2}\overline{y}\nabla y = \vec{a} + i\vec{b}$. We get

$$\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}} = (q-1)\int_{\mathcal{F}_0}|\vec{a}|^2 + \int_{\mathcal{F}_0}|\vec{b}|^2 + (q-2)i\int_{\mathcal{F}_0}\vec{a}\cdot\vec{b} + k^2\int_{\mathcal{F}_0}\frac{|\mathrm{v}|^q}{r^2} + 2\pi k|g|^q.$$

In particular,

$$\Re\left(\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}}\right) \ge (q-1)\|\vec{a}\|^2_{L^2(\mathcal{F}_0)} + \|\vec{b}\|^2_{L^2(\mathcal{F}_0)},$$

whereas

$$|\Im\left(\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}}\right)| \le |q-2|\|\vec{a}\|_{L^2(\mathcal{F}_0)}\|\vec{b}\|_{L^2(\mathcal{F}_0)}.$$

This implies

$$\frac{|\Im\left(\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}}\right)|}{\Re\left(\langle A_qY, Y^*\rangle_{\mathfrak{L}^q,\mathfrak{L}^{q'}}\right)} \le \frac{1}{2}\frac{|q-2|}{\sqrt{q-1}}. \tag{3.23}$$

From Theorem 3.5, $-A_q$ generates a C_0 semigroup of contractions on $\mathfrak{L}^q$ hence Theorem 3.1 in [20, Chapter 1] implies that for all $\lambda > 0$, λ is in the resolvent set of $-A_q$.

For $q \ge 2$, from (3.23), the numerical range $S(-A_q)$ is contained in the sector $\Sigma_{\theta_0} = \{\lambda \in \mathbb{C}\setminus\{0\} : |\arg\lambda| > \pi - \theta_0\}$ where

$$\theta_0 = \arctan\left(\frac{1}{2}\frac{|q-2|}{\sqrt{q-1}}\right) \in [0,\pi/2).$$

In particular, choosing $\theta_1 \in (\theta_0, \pi/2)$, denoting by $\Sigma_{\theta_1} = \{\lambda \in \mathbb{C} \setminus \{0\} \,:\, |\arg \lambda| > \pi - \theta_1\}$ the corresponding sector of $\mathbb{C}$, and using the fact that $\mathbb{R}_+^*$ is in the resolvent set, Theorem 3.9 in [20, Chapter 1] implies the existence of a $C_\theta = C_\theta(q)$ such that

$$\|(\lambda I + A_q)^{-1}\|_{\mathscr{L}(\mathfrak{L}^p)} \leq \frac{C_\theta}{|\lambda|}, \quad \forall\, \lambda \in \mathbb{C} \setminus \Sigma_{\theta_1}. \tag{3.24}$$

Now, the regularizing properties of the semigroup generated by $-A_q$ are a consequence of Theorem 5.2 in [20, Chapter 2] and the above resolvent estimate. However, here again, we need to be careful since Theorem 5.2 in [20, Chapter 2] requires that 0 belongs to the resolvent set of A_q, which is not the case here. Set $\theta_2 \in (\pi/2, \pi - \theta_1)$. For each $\varepsilon > 0$, we introduce the curve Γ_ε, defined for $\varepsilon > 0$ by the path composed as follows:

$$\Gamma_\varepsilon = \begin{cases} -\rho \exp(-i\theta_2), & \rho \in (-\infty, -\varepsilon), \\ \varepsilon \exp(i\theta), & \theta \in (-\theta_2, \theta_2), \\ \rho \exp(i\theta_2), & \rho \in (\varepsilon, \infty), \end{cases}$$

oriented in the increasing directions of the parameters. Then, for $t > 0$, we use the formula

$$S_y(t) = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I + A_q)^{-1}\, \mathrm{d}\lambda.$$

This integral converges due to the resolvent estimates (3.24) and can be differentiated with respect to time since

$$\frac{1}{2\pi} \int_{\Gamma_\varepsilon} e^{-\Re(\lambda) t} \left\| \lambda(\lambda I + A_q)^{-1} \right\|_{\mathscr{L}_c(\mathfrak{L}^p)} \mathrm{d}\lambda \leq \frac{C_{\theta_2}}{t} + C_t \varepsilon,$$

where the constant C_{θ_2} does not depend on t and $\varepsilon > 0$ and the constant C_t depends on t but not on $\varepsilon > 0$. Of course, letting then $\varepsilon \to 0$, this yields

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} e^{\lambda t} \lambda (\lambda I + A_q)^{-1}\, \mathrm{d}\lambda \right\|_{\mathscr{L}_c(\mathfrak{L}^p)} = \|\partial_t S_y(t)\|_{\mathscr{L}_c(\mathfrak{L}^q)} \leq \frac{C_{\theta_2}}{t}.$$

This completes the proof of (3.22) for $q \geq 2$. The case $q \in (1, 2)$ can be deduced by a simple duality argument. ☐

Remark 3.7. Actually, following the proof of Theorem 5.2 in [20, Chapter 2], one can prove that $-A_q$ generates an analytic semigroup on $\mathfrak{L}^q$ for all $q \in (1, \infty)$.

In the two next subsections, we apply the semigroup estimates we have proved to systems (2.28)-(2.30) and (2.12)-(2.14) .

3.3. **Decay of solutions to (2.28)–(2.30).** We first consider the solution $Z \doteq (z, \ell_Z)$ of (2.28)–(2.30). As we noticed previously, this corresponds to the computations of the previous subsection in the case

$$k = 0\,, \quad \tilde{\alpha} = \frac{4\pi}{\pi + m}\,.$$

We obtain in this way the following decay estimates on solutions:

Theorem 3.8. *Given $q \in (1, \infty)$ and radial $Z_0 \in \mathfrak{L}^q$, there exists a unique solution $Z \in \mathcal{C}([0, \infty); \mathfrak{L}^q)$ to (2.28)-(2.30) such that $Z(0, \cdot) = Z_0$. This solution satisfies the further decay estimates:*

- *for all $p \in [q, \infty]$ we have $Z \in \mathcal{C}((0, \infty); \mathfrak{L}^p)$ and there exists a constant $K_{1,1} = K_{1,1}(p, q)$ such that:*

$$\|Z(t, \cdot)\|_{\mathfrak{L}^p} \leq K_{1,1}\, t^{\frac{1}{p} - \frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}\,, \quad \forall\, t > 0\,, \tag{3.25}$$

- *if $q < 2$ for all $p \in [q, 2)$, we have $(\partial_r z, z/r) \in \mathcal{C}((0, \infty); L^p(\mathcal{F}_0))$ and there exists $K_{2,1} = K_{2,1}(p, q)$ such that:*

$$\|\partial_r z(t, \cdot)\|_{L^p(\mathcal{F}_0)} + \left\| \frac{z(t, \cdot)}{r} \right\|_{L^p(\mathcal{F}_0)} \leq K_{2,1}\, t^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}\,, \quad \forall\, t > 0\,, \tag{3.26}$$

- *if $q \in (1,\infty)$, for all $p \in [\max\{2,q\},\infty)$ with $p > 2$ we have $(\partial_r z, z/r) \in \mathcal{C}((0,\infty); L^p(\mathcal{F}_0))$ and there exists $K_{3,1} = K_{3,1}(p,q)$ such that:*

$$\|\partial_r z(t,\cdot)\|_{L^p(\mathcal{F}_0)} + \left\|\frac{z(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)} \leq \left\{ \begin{array}{ll} K_{3,1}\, t^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}\,, & \forall\, t < 1\,, \\ K_{3,1}\, t^{-\frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}\,, & \forall\, t > 1\,. \end{array} \right. \tag{3.27}$$

These decay estimates are also satisfied for $q = 1$ and $p \in (1,\infty) \setminus \{2\}$.

Proof. Existence of solutions and $\mathfrak{L}^p - \mathfrak{L}^q$ decay estimates are straightforward applications of Theorem 3.5 in the case $k = 0$ and $\tilde{\alpha} = 4\pi/(\pi+m) > 0$. We focus on estimates (3.26)–(3.27). Actually, we only need to prove the case $p = q$, as other cases are then obtained by combining the estimates (3.26)–(3.27) for $p = q \neq 2$ between $t/2$ and t with (3.25) between 0 and $t/2$. Indeed it will follow from the semigroup property:

$$\begin{aligned} \|\partial_r z(t,\cdot)\|_{L^p(\mathcal{F}_0)} + \left\|\frac{z(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)} &\leq K_{2,1}(p,p)\, (t/2)^{-\frac{1}{2}} \|Z(t/2,\cdot)\|_{\mathfrak{L}^p} \\ &\leq K_{2,1}(p,p) K_{1,1}(p,q)\, (t/2)^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}. \end{aligned}$$

For radial $Z_0 \in \mathfrak{L}^q$, estimate (3.22) implies

$$\left\|\partial_{rr} z(t) + \frac{\partial_r z(t)}{r}\right\|_{L^q(\mathcal{F}_0)} + |\partial_r z(t,1)| \leq \frac{C}{t} \|Z_0\|_{\mathfrak{L}^q}.$$

We are now in position to apply Proposition 2.6 to $\partial_r z$ which yields that (see (2.36))

$$\|\partial_{rr} z(t)\|_{L^q(\mathcal{F}_0)} + \left\|\frac{\partial_r z(t)}{r}\right\|_{L^q(\mathcal{F}_0)} \leq \frac{C}{t} \|Z_0\|_{\mathfrak{L}^q}\,. \tag{3.28}$$

From the Gagliardo Niremberg inequality in exterior domains, see [4], we have then: for $q \in [1,\infty]$, for all z such that $\partial_{xx} z$, $\partial_{xy} z$ and $\partial_{yy} z$ belong to $L^q(\mathcal{F}_0)$,

$$\|\nabla z\|_{L^q(\mathcal{F}_0)} \leq C \left(\|\partial_{xx} z\|_{L^q(\mathcal{F}_0)} + \|\partial_{xy} z\|_{L^q(\mathcal{F}_0)} + \|\partial_{yy} z\|_{L^q(\mathcal{F}_0)}\right)^{1/2} \|z\|_{L^q(\mathcal{F}_0)}^{1/2}. \tag{3.29}$$

Since we are focusing on the case of radial solutions, estimates (3.28)–(3.29) and the fact that for radial functions

$$\|\partial_{xx} z\|_{L^q(\mathcal{F}_0)} + \|\partial_{xy} z\|_{L^q(\mathcal{F}_0)} + \|\partial_{yy} z\|_{L^q(\mathcal{F}_0)} \leq C \left(\|\partial_{rr} z\|_{L^q(\mathcal{F}_0)} + \left\|\frac{\partial_r z}{r}\right\|_{L^q(\mathcal{F}_0)} \right),$$

imply

$$\|\partial_r z(t)\|_{L^q(\mathcal{F}_0)} \leq \frac{C}{\sqrt{t}} \|Z_0\|_{\mathfrak{L}^q}.$$

To conclude the proof of Theorem 3.8, we prove the boundedness of the mapping $\partial_r z \mapsto z/r$. As $z \in L^q(\mathcal{F}_0)$, this is already contained in Proposition 2.7, provided we get a suitable estimate on $z(t,1) = \ell_Z(t)$. But, using (3.25) for $p = \infty$, we get

$$|\ell_Z(t)| = |z(t,1)| \leq C_q t^{-\frac{1}{q}} \|Z_0\|_{\mathfrak{L}^q}.$$

Thus, (2.37) implies:

$$\left\|\frac{z(t)}{r}\right\|_{L^q(\mathcal{F}_0)} \leq C_q \left(t^{-\frac{1}{2}} + \varepsilon_q t^{-\frac{1}{q}}\right) \|Z_0\|_{\mathfrak{L}^q}$$

where $\varepsilon_q = 1$ if $q > 2$ and $\varepsilon_q = 0$ if $q < 2$. We obtain (3.26) and (3.27) comparing the size of the different terms on the right-hand side depending on $q < 2$ or $q > 2$ and $t \geq 1$ or $t < 1$. ☐

3.4. **Decay of solutions to (2.12)–(2.14).** The equation (2.12)–(2.14) of w is linked to the computations in Section 3.2 in the case $k=1$ and $\tilde{\alpha}=\mathcal{J}/2\pi$. Thus, we compute the following time-decay of solutions:

Theorem 3.9. *Given $q\in(1,\infty)$ and radial $W_0\in\mathfrak{L}^q$, there exists a unique solution $W\in\mathcal{C}([0,\infty);\mathfrak{L}^q)$ to (2.12)-(2.14) such that $W(0,\cdot)=W_0$. This solution satisfies the further decay estimates:*

- *for all $p\in[q,\infty]$ we have $W\in\mathcal{C}((0,\infty);\mathfrak{L}^p)$ and there exists a constant $K_{1,0}=K_{1,0}(p,q)$ such that:*
$$\|W(t,\cdot)\|_{\mathfrak{L}^p}\le K_{1,0}\,t^{\frac1p-\frac1q}\|W_0\|_{\mathfrak{L}^q}\,,\quad\forall\,t>0\,,\tag{3.30}$$
- *if $q<2$ for all $p\in[q,2)$, we have $(\partial_r w, w/r)\in\mathcal{C}((0,\infty);L^p(\mathcal{F}_0))$ and there exists $K_{2,0}=K_{2,0}(p,q)$ such that:*
$$\|\partial_r w(t,\cdot)\|_{L^p(\mathcal{F}_0)}+\left\|\frac{w(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)}\le K_{2,0}\,t^{-\frac12+\frac1p-\frac1q}\|W_0\|_{\mathfrak{L}^q}\,,\quad\forall\,t>0\,,\tag{3.31}$$
- *if $q\in(1,\infty)$, for all $p\in[\max\{2,q\},\infty)$ satisfying $p>2$ we have $(\partial_r w,w/r)\in\mathcal{C}((0,\infty);L^p(\mathcal{F}_0))$ and there exists $K_{3,0}=K_{3,0}(p,q)$ such that:*
$$\|\partial_r w(t,\cdot)\|_{L^p(\mathcal{F}_0)}+\left\|\frac{w(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)}\le\begin{cases}K_{3,0}\,t^{-\frac12+\frac1p-\frac1q}\|W_0\|_{\mathfrak{L}^q}\,, & \forall\,t<1\,,\\ K_{3,0}\,t^{-\frac1q}\|W_0\|_{\mathfrak{L}^q}\,, & \forall\,t>1\,.\end{cases}\tag{3.32}$$

Proof. Again, existence of solutions and $\mathfrak{L}^p-\mathfrak{L}^q$ decay estimates are straightforward applications of Theorem 3.5 in the case $k=1$ and $\tilde{\alpha}=\mathcal{J}/2\pi>0$. We focus now on gradient estimates in the case $p=q\neq2$, the estimates (3.31)–(3.32) with $p\ge q$, $p\neq2$ being a simple consequence of the semigroup property.

Let $q\in(1,\infty)$. We note that $w\in\mathcal{C}^1((0,\infty);L^q(\mathcal{F}_0))$ with $\|\partial_t W(t)\|_{\mathfrak{L}^q}\le C/t\|W_0\|_{\mathfrak{L}^q}$ yields
$$\partial_{rr}w(t)+\frac{\partial_r w(t)}{r}-\frac{w(t)}{r^2}\in\mathcal{C}((0,\infty);L^q(\mathcal{F}_0))$$
$$\left\|\partial_{rr}w(t)+\frac{\partial_r w(t)}{r}-\frac{w(t)}{r^2}\right\|_{L^q(\mathcal{F}_0)}+|\partial_r w(t,1)-w(t,1)|\le\frac{C}{t}\|W_0\|_{\mathfrak{L}^q}\,.$$
Recalling estimates (2.38) and (2.39), for all $t>0$,
$$\|\partial_{rr}w(t)\|_{L^q(\mathcal{F}_0)}\le\frac{C}{t}\|W_0\|_{\mathfrak{L}^q},$$
$$\left\|\frac{\partial_r w(t)}{r}\right\|_{L^q((1,\infty),r\mathrm{d}r)}+\left\|\frac{w(t)}{r^2}\right\|_{L^q((1,\infty),r\mathrm{d}r)}\le\frac{C}{t}\|W_0\|_{\mathfrak{L}^q}+C(q)\varepsilon_q|w(t,1)|,$$
with $\varepsilon_q=1$ if $q>2$ and $\varepsilon_q=0$ if $q<2$. But, for $q>2$, estimate (3.17) with $p=\infty$ yields
$$|\ell_W(t)|=|w(t,1)|\le Ct^{-2/q}\left\|\left(\frac{w_0}{r},\ell_{W_0}\right)\right\|_{\mathfrak{L}^q(\mathbb{R}^4)}\le Ct^{-2/q}\|W_0\|_{\mathfrak{L}^q},$$
where the last estimate is a consequence of $q>2$. Hence
$$\left\|\frac{\partial_r w(t)}{r}\right\|_{L^q((1,\infty),r\mathrm{d}r)}+\left\|\frac{w(t)}{r^2}\right\|_{L^q((1,\infty),r\mathrm{d}r)}\le C\left(t^{-1}+\varepsilon_q t^{-2/q}\right)\|W_0\|_{\mathfrak{L}^q}.$$
We can then bound $\|\partial_{xx}w\|_{L^q(\mathcal{F}_0)}$, $\|\partial_{xy}w\|_{L^q(\mathcal{F}_0)}$ and $\|\partial_{yy}w\|_{L^q(\mathcal{F}_0)}$ in the same way as in the previous proof. Applying interpolation inequality (3.29) to W we then obtain that $\partial_r w\in\mathcal{C}((0,\infty);L^q(\mathcal{F}_0))$ with:
$$\|\partial_r w(t)\|_{L^q(\mathcal{F}_0)}\le C\left(t^{-1/2}+\varepsilon_q t^{-1/q}\right)\|W_0\|_{\mathfrak{L}^q}.$$
To get the decay of w/r, we then simply use that
$$\left\|\frac{w(t)}{r}\right\|^2_{L^q((1,\infty),r\mathrm{d}r)}\le\left\|\frac{w(t)}{r^2}\right\|_{L^q((1,\infty),r\mathrm{d}r)}\|W(t)\|_{\mathfrak{L}^q}.$$
□

3.5. **Decay estimates of solutions to the Stokes system.** It remains now to combine together the results obtained in Subsections 3.1, 3.3 and 3.4 to prove our main results regarding the long-time behavior of Stokes solutions.

3.5.1. *Proof of Theorem 1.1.*

Proof. Given $q \in (1,\infty)$, as $\mathcal{C}_c^\infty(\mathbb{R}^2) \cap \mathcal{L}^2$ is dense in $\mathcal{L}^q$ we remark that it is sufficient to prove decay estimate for initial data $V_0 \in \mathcal{L}^2 \cap \mathcal{C}_c^\infty(\mathbb{R}^2)$. We emphasize $V_0 \in \mathcal{L}^q$ for all $q \in [1,\infty]$ under these assumptions. We denote $(W_0, \Phi_0, \Psi_0, V_{R,0})$ the spherical harmonic decomposition of this initial data. We already know that there exists a unique solution to (1.22)–(1.29) in $\mathcal{C}([0,\infty);\mathcal{L}^2)$ for such an initial data.

First, we decompose the solution $S(t)V_0$ of (1.22)–(1.29) into the spherical-harmonic decomposition (W, Φ, Ψ, V_R) of Proposition 2.1. According to Proposition 2.2, this decomposition satisfies:

- $W \in \mathcal{C}([0,\infty);\mathfrak{L}^2)$ and is a solution to (2.12)–(2.14);
- $V_R \in \mathcal{C}([0,\infty); L^2_\sigma(\mathcal{F}_0))$ and is a solution to (2.23)–(2.25).

According to Theorem 3.9 and Theorem 3.1, these are the respective unique solutions to (2.12)–(2.14) and (2.23)–(2.25) in these spaces, with respective initial data W_0 and $v_{R,0}$. We can then apply the decay estimates of Theorem 3.9 and Theorem 3.1 to these solutions.

Referring moreover to Proposition 2.3 and Remark 2.4, we have:

- $Z_\Phi \in \mathcal{C}([0,\infty);\mathfrak{L}^2)$ and is a solution to (2.28)–(2.30);
- $Z_\Psi \in \mathcal{C}([0,\infty);\mathfrak{L}^2)$ and is a solution to (2.28)–(2.30).

Consequently, applying Theorem 3.8, these are the unique solutions to (2.28)–(2.30) in these spaces, with respective initial data $Z_{\Phi,0} = -(\partial_r \Phi_0 + \Phi_0/r)$ and $Z_{\Psi,0} = \partial_r \Psi_0 + \Psi_0/r$ and the decay estimates of Theorem 3.8 are also satisfied by Z_Φ and Z_Ψ.

We proceed with $L^p - L^q$ estimates. Applying (2.3) we have:

$$\|W_0\|_{\mathfrak{L}^q} + \|Z_{\Phi,0}\|_{\mathfrak{L}^q} + \|Z_{\Psi,0}\|_{\mathfrak{L}^q} + \|V_{R,0}\|_{L^q_\sigma(\mathcal{F}_0)} \le C_q \|V_0\|_{\mathcal{L}^q}\,.$$

Then, combining decay estimates of the different components in the spherical-harmonic decomposition obtained in (3.1), (3.25), and (3.30), we have for $t > 0$:

$$\|W(t)\|_{\mathfrak{L}^p} + \|Z_\Phi(t)\|_{\mathfrak{L}^p} + \|Z_\Psi(t)\|_{\mathfrak{L}^p} + \|V_R(t)\|_{L^p_\sigma(\mathcal{F}_0)} \le C_q t^{\frac{1}{p}-\frac{1}{q}} \|V_0\|_{\mathcal{L}^q}\,.$$

On the left hand side, we have for instance $Z_\Psi(t) = \partial_r \Psi(t) + \Psi(t)/r$ with $\Psi(t,1) = Z_\Psi(t,1)/2$ so that:

$$|\Psi(t,1)| \le C \|Z_\Psi(t)\|_{\mathfrak{L}^p}\,.$$

Hence, applying Proposition 2.6 we obtain:

$$\|\partial_r \Psi\|_{L^p((0,\infty);r\mathrm{d}r)} + \left\|\frac{\Psi}{r}\right\|_{L^p((0,\infty);r\mathrm{d}r)} \le C\|Z_\Psi\|_{\mathfrak{L}^p}\,.$$

Applying similar argument to bound Φ, we finally obtain that:

$$\begin{aligned}\|W(t)\|_{\mathfrak{L}^p} + \|V_R(t)\|_{L^p_\sigma(\mathcal{F}_0)} \quad &+ \quad \|\partial_r \Psi(t)\|_{L^p((0,\infty);r\mathrm{d}r)} + \left\|\frac{\Psi(t)}{r}\right\|_{L^p((0,\infty);r\mathrm{d}r)} \\ &+ \quad \|\partial_r \Phi(t)\|_{L^p((0,\infty);r\mathrm{d}r)} + \left\|\frac{\Phi(t)}{r}\right\|_{L^p((0,\infty);r\mathrm{d}r)} \le C_q t^{\frac{1}{p}-\frac{1}{q}} \|V_0\|_{\mathcal{L}^q}\,.\end{aligned}$$

Noting that $\|W(t)\|_{\mathfrak{L}^p}$ is equivalent to $\|W(t)\|_{L^p((0,\infty),r\mathrm{d}r)}$, we apply (2.4) and conclude immediately

$$\|V(t)\|_{\mathcal{L}^p} \le C_{p,q} t^{\frac{1}{p}-\frac{1}{q}} \|V_0\|_{\mathcal{L}^q}\,.$$

We now proceed with the gradient estimates. Let us recall that the case $p = 2$ is a straightforward consequence of [22] and the previous inequality:

$$\|\nabla v(t)\|_{L^2(\mathcal{F}_0)} \le C t^{-1/2} \|V(t/2)\|_{\mathcal{L}^2} \le C_q t^{\frac{1}{2}-\frac{1}{q}-\frac{1}{2}} \|V_0\|_{\mathcal{L}^q}\,.$$

So we focus on the case $q \in (1,\infty)$ and $p \in [q,\infty)$ with $p \neq 2$.

Similarly to the previous computations, the method is then an application of Proposition 2.1 and the decay estimates obtained in Theorems 3.1, 3.8, and 3.9. The only difference we detail now is the computation of

$$\|\partial_{rr}\psi\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{\partial_r\psi}{r}\right\|_{L^p((1,\infty);r\mathrm{d}r)} + \left\|\frac{\psi}{r^2}\right\|_{L^p((1,\infty);r\mathrm{d}r)}$$

from the estimates for $\|\partial_r z_\psi\|_{L^p(\mathcal{F}_0)}$ and $\|z_\psi/r\|_{L^p(\mathcal{F}_0)}$ as given in (3.26) and (3.27). We focus on ψ, the problem being completely similar for φ. Differentiating the definition of z_ψ, we remark that ψ satisfies:

$$\begin{aligned} \partial_{rr}\psi + \frac{\partial_r\psi}{r} - \frac{\psi}{r^2} &= \partial_r z_\psi\,, \qquad \text{for } r\in(1,\infty)\,,\\ \partial_r\psi(1) - \psi(1) &= 0\,. \end{aligned}$$

Consequently, we apply Proposition 2.8 which yields:

$$\|\partial_{rr}\psi\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{\partial_r\psi}{r} - \frac{\psi}{r^2}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \le C\|\partial_r z_\psi\|_{L^p(\mathcal{F}_0)}\,.$$

On the other hand, we have, by definition of z_ψ,

$$\left\|\frac{\partial_r\psi}{r} + \frac{\psi}{r^2}\right\|_{L^p((1,\infty),r\mathrm{d}r)} \le C\left\|\frac{z_\psi}{r}\right\|_{L^p(\mathcal{F}_0)}.$$

So that, we finally get:

$$\|\partial_{rr}\psi\|_{L^p((1,\infty),r\mathrm{d}r)} + \left\|\frac{\partial_r\psi}{r}\right\|_{L^p((1,\infty);r\mathrm{d}r)} + \left\|\frac{\psi}{r^2}\right\|_{L^p((1,\infty);r\mathrm{d}r)} \le C\left(\|\partial_r z_\psi\|_{L^p(\mathcal{F}_0)} + \left\|\frac{z_\psi}{r}\right\|_{L^p(\mathcal{F}_0)}\right).$$

This ends the proof of Theorem 1.1. □

3.5.2. *Duality decay estimates.* For later use, based on Theorem 1.1, we derive here additional estimates on the behavior of the semigroup corresponding to (1.22)-(1.29):

Corollary 3.10. *Assume that $1<q\le p<\infty$ and let $F\in L^q(\mathbb{R}^2;M_{2\times2}(\mathbb{R}))$ satisfying $F=0$ on B_0. The following decay estimates hold true:*

- *if $2\le q\le p<\infty$, there exists $K_4=K_4(p,q)>0$ such that:*

$$\|S(t)\mathbb{P}\mathrm{div}\,F\|_{\mathcal{L}^p} \le K_4(\nu t)^{-\frac12+\frac1p-\frac1q}\|F\|_{L^q(\mathbb{R}^2)} \qquad \text{for all} \quad t>0. \tag{3.33}$$

- *if $1<q\le p$ and $q\le 2$, there exists $K_5=K_5(p,q)>0$ such that:*

$$\|S(t)\mathbb{P}\mathrm{div}\,F\|_{\mathcal{L}^p} \le \begin{cases} K_5(\nu t)^{-\frac12+\frac1p-\frac1q}\|F\|_{L^q(\mathbb{R}^2)} & \text{for all} \quad 0<t<\dfrac1\nu,\\ K_5(\nu t)^{-1+\frac1p}\|F\|_{L^q(\mathbb{R}^2)} & \text{for all} \quad t\ge\dfrac1\nu. \end{cases} \tag{3.34}$$

In this corollary the divergence div is computed along rows of the matrix F.

Before going into the proof, let us emphasize that, in our case $\nabla S(t)$ is not the dual operator of $S(t)\mathbb{P}\mathrm{div}$. Indeed if F is smooth with compact support, there holds, for all $\Phi\in\mathcal{L}^2\cap\mathcal{C}^\infty_c(\mathbb{R}^2)$:

$$\begin{aligned} (\nabla S(t)\varphi, F) &:= \frac{m}{\pi}\int_{B_0}\nabla S(t)\Phi : F + \int_{\mathcal{F}_0}\nabla S(t)\Phi : F\\ &= \left(1-\frac{m}{\pi}\right)\int_{\partial B_0} S(t)\Phi\cdot Fn\,\mathrm{d}\sigma + (S(t)\Phi, \mathrm{div}\,F)\\ &= \left(1-\frac{m}{\pi}\right)\int_{\partial B_0} S(t)\Phi\cdot Fn\,\mathrm{d}\sigma + (\Phi, S(t)\mathbb{P}\mathrm{div}\,F)\,. \end{aligned}$$

Hence Corollary 3.10 only concerns the restriction of the dual of $\nabla S(t)$ to functions F which vanish at the boundary.

Proof. The following proof contains a construction of the operator $S(t)\mathbb{P}\mathrm{div}$ on the closed subset of $L^q(\mathbb{R}^2; M_{2\times 2}(\mathbb{R}))$ of functions vanishing on B_0. We prove our result in the case $2 \le q \le p < \infty$ only. The other cases can be done similarly.

Let $F \in L^q(\mathbb{R}^2; M_{2\times 2}(\mathbb{R}^2))$ such that $F = 0$ on B_0. Up to a regularizing argument, we assume that $F \in \mathcal{C}^\infty_c(\mathcal{F}_0; M_{2\times 2}(\mathbb{R}))$. Then, $V(t) := S(t)\mathbb{P}\mathrm{div}\, F \in \mathcal{C}((0,\infty); \mathcal{L}^p)$ for all $p \in (1,\infty)$ by a straightforward application of Theorem 1.1. For all $t > 0$ and $\tilde{V} \in \mathcal{L}^2 \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$, we have, as S is self-adjoint with respect to the scalar product $(\cdot,\cdot)$ we introduced on $\mathcal{L}^2$ (see (1.32)):

$$\begin{aligned}\langle V(t), \tilde{V}\rangle_{\mathcal{L}^p,\mathcal{L}^{p'}} &= (\mathbb{P}\mathrm{div}\, F, S(t)\tilde{V})\,,\\ &= \int_{\mathcal{F}_0} \mathrm{div}\, F \cdot S(t)\tilde{V}\,, \qquad \text{(as } F \text{ vanishes on } B_0)\,,\\ &= -\int_{\mathcal{F}_0} F : \nabla S(t)\tilde{V} \qquad \text{(as } Fn \text{ vanishes on } \partial B_0)\,.\end{aligned}$$

Finally, we obtain:

$$\left|\langle V(t), \tilde{V}\rangle_{\mathcal{L}^p,\mathcal{L}^{p'}}\right| \le \|F\|_{L^q(\mathbb{R}^2)} \|\nabla S(t)\tilde{V}\|_{L^{q'}(\mathcal{F}_0)}\,,$$

where we apply decay estimates we obtained in Theorem 1.1: as $p' \le q' < 2$ we have from (1.35)

$$\|\nabla S(t)\tilde{V}\|_{L^{q'}(\mathcal{F}_0)} \le Ct^{-\frac{1}{2}+\frac{1}{q'}-\frac{1}{p'}} \|\tilde{V}\|_{\mathfrak{L}^{p'}} \le Ct^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|\tilde{V}\|_{\mathcal{L}^{p'}}.$$

So that, we obtain:

$$\left|\langle V(t), \tilde{V}\rangle_{\mathcal{L}^p,\mathcal{L}^{p'}}\right| \le C\|F\|_{L^q(\mathbb{R}^2)} t^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|\tilde{V}\|_{\mathcal{L}^{p'}}.$$

As $\mathcal{L}^{p'} \cap \mathcal{C}^\infty_c(\mathbb{R}^2)$ is dense in $\mathcal{L}^{p'}$, this inequality implies by duality that $V(t) \in \mathcal{L}^p$ with norm lower than $Ct^{-\frac{1}{2}+\frac{1}{p}-\frac{1}{q}} \|F\|_{L^q(\mathbb{R}^2)}$. ☐

In the previous corollary we restrict p to finite values. In the case $p = \infty$, we do not obtain a control of the whole solution. Nevertheless, we can obtain a result that would correspond to the case $p = \infty$ in (3.33) for the translation speed $\ell_{V(t)}$. This result is a new application of the added mass effect and relies on the fact that Kirchoff potentials are easily computed in our case.

Corollary 3.11. *Let $q \in [2,\infty)$ and $F \in L^q(\mathbb{R}^2; M_{2\times 2}(\mathbb{R}))$ satisfying $F = 0$ on B_0, The following decay estimate holds true for $V(t) := S(t)\mathbb{P}\mathrm{div}\, F$:*

$$|\ell_{V(t)}| \le K_\ell(q)(\nu t)^{-\left(\frac{1}{2}+\frac{1}{q}\right)} \|F\|_{L^q(\mathbb{R}^2)}\,, \quad \forall\, t > 0 \tag{3.35}$$

where $K_\ell(q)$ depends only on q.

Proof. Let the assumptions of the corollary be satisfied. At first, we recall that we have $V(t) \in \mathcal{L}^q$ for all $t \in (0,\infty)$ as has been shown in the previous corollary. We show how to prove that the first component $\ell_{V,1}$ of $\ell_V(t)$ satisfies (3.35). Similar estimate for the other component $\ell_{V,2}$ is obtained applying comparable arguments.

Let $\bar{\psi} \in \mathcal{C}^\infty(\mathcal{F}_0)$ be given in polar coordinates by:

$$\bar{\psi}(r,\theta) = \frac{\cos(\theta)}{r}\,, \quad \forall\, (r,\theta) \in (1,\infty) \times (-\pi,\pi)\,.$$

Given $t > 0$ we note that $V := V(t) \doteq (v(t), \ell_V(t))$ is divergence free on any subdomain $B(0,R) \setminus B_0$ of $\mathcal{F}_0$. This yields:

$$\int_{\partial B(0,R)} v \cdot n\, \bar{\psi}\, \mathrm{d}\sigma + \int_{\partial B_0} v \cdot n\, \bar{\psi}\, \mathrm{d}\sigma = \int_{B(0,R)\setminus B_0} v \cdot \nabla \bar{\psi}$$

Letting $R \to \infty$, we obtain (the exterior boundary term vanishes as $v \in \mathcal{L}^q$):

$$-\int_0^{2\pi} v_r(1,\theta)\cos(\theta)\, \mathrm{d}\theta = \int_{\mathcal{F}_0} v \cdot \nabla \bar{\psi}\,. \tag{3.36}$$

We observe then that, on the one hand, we have $\nabla\bar{\psi} = \nabla^{\perp}\bar{\varphi}$ where

$$\bar{\varphi}(r,\theta) = \frac{\sin(\theta)}{r}\,, \quad \forall\,(r,\theta) \in (1,\infty)\times(-\pi,\pi)\,,$$

on the other hand:

$$\int_0^{2\pi} v_r(t,1,\theta)\cos(\theta)\,\mathrm{d}\theta = \pi\ell_{V,1}(t)\,.$$

Setting finally:

$$\Xi := 1_{\mathcal{F}_0}\nabla\bar{\psi} - 1_{B_0}e_1$$

we have that $\Xi \in \mathcal{L}^p$ for arbitrary $p > 1$ and that (3.36) reads:

$$-(\pi+m)\ell_{V,1}(t) = m\ell_V(t)\cdot\ell_\Xi + \int_{\mathcal{F}_0} v(t)\cdot\xi = (V(t),\Xi)\,, \quad \forall\,t>0\,.$$

Given this identity, we reproduce the computations done in the proof of the previous corollary. We obtain:

$$-\,(\pi+m)\ell_{V,1}(t) = \int_{\mathcal{F}_0} F : \nabla S(t)\Xi\,, \tag{3.37}$$

which implies

$$|\ell_{V,1}(t)| \le \frac{1}{|\pi+m|}\|F\|_{L^q(\mathbb{R})}\|\nabla S(t)\Xi\|_{L^{q'}(\mathcal{F}_0)}\,.$$

The proof now reduces to find a bound on $\|\nabla S(t)\Xi\|_{L^{q'}(\mathcal{F}_0)}$. To this end, we remark that the spherical-harmonic decomposition of Ξ reduces to the first mode $\bar{\varphi}(r,\theta) = \min\{r,1/r\}\sin(\theta)$. Going back to the computation of Section 2, we note that $S(t)\Xi$ is given by its first-mode, corresponding to $Z_\Phi = -\partial_r\Phi - \Phi/r$ where $\Phi(r) = \min\{r,1/r\}$. This mode satisfies (2.28)-(2.30) with initial condition:

$$Z_{\Phi,0} = -2\cdot 1_{B_0}\,.$$

Consequently, for $q>2$, $Z_{\Phi,0}\in\mathcal{L}^1$ and we apply Theorem 3.8 with "p"$= q'$ and "q"$= 1$, which yields (see (3.26))

$$\|\partial_r z_\Phi(t,\cdot)\|_{L^{q'}(\mathcal{F}_0)} + \left\|\frac{z_\Phi(t,\cdot)}{r}\right\|_{L^{q'}(\mathcal{F}_0)} \le K\,(\nu t)^{-\frac{1}{2}+\frac{1}{q'}-1} = K\,(\nu t)^{-\frac{1}{2}-\frac{1}{q}}\,, \quad \forall\,t>0\,.$$

We go back to Ξ as in the proof of Theorem 1.1 so that:

$$\|\nabla S(t)\Xi\|_{L^{q'}(\mathcal{F}_0)} \le K\,(\nu t)^{-\frac{1}{2}-\frac{1}{q}}\,, \quad \forall\,t>0\,.$$

Plugging this estimate in (3.37) we obtain the expected result for $q>2$.

The case $q=2$, corresponding to $q'=2$, does not immediately follows from Theorem 3.8, but rather from the fact that

$$\|\nabla^{\perp}\left(\Phi\left(\frac{t}{2},r\right)\sin(\theta)\right)\|_{\mathcal{L}^2} \le C\|Z_\Phi\left(\frac{t}{2},\cdot\right)\|_{\mathfrak{L}^2} \le C(\nu t)^{-1/2}\|Z_{\Phi,0}\|_{\mathfrak{L}^1}$$

and from the $\mathcal{L}^2$ decay estimates on the gradient obtained in Theorem 1.1:

$$\|\nabla S(t/2)\left(\nabla^{\perp}\left(\Phi\left(\frac{t}{2},r\right)\sin(\theta)\right)\right)\|_{L^2(\mathcal{F}_0)} \le C(\nu t)^{-1/2}\|\nabla^{\perp}\left(\Phi\left(\frac{t}{2},r\right)\sin\theta\right)\|_{\mathcal{L}^2}.$$

□

3.6. **Asymptotic expansion of solutions to the Stokes system.** This section aims at proving Theorem 1.2. We first show that the solutions W and V_R corresponding to the modes $k\neq 1$ decay faster than the modes corresponding to $k=1$. In a second step, we derive precisely the first-order in the long-time behavior of this first mode.

3.6.1. *Faster decay on W.*

Theorem 3.12. *Given a radial $W_0 \in \mathfrak{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$, the unique solution to (2.12)–(2.14) satisfies $W(t) \in \mathfrak{L}^p$ for all $t > 0$ and $p \in [1, \infty]$. Furthermore, for all $p \in [2, \infty]$, there exists a constant $C_p > 0$ such that*

$$\|W(t)\|_{\mathfrak{L}^p} \leq C_p t^{\frac{1}{p} - \frac{3}{2}} \|W_0 r\|_{\mathfrak{L}^1} ,$$

and there exists a constant C such that

$$|\ell_{W(t)}| \leq C t^{-2} \|W_0 r\|_{\mathfrak{L}^1}.$$

Proof. Let us first remark that $W_0 \in \mathfrak{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$ obviously implies that $W_0 r \in \mathfrak{L}^1$.

We first focus on the decay estimate, as $W(t) \in \mathfrak{L}^p$ for $t > 0$ is obvious. Given $t > 0$ and $p \geq 2$ we apply (3.30) with $q = 2$ between $t/2$ and t and then, we apply (3.17) between 0 and $t/2$:

$$\|W(t)\|_{\mathfrak{L}^p} \leq C t^{\frac{1}{p} - \frac{1}{2}} \|W(t/2)\|_{\mathfrak{L}^2} \leq C t^{\frac{1}{p} - \frac{3}{2}} \left\| \frac{W_0}{r} \right\|_{\mathfrak{L}^1(\mathbb{R}^4)} \leq C t^{\frac{1}{p} - \frac{3}{2}} \|W_0 r\|_{\mathfrak{L}^1} .$$

Concerning the second estimate, it suffices to use (1.31) and (3.17):

$$|\ell_{W(t)}| \leq \left\| \frac{W(t)}{r} \right\|_{\mathfrak{L}^\infty(\mathbb{R}^4)} \leq C t^{-2} \left\| \frac{W_0}{r} \right\|_{\mathfrak{L}^1(\mathbb{R}^4)} = C t^{-2} \|W_0 r\|_{\mathfrak{L}^1}.$$

This ends the proof. □

As a consequence, we can already note that $\omega_{S(t)V_0} = \ell_{W(t)}$ (see Proposition 2.1) verifies (1.38).

3.6.2. *Faster decay on V_R.*

Theorem 3.13. *Given $V_{R,0} \in L^2_\sigma(\mathcal{F}_0) \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$, for all $p \in [2, \infty]$, there exists a constant $C = C(p, v_{R,0})$ such that:*

$$\|V_R(t, \cdot)\|_{L^p_\sigma(\mathcal{F}_0)} \leq C \frac{|\log(t)|}{t^{3/2 - 1/p}} , \quad \forall\, t > 1 .$$

Proof. In order to prove Theorem 3.13, we expand V_R solution of (2.23)–(2.25) on its Fourier basis:

$$V_R(t, r, \theta) = \nabla^\perp \left[\sum_{k \geq 2} (\psi_k(t, r) \cos(k\theta) + \varphi_k(t, r) \sin(k\theta)) \right] ,$$

where $\psi_k(t, 1) = \partial_r \psi_k(t, 1) = \varphi_k(t, 1) = \partial_r \varphi_k(t, 1) = 0$ thanks to the homogeneous Dirichlet boundary conditions satisfied by the restriction v_R of V_R on $\mathcal{F}_0$. Note that v_R does not contain any 0 or 1 mode due to the orthogonality condition (2.2).

As in the case $k = 0, 1$, we can show that for all $k \geq 2$ the new unknown $z_k = z_{\psi,k}(t, r) := 1/r^k \partial_r [r^k \psi_k(t, r)]$ or $z_k = z_{\varphi,k}(t, r) = -1/r^k \partial_r [r^k \varphi_k(t, r)]$ satisfies:

$$\begin{aligned} \partial_t z_k + \left(-\frac{1}{r} \partial_r (r \partial_r z_k) + \frac{(k-1)^2}{r^2} z_k \right) &= 0 \qquad && \text{for } (t, r) \in (0, \infty) \times (1, \infty); \\ z_k(t, 1) &= 0 && \text{for } t \in (0, \infty). \end{aligned}$$

One can then use the asymptotic formula given by Theorem 3.3 for

$$\tilde{v}_R(t, r, \theta) = \sum_{k \geq 2} (z_{\psi,k}(t, r) \cos((k-1)\theta) + z_{\varphi,k}(t, r) \sin((k-1)\theta))$$

which is a solution of (3.7) for $n = 2$, arbitrary $\alpha > 0$ and vanishing initial mass $M = 0$. This immediately yields that, provided

$$\tilde{v}_R(0) \in L^2(\mathcal{F}_0, \exp(|x|^2/4)\, \mathrm{d}x),$$

which holds true since $V_{R,0}$ is assumed to belong to $L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$, we have

$$t^{1 - 1/p} \|\tilde{v}_R(t, \cdot)\|_{L^p(\mathcal{F}_0)} \leq C R_{1,p}(t).$$

In particular, for $p = 2$, this implies that

$$t\sum_{k\geq 2}\int_1^\infty r(|z_{\psi,k}(t,r)|^2 + |z_{\varphi,k}(t,r)|^2\,dr \leq CR_{1,2}(t)^2.$$

But recall that, for $k \geq 2$,

$$z_{\psi,k} = \partial_r\psi_k + \frac{k}{r}\psi_k,$$

and $\psi_k(t,1) = 0$. Hence for all $R > 1$,

$$\int_1^R |z_{\psi,k}|^2\, rdr = \int_1^R |\partial_r\psi_k|^2 rdr + k^2\int_1^R \frac{|\psi_k|^2}{r}\, rdr + k\int_1^R \partial_r\left(|\psi_k|^2\right)\, dr.$$

As $\psi_k(1) = 0$, passing to the limit $R \to \infty$, we get

$$\int_1^\infty |\partial_r\psi_k|^2 rdr + k^2\int_1^\infty \frac{|\psi_k|^2}{r}\, rdr \leq \int_1^\infty |z_{\psi,k}|^2\, rdr,$$

and thus,

$$t^{1/2}\,\|v_R(t)\|_{L^2(\mathcal{F}_0)} \leq CR_{1,2}(t).$$

Using then the semigroup estimates (3.1), we get, for $p \geq 2$,

$$t^{1-1/p}\,\|v_R(t)\|_{L^p(\mathcal{F}_0)} \leq CR_{1,2}(t).$$

This concludes the proof of Theorem 3.13, as V_R simply vanishes in $B(0,1)$. □

3.6.3. *Proof of Theorem 1.2.*

Proof. Let $V_0 \in \mathcal{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$ and $V(t)$ the unique associated solution to (1.22)–(1.29). Note that $V_0 \in \mathcal{L}^q$ for all $q \in (1,\infty)$ so that we already know that $V(t) \in \mathcal{L}^p$ for all $p \in [2,\infty)$ for all $t > 0$ from Theorem 1.1. Let now $p \in [2,\infty)$ and (W, Φ, Ψ, V_R) the spherical-harmonic decomposition of V.

The components Φ, Ψ and W are computed as means of V in θ so that they inherit the asymptotic decay in r of the data V_0. Combining this remark with Proposition 2.1, this yields that:

$$W(0,\cdot) \in \mathfrak{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\,\mathrm{d}x)\,, \qquad V_R(0,\cdot) \in L^2_\sigma(\mathcal{F}_0) \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\mathrm{d}x)$$
$$(Z_\Phi(0,\cdot), Z_\Psi(0,\cdot)) \in \mathfrak{L}^1 \cap L^2(\mathbb{R}^2, \exp(|x|^2/4)\,\mathrm{d}x)\,.$$

Consequently, Theorems 3.13 and 3.12 imply respectively:

$$\|V_R(t,\cdot)\|_{L^p_\sigma(\mathcal{F}_0^2)} = \mathcal{O}\left(\frac{|\log(t)|}{t^{3/2-1/p}}\right)\,, \qquad \|W(t,\cdot)\|_{\mathfrak{L}^p} = \mathcal{O}(t^{1/p-3/2})\,.$$

We focus now on Z_Φ and Z_Ψ. Using Theorem 3.3 with $\alpha = 2/(\pi + m)$, we immediately get:

$$t^{1-1/p}\,\|z_\Phi(t,\cdot) - M_\Phi G(t)\|_{L^p(\mathcal{F}_0)} \leq C_p R_{1,p}(t), \qquad t\left|\ell_{Z_\Phi}(t) - \frac{M_\varphi}{4\pi t}\right| \leq CR_2(t), \tag{3.38}$$

$$t^{1-1/p}\,\|z_\Psi(t,\cdot) - M_\psi G(t)\|_{L^p(\mathcal{F}_0)} \leq C_p R_{1,p}(t), \qquad t\left|\ell_{Z_\Psi}(t) - \frac{M_\psi}{4\pi t}\right| \leq CR_2(t), \tag{3.39}$$

with G and $(R_{1,p}, R_2)$ as given in Theorem 3.3 in the case $n = 2$ and

$$M_\Phi \;:=\; 2\pi\int_1^\infty Z_\Phi(0,r)r\mathrm{d}r + (\pi+m)\int_0^1 Z_\Phi(0,r)r\mathrm{d}r\,,$$
$$M_\Psi \;:=\; 2\pi\int_1^\infty Z_\Psi(0,r)r\mathrm{d}r + (\pi+m)\int_0^1 Z_\Psi(0,r)r\mathrm{d}r\,.$$

Recalling that

$$\begin{aligned}
&Z_\Phi(t,\cdot) = -\frac{1}{r}\partial_r(r\Phi(t,\cdot)) \quad \text{ for } (t,r) \in [0,\infty)\times(0,\infty);\\
&Z_\Phi(t,r) = \ell_{Z_\Phi}(t) = -2\Phi(t,r)/r = 2\ell_1(t) \quad \text{ for } (t,r) \in (0,\infty)\times(0,1),\\
&\Phi(0,1) = -\ell_1(0), \quad \text{(by the continuity of } \Phi)
\end{aligned}$$

and using $\Phi(0,\cdot)/r, \partial_r\Phi(0,\cdot) \in L^1 \cap L^2((0,\infty), \exp(|r|^2/4) r\mathrm{d}r)$, which implies the existence of a sequence $R_n \to \infty$ such that $R_n\Phi(0,R_n)$ goes to 0 as $n\to\infty$,

$$M_\Phi = (\pi - m)\Phi(0,1) = (m-\pi)\ell_1(0).$$

Similarly,

$$M_\Psi = (m-\pi)\Psi(0,1) = (m-\pi)\ell_2(0).$$

By Proposition 2.1 , we recall for $r \in (0,1)$ that $\ell_{S(t)V_0,1} = -\Phi(t,r)/r = \ell_{Z_\Phi}(t)/2$. From (3.38)-(3.39) and the previous formulas, we have obtained (1.37) and (1.39).

Solving Φ and Ψ in terms of $Z_\Phi(t,\cdot) \doteq (z_\Phi(t,\cdot), \ell_{Z_\Phi}(t))$ and $Z_\Psi(t,\cdot) \doteq (z_\Psi(t,\cdot), \ell_{Z_\Psi}(t))$, we are then led to define $\hat\Psi(t,r)$ on $t\geq 0$, $r \in (0,\infty)$ as the extension of $\hat\psi$ solution of

$$\frac{1}{r}\partial_r(r\hat\psi(t,r)) = G(t,r) \quad \text{ for } (t,r)\in(0,\infty)\times(1,\infty), \qquad \hat\psi(t,1) = \frac{1}{8\pi t} \quad \text{ for } t\in(0,\infty).$$

by

$$\hat\Psi(t,r) = \frac{r}{8\pi t} \quad \text{ for } (t,r)\in(0,\infty)\times(0,1).$$

Note that this function can be computed explicitly:

$$\hat\Psi(t,r) = \begin{cases} \dfrac{1}{2\pi r}\left(\exp\left(-\dfrac{1}{4t}\right) - \exp\left(-\dfrac{r^2}{4t}\right) + \dfrac{1}{4t}\right) & \text{ for } (t,r)\in(0,\infty)\times(1,\infty),\\ \dfrac{r}{8\pi t} & \text{ for } (t,r)\in(0,\infty)\times(0,1), \end{cases}$$

Using Proposition 2.6, we get for all $p>1$,

$$\begin{aligned}
\|\partial_r\psi(t,\cdot) - M_\Psi\partial_r\hat\psi(t,\cdot)\|_{L^p(\mathcal{F}_0)} + \left\|\frac{\psi(t,\cdot) - M_\Psi\hat\Psi(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)} &\leq C_p\Big(R_{1,p}(t)t^{1/p-1} + R_2(t)t^{-1}\Big),\\
\|-\partial_r\varphi(t,\cdot) - M_\Phi\partial_r\hat\psi(t,\cdot)\|_{L^p(\mathcal{F}_0)} + \left\|\frac{-\varphi(t,\cdot) - M_\Phi\hat\Psi(t,\cdot)}{r}\right\|_{L^p(\mathcal{F}_0)} &\leq C_p\Big(R_{1,p}(t)t^{1/p-1} + R_2(t)t^{-1}\Big).
\end{aligned}$$

With the expression of $R_{1,p}$ and R_2, we can check that $-\frac{1}{2}+\theta_{2,p}+\frac{1}{p}-1 > -\frac{1}{4}-1$ for all $p\in[2,\infty]$. Hence for $t>1$, we have:

$$\begin{aligned}
t^{1-1/p}\|\nabla^\perp\Big(\psi(t,\cdot)\cos(\theta) - M_\Psi\hat\psi(t,\cdot)\cos(\theta)\Big)\|_{L^p(\mathcal{F}_0)} &\leq 2C_pR_{1,p}(t),\\
t^{1-1/p}\|\nabla^\perp\Big(\varphi(t,\cdot)\sin(\theta) + M_\Phi\hat\psi(t,\cdot)\sin(\theta)\Big)\|_{L^p(\mathcal{F}_0)} &\leq 2C_pR_{1,p}(t).
\end{aligned}$$

Remark then that, denoting

$$\tilde\psi(t,r) = \frac{1}{2\pi r}\left(1-\exp\left(-\frac{r^2}{4t}\right)\right), \qquad \text{ for } (t,r)\in(0,\infty)\times(1,\infty),$$

we have for all $p\in(1,\infty]$,

$$\|\nabla^\perp((\tilde\psi(t,r) - \hat\psi(t,r))\cos(\theta))\|_{L^p(\mathcal{F}_0)} \leq C\left|1-\exp\left(-\frac{1}{4t}\right) - \frac{1}{4t}\right| \leq \frac{C_p}{t^2}.$$

We then obtain

$$t^{1-1/p}\|v(t) - \nabla^\perp\Big[(m-\pi)\tilde\psi(t,\cdot)(\ell_2(0)\cos(\theta) - \ell_1(0)\sin(\theta))\Big]\|_{L^p(\mathcal{F}_0)} \leq C_pR_{1,p}(t).$$

This yields the expected result. □

4. Long-time behavior of solutions to the Navier-Stokes problem

In this section, we prove Theorem 1.3 and Theorem 1.4. We first apply Kato's method [14] of successive approximations yielding decay estimates for initial data $V_0 \in \mathcal{L}^2$. In a second subsection, we then extend these estimates to the case of initial data $V_0 \in \mathcal{L}^q$ with $q \in (1,2]$ in order to get Theorem 1.3. We finally explain how a bootstrap argument yields Theorem 1.4.

To simplify notations, we replace the constants $K_1(p,q)$, $K_4(p,q)$ and $K_\ell(q)$ defined respectively in (1.33), (3.33) and (3.35) by $K_1(p,q)\nu^{1/p-1/q}$, $K_4(p,q)\nu^{-1/2+1/p-1/q}$, and $K_\ell(q)\nu^{-1/2-1/q}$, so that the viscosity parameter will not appear in our computations.

4.1. $\mathcal{L}^p$ decay estimates for $\mathcal{L}^2$ initial data.

We recall that we transferred our system in the body frame applying the change of variable (1.10). So, the equations (1.1)-(1.8) became (1.11)-(1.18). Our first proposition reads:

Proposition 4.1. *Let $V_0 \in \mathcal{L}^2$. There exists $\lambda_0 > 0$ such that, if*

$$\|V_0\|_{\mathcal{L}^2} \leq \lambda_0, \tag{4.1}$$

then, the unique global weak solution V of (1.11)-(1.18) *satisfies the following: for all $p \in [2,\infty)$, there exists constants $H(p,\lambda_0)$ and $H_\ell(\lambda_0)$ such that*

$$\sup_{t>0} t^{\frac{1}{2}-\frac{1}{p}} \|V(t)\|_{\mathcal{L}^p} \leq H(p,\lambda_0), \qquad \text{and} \qquad \sup_{t>0} t^{\frac{1}{2}} |\ell_V(t)| \leq H_\ell(\lambda_0),$$

Proof. We split the proof of Proposition 4.1 into six steps.

Step 1: integral formulation. Following [22], we rewrite the Navier-Stokes equations (1.11)-(1.18) in the following abstract form:

$$\partial_t V + AV = \mathbb{P}F$$

where

$$F(V) = \begin{cases} (\ell_V - V)\cdot\nabla V & \text{on } \mathcal{F}_0 \\ 0 & \text{on } B_0, \end{cases}$$

$\mathbb{P}$ denotes the continuous projector from L^p to $\mathcal{L}^p$, and ℓ_V is defined for $V \in \mathcal{L}^p$ by (1.31). Then, Duhamel formula gives the following integral formulation of the above equations:

$$V(t) = S(t)V_0 + \int_0^t S(t-s)\mathbb{P}F(V(s))\,\mathrm{d}s. \tag{4.2}$$

T. Kato suggests to construct a solution by successive approximations: let the sequence $(y_n)_{\in\mathbb{N}}$ be defined by

$$\begin{cases} Y_0 &:= & S(t)V_0\,, \\ Y_n &:= & S(t)V_0 + \mathcal{K}Y_{n-1}\,, \quad \forall\, n \in \mathbb{N}\,, \end{cases} \qquad \text{where} \qquad \mathcal{K}Y(t) = \int_0^t S(t-s)\mathbb{P}F(Y)(s)\,\mathrm{d}s. \tag{4.3}$$

Our aim is to prove that this sequence satisfies uniformly estimates of Proposition 4.1 and converges for small initial data. To simplify notations, in the following we set $\ell_{Y_n} = \ell_n$.

Concerning the nonlinear term, we note that, $\mathbb{P}F(Y)$ is well-defined as soon as $Y \in \mathcal{L}^p$ for $p > 2$ satisfies $\nabla Y \in L^2(\mathcal{F}_0)$. Indeed, we can then split $F(Y)|_{\mathcal{F}_0} = -Y\cdot\nabla Y + \ell_Y \cdot \nabla Y$, the first term being in $L^q(\mathcal{F}_0)$ (where $q = 2p/(2+p)$) and the second one in $L^2(\mathcal{F}_0)$. We have then :

$$\mathbb{P}F(Y) = -\mathbb{P}_q[1_{\mathcal{F}_0} Y\cdot\nabla Y] + \mathbb{P}_2[1_{\mathcal{F}_0}\ell_Y\cdot\nabla Y].$$

Furthermore, we remark that, if $Y \in \mathcal{L}^{p_0}$ (with $p_0 \in [1,\infty)$) satisfies $y \in H^1(\mathcal{F}_0)$ then:

$$F(Y) = \operatorname{div}\tilde{F}(Y) \text{ where } \tilde{F}(Y) = \begin{cases} (\ell_Y - Y)\otimes Y & \text{on } \mathcal{F}_0\,, \\ 0 & \text{on } B_0\,. \end{cases}$$

This property is satisfied since $\tilde{F}(Y)n$ vanishes on ∂B_0 as B_0 is a disk. The operator $\mathcal{K}$ can then be defined indifferently as:

$$\int_0^t S(t-s)[\mathbb{P}F(Y)(s)]\,\mathrm{d}s \quad \text{ or } \quad \int_0^t [S(t-s)\mathbb{P}\mathrm{div}\,]\tilde{F}(Y)(s)\,\mathrm{d}s\,,$$

where $S(t-s)\mathbb{P}\mathrm{div}$ is defined by duality. In order to get uniform estimates on the functions Y_n and their limit, we work with the second form (Step 2 to 5). In Step 6, we apply the first form to prove that our construction coincides with the unique global weak solution constructed in [22].

Step 2: estimates of $t^{\frac{3}{8}}\|Y_n\|_{\mathcal{L}^8},\ \|Y_n\|_{\mathcal{L}^2},\ t^{\frac{1}{2}}|\ell_n(t)|$. The goal of this step is to show the following Lemma:

Lemma 4.2. *There exists a constant $\lambda_0 > 0$ such that for all $V_0 \in \mathcal{L}^2$ satisfying (4.1) there exists $\mu_0 > 0$ such that:*

$$\sup_{t>0}\{t^{\frac{3}{8}}\|Y_n(t)\|_{\mathcal{L}^8}\} \leq \mu_0, \quad \sup_{t>0}\{\|Y_n(t)\|_{\mathcal{L}^2}\} \leq \mu_0\,, \quad \sup_{t>0} t^{\frac{1}{2}}|\ell_n(t)| \leq \mu_0.$$

Besides, μ_0 can be chosen arbitrary small, independent of V_0, up to restrict the size of λ_0.

Proof. We are going to find by induction a sequence G_n such that for all n,

$$\sup_{t>0}\{t^{\frac{1}{2}-\frac{1}{8}}\|Y_n(t)\|_{\mathcal{L}^8}\} = \sup_{t>0} t^{\frac{3}{8}}\|Y_n(t)\|_{\mathcal{L}^8} \quad\leq\quad G_n, \tag{4.4}$$

$$\sup_{t>0}\{t^{\frac{1}{2}-\frac{1}{2}}\|Y_n(t)\|_{\mathcal{L}^2}\} = \sup_{t>0}\|Y_n(t)\|_{\mathcal{L}^2} \quad\leq\quad G_n. \tag{4.5}$$

$$\sup_{t>0}\{t^{\frac{1}{2}}|\ell_n(t)|\} \quad\leq\quad G_n. \tag{4.6}$$

It is clear from (1.33) that (4.4)-(4.6) is verified for Y_0, where

$$G_0 = \max\{K_1(8,2), K_1(2,2), K_1(\infty,2)\}\|V_0\|_{\mathcal{L}^2}. \tag{4.7}$$

In the sequel, we denote by C_0 the following positive constants:

$$C_0 := \max\Big(K_4(8,4), K_4(2,2), K_\ell(4)\Big)\int_0^1 (1-\tau)^{-\frac{5}{8}}\tau^{-\frac{3}{4}}\,\mathrm{d}\tau,$$

where K_4 is the constant in (3.33) and K_ℓ in (3.35).

Next, we assume that the properties are true for the rank n, and we show it for rank $n+1$: using (3.33) with $p = 8$, $q = 4$, we get

$$t^{\frac{3}{8}}\|Y_{n+1}(t)\|_{\mathcal{L}^8} \leq G_0 + t^{\frac{3}{8}}K_4(8,4)\int_0^t (t-s)^{-\frac{5}{8}}\|Y_n(s)\|^2_{\mathcal{L}^8}\,\mathrm{d}s + t^{\frac{3}{8}}K_4(8,4)\int_0^t (t-s)^{-\frac{5}{8}}|\ell_n(s)|\|Y_n(s)\|_{\mathcal{L}^4}\,\mathrm{d}s.$$

By interpolation, we have:

$$\|Y_n\|_{\mathcal{L}^4} \leq \|Y_n\|^{1/3}_{\mathcal{L}^2}\|Y_n\|^{2/3}_{\mathcal{L}^8}\,. \tag{4.8}$$

So, we use that:

$$\|Y_n(s)\|^2_{\mathcal{L}^8} \leq (s^{-\frac{3}{8}}G_n)^2, \qquad \|Y_n(s)\|_{\mathcal{L}^4} \leq s^{-\frac{1}{4}}G_n, \qquad |\ell_n(s)| \leq s^{-\frac{1}{2}}G_n\,,$$

to get

$$\begin{aligned} t^{\frac{3}{8}}\|Y_{n+1}(t)\|_{\mathcal{L}^8} &\leq G_0 + t^{\frac{3}{8}}K_4(8,4)\int_0^t (t-s)^{-\frac{5}{8}}s^{-\frac{3}{4}}|G_n|^2\,\mathrm{d}s + t^{\frac{3}{8}}K_4(8,4)\int_0^t (t-s)^{-\frac{5}{8}}s^{-\frac{3}{4}}|G_n|^2\,\mathrm{d}s \\ &\leq G_0 + 2|G_n|^2K_4(8,4)\int_0^1 (1-\tau)^{-\frac{5}{8}}\tau^{-\frac{3}{4}}\,\mathrm{d}\tau \\ &\leq G_0 + 2C_0|G_n|^2\,. \end{aligned}$$

Writing the same computation and using (3.33) with $p = q = 2$ gives

$$\|Y_{n+1}(t)\|_{\mathcal{L}^2} \leq G_0 + K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}}\|Y_n(s)\|^2_{\mathcal{L}^4}\,\mathrm{d}s + K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}}|\ell_n(s)|\|Y_n(s)\|_{\mathcal{L}^2}\,\mathrm{d}s$$

Thus, we obtain

$$\begin{aligned}\|Y_{n+1}(t)\|_{\mathcal{L}^2} &\leq G_0 + 2K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}|G_n|^2\,\mathrm{d}s\\ &\leq G_0 + 2C_0|G_n|^2.\end{aligned}$$

Finally, we apply Corollary 3.11 with $q=4$, which yields:

$$\begin{aligned}t^{\frac{1}{2}}|\ell_{n+1}(t)| &\leq G_0 + t^{\frac{1}{2}}K_\ell(4)\int_0^t (t-s)^{-\frac{3}{4}}\|Y_n(s)\|^2_{\mathcal{L}^8}\,\mathrm{d}s + t^{\frac{1}{2}}K_\ell(4)\int_0^t (t-s)^{-\frac{3}{4}}|\ell_n(s)|\|Y_n(s)\|_{\mathcal{L}^4}\,\mathrm{d}s\\ &\leq G_0 + 2C_0|G_n|^2\,.\end{aligned}$$

Hence, we can take

$$G_{n+1} = G_0 + 2C_0|G_n|^2,$$

in (4.4)–(4.6) with G_0 given by (4.7). Choosing λ_0 such that $G_0 \leq 1/(8C_0)$, we easily get by induction that for all $n\in\mathbb{N}$

$$G_n \leq \frac{1-(1-8C_0G_0)^{\frac{1}{2}}}{4C_0} =: \mu_0. \tag{4.9}$$

Therefore, (G_n) is bounded by μ_0 which implies that (4.4)-(4.6) are uniform estimates. This ends the proof of Lemma 4.2. According to (4.7)–(4.9), μ_0 can be chosen arbitrarily small by taking $\lambda_0>0$ small enough. □

Step 3: convergence of Y_n. The goal of this step is to show that the sequence Y_n constructed in the previous step strongly converges in $L^\infty(0,\infty;\mathcal{L}^2)\cap L^\infty_{loc}(0,\infty;\mathcal{L}^8)$, endowed with the norm:

$$\|\cdot\|_{L^\infty(0,\infty;\mathcal{L}^2)} + \|t^{3/8}\cdot\|_{L^\infty(0,\infty;\mathcal{L}^8)} + \|t^{\frac{1}{2}}\ell.\|_{L^\infty(0,\infty)}\,,$$

and that the limit V solves the integral formulation (4.2) of the Navier-Stokes equations (1.11)–(1.18).

The main idea here comes from [15]: let us define

$$\begin{aligned}&W_{n+1}(t) := Y_{n+1}(t) - Y_n(t)\\ &= \int_0^t S(t-s)\mathbb{P}\mathrm{div}\,1_{\mathcal{F}_0}\Big((\ell_n - Y_n)(s)\otimes(Y_n - Y_{n-1})(s) + (\ell_n - \ell_{n-1} + Y_{n-1} - Y_n)(s)\otimes Y_{n-1}(s)\Big)\,\mathrm{d}s.\end{aligned}$$

Again, we construct a sequence a_n such that for all n,

$$a_n \geq \max\left\{\sup_{t>0} t^{\frac{3}{8}}\|W_n(t)\|_{\mathcal{L}^8}, \sup_{t>0}\|W_n(t)\|_{\mathcal{L}^2}, \sup_{t>0} t^{\frac{1}{2}}|\ell_{W_n}(t)|\right\}.$$

Indeed, we have:

$$\begin{aligned}t^{\frac{3}{8}}\|W_{n+1}(t)\|_{\mathcal{L}^8} &\leq K_4(8,4)t^{\frac{3}{8}}\Big(\int_0^t (t-s)^{-\frac{5}{8}}(\|Y_n(s)\|_{\mathcal{L}^8} + \|Y_{n-1}(s)\|_{\mathcal{L}^8})\|W_n(s)\|_{\mathcal{L}^8}\,\mathrm{d}s\\ &\quad + \int_0^t (t-s)^{-\frac{5}{8}}(|\ell_n(s)|\|W_n(s)\|_{\mathcal{L}^4} + |\ell_n(s)-\ell_{n-1}(s)|\|Y_{n-1}(s)\|_{\mathcal{L}^4})\,\mathrm{d}s\Big)\\ &\leq 4K_4(8,4)\int_0^t (t-s)^{-\frac{5}{8}}s^{-\frac{3}{4}}\mu_0 a_n\,\mathrm{d}s\\ &\leq 4C_0\mu_0 a_n.\end{aligned}$$

Here and in what follows, we always estimate L^4-norms by interpolating the L^2-norm and L^8-norm (see (4.8)). In the same manner, we have

$$\begin{aligned}\|W_{n+1}(t)\|_{\mathcal{L}^2} &\leq K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}}(\|Y_n(s)\|_{\mathcal{L}^4} + \|Y_{n-1}(s)\|_{\mathcal{L}^4})\|W_n(s)\|_{\mathcal{L}^4}\,\mathrm{d}s \qquad (4.10)\\ &\quad + K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}}(|\ell_n(s)|\|W_n(s)\|_{\mathcal{L}^2} + |\ell_n(s)-\ell_{n-1}(s)|\|Y_{n-1}(s)\|_{\mathcal{L}^2})\,\mathrm{d}s\,,\end{aligned}$$

which implies:

$$\begin{aligned}\|W_{n+1}(t)\|_{\mathcal{L}^2} &\leq 4K_4(2,2)\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \mu_0 a_n \,\mathrm{d}s \\ &\leq 4C_0\mu_0 a_n \,. \end{aligned} \tag{4.11}$$

Finally, we have:

$$\begin{aligned} t^{\frac{1}{2}}|\ell_{n+1}(t)-\ell_n(t)| &\leq K_\ell(4) t^{\frac{1}{2}}\Big(\int_0^t (t-s)^{-\frac{3}{4}}(\|Y_n(s)\|_{\mathcal{L}^8}+\|Y_{n-1}(s)\|_{\mathcal{L}^8})\|W_n(s)\|_{\mathcal{L}^8}\,\mathrm{d}s \qquad (4.12)\\ &+ \int_0^t (t-s)^{-\frac{3}{4}}(|\ell_n(s)|\|W_n(s)\|_{\mathcal{L}^4}+|\ell_n(s)-\ell_{n-1}(s)|\|Y_{n-1}(s)\|_{\mathcal{L}^4})\,\mathrm{d}s\Big) \\ &\leq 4K_\ell(4)t^{\frac{1}{2}}\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}}\mu_0 a_n\,\mathrm{d}s \\ &\leq 4C_0\mu_0 a_n. \end{aligned}$$

Therefore, we can take $a_n = (4C_0\mu_0)^{n-1}a_1$ where a_1 can be easily estimated thanks to Lemma 4.2. According to Lemma 4.2 again, one can choose $\lambda_0 > 0$ such that $\mu_0 < 1/(4C_0)$. With this choice, $\sum(\|W_n\|_{L^\infty(0,\infty;\mathcal{L}^2)} + \|t^{1/8}W_n\|_{L^\infty(0,\infty;\mathcal{L}^8)} + \|t^{1/2}\ell_{W_n}(t)\|_{L^\infty(0,\infty)})$ converges uniformly and there exists a function $V \in L^\infty(0,\infty;\mathcal{L}^2)\cap L^\infty_{loc}(0,\infty;\mathcal{L}^8)$ such that

$$Y_n \to V \text{ strongly in } L^\infty(0,\infty;\mathcal{L}^2)\cap L^\infty_{loc}(0,\infty;\mathcal{L}^8)\,,\quad \ell_n\to\ell_V \text{ in } L^\infty_{loc}(0,\infty)\,.$$

By construction V satisfies the decay estimates of Lemma 4.2:

$$\sup_{t>0}\{t^{\frac{3}{8}}\|V(t)\|_{\mathcal{L}^8}\}\leq\mu_0,\qquad \sup_{t>0}\{\|V(t)\|_{\mathcal{L}^2}\}\leq\mu_0,\qquad \sup_{t>0}\{t^{\frac{1}{2}}|\ell_V(t)|\}\leq\mu_0. \tag{4.13}$$

The last point to check is that V indeed is a solution of the integral equation (4.2), *i.e.* we have to check that $\mathcal{K}Y_n \to \mathcal{K}V$. This computation is exactly the previous one:

$$\mathcal{K}V(t)-\mathcal{K}Y_n(t)=\int_0^t S(t-s)\mathbb{P}\mathrm{div}\,1_{\mathcal{F}_0}\Big((\ell_V-V)\otimes(V-Y_n)+(\ell_V-\ell_n+Y_n-V)\otimes Y_n\Big)(s)\,\mathrm{d}s.$$

Doing as in (4.10) and using that $\sup_{t>0} t^{\frac{3}{8}}\|V-Y_n(t)\|_{\mathcal{L}^8}$, $\sup_{t>0}\|V(t)-Y_n(t)\|_{\mathcal{L}^2}$ and $\sup_{t>0}\{t^{\frac{1}{2}}|\ell_V(t)-\ell_n(t)|\}$ tend to zero as $n\to\infty$, one easily shows that $\mathcal{K}V-\mathcal{K}Y_n$ converges to 0 in $L^\infty(0,\infty;\mathcal{L}^2)$. This shows that the limit V of the sequence Y_n solves the integral formulation (4.2) of the Navier-Stokes equations (1.11)–(1.18).

Step 4: The limit V is the unique weak solution of (1.11)–(1.18) **when** $V_0 \in H^1(\mathbb{R}^2)$**.** In the previous steps, we have constructed a solution to the integral formulation (4.2) of the Navier-Stokes equations (1.11)–(1.18) verifying the $L^p - L^q$ decay estimates (4.13). The last point that we have to check is that this solution V is the unique solution from the well-posedness theory of [22]. In [22], uniqueness is obtained in the framework $V\in L^\infty(0,T;\mathcal{L}^2)\cap L^2(0,T;H^1(\mathbb{R}^2))$.

Of course, our solution satisfies by construction $V\in L^\infty(0,T;\mathcal{L}^2)$, and thus we only have to check that $V\in L^2(0,T;H^1(\mathbb{R}^2))$.

We focus on the case of initial data $V_0\in H^1(\mathbb{R}^2)$ (*i.e.*, $V_0\in\mathcal{D}(A^{1/2})$, see (2.1)). In that case, we prove that the solution V constructed above is the unique solution in $L^\infty(0,T;\mathcal{L}^2)\cap L^2(0,T;H^1(\mathbb{R}^2))$. The main issue is to show that the sequence $\|\nabla y_n\|_{L^\infty(0,T;L^2(\mathcal{F}_0))}$ is uniformly bounded in n for any arbitrary $T>0$ fixed. For T fixed, it is proved in [22, Cor. 4.3] that $S(t)V_0$ belongs to $\mathcal{C}([0,T];H^1(\mathbb{R}^2))$ when $V_0\in H^1(\mathbb{R}^2)$, which implies that there exists $J_0>0$ such that

$$\|\nabla y_0\|_{L^\infty(0,T;L^2(\mathcal{F}_0))}\leq J_0.$$

Next, we construct by induction a sequence J_n such that for all $n\in\mathbb{N}$

$$\|\nabla y_n\|_{L^\infty(0,T;L^2(\mathcal{F}_0))}\leq J_n.$$

Using (1.34) with $p=2$, $q=8/5$ and $p=q=2$, for all $t\in[0,T]$

$$\begin{aligned}
\|\nabla y_{n+1}(t)\|_{L^2(\mathcal{F}_0)} &\leq J_0 + C_{8/5}K_2(2,8/5)\int_0^t (t-s)^{-\frac12+\frac12-\frac58}\||y_n(s)||\nabla y_n(s)|\|_{L^{8/5}(\mathcal{F}_0)}\,\mathrm{d}s\\
&\quad + K_2(2,2)\int_0^t (t-s)^{-\frac12}|\ell_n(s)|\|\nabla y_n(s)\|_{L^2(\mathcal{F}_0)}\,\mathrm{d}s\\
&\leq J_0 + C_{8/5}K_2(2,8/5)\int_0^t (t-s)^{-\frac58}\|Y_n(s)\|_{\mathcal{L}^8}\|\nabla y_n(s)\|_{L^2(\mathcal{F}_0)}\,\mathrm{d}s\\
&\quad + K_2(2,2)\int_0^t (t-s)^{-\frac12}|\ell_n(s)|\|\nabla y_n(s)\|_{L^2(\mathcal{F}_0)}\,\mathrm{d}s\\
&\leq J_0 + C_{8/5}K_2(2,8/5)\int_0^t (t-s)^{-\frac58}s^{-\frac38}\mu_0 J_n\,\mathrm{d}s\\
&\quad + K_2(2,2)\int_0^t (t-s)^{-\frac12}s^{-\frac12}\mu_0 J_n\,\mathrm{d}s\\
&\leq J_0 + \tilde C_0\mu_0 J_n =: J_{n+1},
\end{aligned}$$

where $C_{8/5} := \|\mathbb{P}_{8/5}\|_{\mathscr{L}_c(L^{8/5}(\mathbb{R}^2)\to\mathcal{L}^{8/5})}$ and $\tilde C_0 := (C_{8/5}K_2(2,8/5)+K_2(2,2))\int_0^1(1-\tau)^{-\frac58}\tau^{-\frac12}\,\mathrm{d}\tau$. Taking $\lambda_0>0$ small enough so that $\tilde C_0\mu_0\leq 1/2$, there holds:

$$J_n = J_0\sum_{k=0}^n\left(\tilde C_0\mu_0\right)^k \leq J_0\frac{1}{1-\tilde C_0\mu_0}\leq 2J_0.$$

Hence we have, for all $n\in\mathbb{N}$,

$$\|\nabla y_n\|_{L^\infty(0,T;L^2(\mathcal{F}_0))}\leq 2J_0,$$

which implies that ∇v verifies the same estimate. Inside $B(0,1)$, we have

$$|\nabla Y_n| = |\omega_{Y_n}| = \left|\int_{B(0,1)} Y_n\cdot x^\perp\,\mathrm{d}x\right|\leq C\|Y_n\|_{\mathcal{L}^2}$$

which is uniformly bounded in time and n.

Note that this is not enough to conclude that $Y_n\in L^\infty([0,T];H^1(\mathbb{R}^2))$ for all n and one should be careful that the boundary conditions are compatible on ∂B_0. In order to do that, for all $n\in\mathbb{N}$, we introduce, for all $\epsilon\in(0,1)$ and $t>0$,

$$Y^\varepsilon_{n+1}(t) = S(t)V_0 + \int_0^{t(1-\varepsilon)} S(t-s)\mathbb{P}F(Y_n(s))\,\mathrm{d}s.$$

Of course, arguing as above, Y^ε_{n+1} satisfy exactly the same estimate as Y_{n+1}, uniformly with respect to $\varepsilon>0$ and $n\in\mathbb{N}$:

$$\|\nabla Y^\varepsilon_{n+1}\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} = \frac{\pi}{2}\|\omega_{Y^\varepsilon_{n+1}}\|_{L^\infty(0,T)} + \|\nabla y^\varepsilon_{n+1}\|_{L^\infty(0,T;L^2(B_0))}\leq C. \tag{4.14}$$

But, since the semigroup $S(t)$ is analytic on $\mathcal{L}^2$, for $t>0$, $S(t)V_0\in\mathcal{D}(A)$ (see (2.1)) and

$$\begin{aligned}
\int_0^{t(1-\varepsilon)} S(t-s)\mathbb{P}F(Y_n(s))\,\mathrm{d}s &= S(t\varepsilon)\int_0^{t(1-\varepsilon)} S(t(1-\varepsilon)-s)\mathbb{P}F(Y_n(s))\,\mathrm{d}s\\
&= S(t\varepsilon)\left(Y_{n+1}(t(1-\varepsilon)) - S(t(1-\varepsilon))V_0\right)\\
&= S(t\varepsilon)Y_{n+1}(t(1-\varepsilon)) - S(t)V_0.
\end{aligned}$$

Since for all $t>0$, $Y_{n+1}(t)\in\mathcal{L}^2$, this implies that for all $t>0$, $Y^\varepsilon_{n+1}(t)$ belongs to $\mathcal{D}(A)$ for all $t>0$.

Besides, as $\varepsilon \to 0$, Y^ε_{n+1} converges to Y_{n+1} in $L^\infty_{loc}([0,\infty);\mathcal{L}^2)$ since

$$\begin{aligned}
&\|Y^\varepsilon_{n+1}(t) - Y_{n+1}(t)\|_{\mathcal{L}^2}\\
&\leq \int_{t(1-\varepsilon)}^t (K_1(2,4/3)(t-s)^{-1/4}\|y_n(s)\|_{\mathcal{L}^4}\|\nabla y_n(s)\|_{L^2(\mathcal{F}_0)} + K_1(2,2)|\ell_n(s)|\nabla y_n(s)\|_{L^2(\mathcal{F}_0)})\,\mathrm{d}s\\
&\leq t^{1/2}\left(K_1(2,4/3)\mu_0 J_0\int_{1-\varepsilon}^1 (1-\tau)^{-1/4}\tau^{-1/4}\,\mathrm{d}\tau + K_1(2,2)\mu_0 J_0 \int_{1-\varepsilon}^1 \tau^{-1/2}\,\mathrm{d}\tau\right).
\end{aligned}$$

Hence Y_{n+1} is the strong limit in $L^\infty((0,T);\mathcal{L}^2)$ of the sequence of functions Y^ε_{n+1} satisfying (4.14) and the fact that for all $\varepsilon > 0$ and $t > 0$, $Y^\varepsilon_{n+1}(t) \in \mathcal{D}(A)$. Therefore, Y_{n+1} belongs to $L^2([0,T];H^1(\mathbb{R}^2))$.

Besides, since the bound in (4.14) is uniform in $\epsilon > 0$ and $n \in \mathbb{N}$, V also belongs to $L^2((0,T);H^1(\mathbb{R}^2))$. According to [22], when the initial data V_0 belongs to $H^1(\mathbb{R}^2)$, the solution V constructed in the above steps is the unique weak solution of (1.11)–(1.18).

Step 5: Sensitivity of V to the initial data. So far, given $V_0 \in \mathcal{L}^2$ satisfying the smallness condition (4.1), we have constructed a solution V of the integral equation (4.2). In this step, we show that the map $V_0 \mapsto V$ is continuous from the ball of $\mathcal{L}^2$ with radius λ_0 to $L^\infty((0,\infty);\mathcal{L}^2)$.

Let us consider V^a_0 and V^b_0 two elements of $\mathcal{L}^2$ satisfying the smallness condition (4.1), and Y^a_n and Y^b_n the corresponding sequences in (4.3). We set $Z_n = Y^a_n - Y^b_n$, which satisfies by construction

$$Z_{n+1}(t) = S(t)(V^a_0 - V^b_0) + \mathcal{K}Y^a_n(t) - \mathcal{K}Y^b_n(t) = Z_0(t) + \mathcal{K}Y^a_n(t) - \mathcal{K}Y^b_n(t).$$

Similarly as in Step 3, we are going to construct a sequence b_n such that for all n,

$$b_n \geq \max\left\{\sup_{t>0}\{t^{3/8}\|Z_n(t)\|_{\mathcal{L}^8}\},\ \sup_{t>0}\{\|Z_n(t)\|_{\mathcal{L}^2}\},\ \sup_{t>0}\{|t^{\frac12}\ell_{Z_n}(t)|\}\right\}.$$

Of course, by Theorem 1.1, one can take b_0 proportional to $\|V^a_0 - V^b_0\|_{\mathcal{L}^2}$. Since

$$\begin{aligned}
\mathcal{K}Y^a_n(t) - \mathcal{K}Y^b_n(t) &= \int_0^t S(t-s)\mathbb{P}\mathrm{div}\,1_{\mathcal{F}_0}\left((\ell^a_n - Y^a_n)(s)\otimes Y^a_n(s) - (\ell^b_n - Y^b_n)(s)\otimes Y^b_n(s)\right)\mathrm{d}s\\
&= \int_0^t S(t-s)\mathbb{P}\mathrm{div}\,1_{\mathcal{F}_0}\Big((Y^b_n - Y^a_n)(s)\otimes Y^b_n(s) + Y^a_n(s)\otimes(Y^b_n(s) - Y^a_n(s))\\
&\qquad + (\ell^a_n - \ell^b_n)(s)\otimes Y^a_n(s) + \ell^b_n(s)\otimes(Y^a_n - Y^b_n)(s)\Big)\,\mathrm{d}s,
\end{aligned}$$

arguing as in Step 3,

$$\begin{aligned}
t^{\frac38}\|\mathcal{K}Y^a_n(t) - \mathcal{K}Y^b_n(t)\|_{\mathcal{L}^8} &\leq K_4(8,4)t^{\frac38}\Big(\int_0^t (t-s)^{-\frac58}(\|Y^a_n(s)\|_{\mathcal{L}^8} + \|Y^b_n(s)\|_{\mathcal{L}^8})\|Z_n(s)\|_{\mathcal{L}^8}\,\mathrm{d}s\\
&\quad + \int_0^t (t-s)^{-\frac58}(|\ell^b_n(s)|\|Z_n(s)\|_{\mathcal{L}^4} + |\ell^a_n(s) - \ell^b_n(s)|\|Y^a_n(s)\|_{\mathcal{L}^4})\,\mathrm{d}s\Big)\\
&\leq 4K_4(8,4)t^{\frac38}\int_0^t (t-s)^{-\frac58}s^{-\frac34}\mu_0 b_n\,\mathrm{d}s\\
&\leq 4C_0\mu_0 b_n,
\end{aligned}$$

where μ_0 is the constant in Lemma 4.2. And similarly as in (4.10),

$$\begin{aligned}
\|\mathcal{K}Y^a_n(t) - \mathcal{K}Y^b_n(t)\|_{\mathcal{L}^2} &\leq K_4(2,2)\int_0^t (t-s)^{-\frac12}(\|Y^a_n(s)\|_{\mathcal{L}^4} + \|Y^b_n(s)\|_{\mathcal{L}^4})\|Z_n(s)\|_{\mathcal{L}^4}\,\mathrm{d}s\\
&\quad + K_4(2,2)\int_0^t (t-s)^{-\frac12}(|\ell^b_n(s)|\|Z_n(s)\|_{\mathcal{L}^2} + |\ell^a_n(s) - \ell^b_n(s)|\|Y^a_n(s)\|_{\mathcal{L}^2})\,\mathrm{d}s,\\
&\leq 4K_4(2,2)\int_0^t (t-s)^{-\frac12}s^{-\frac12}\mu_0 b_n\,\mathrm{d}s\\
&\leq 4C_0\mu_0 b_n.
\end{aligned}$$

Finally, we prove as in (4.12) that:

$$\begin{aligned} t^{\frac12}|\ell_n^a(t)-\ell_n^b(t)| &\leq K_\ell(4)t^{\frac12}\Big(\int_0^t (t-s)^{-\frac34}(\|Y_n^a(s)\|_{\mathcal{L}^8}+\|Y_n^b(s)\|_{\mathcal{L}^8})\|Z_n(s)\|_{\mathcal{L}^8}\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{-\frac34}(|\ell_n^b(s)|\|Z_n(s)\|_{\mathcal{L}^4}+|\ell_n^a(s)-\ell_n^b(s)|\|Y_n^a(s)\|_{\mathcal{L}^4})\,\mathrm{d}s\Big) \\ &\leq 4K_\ell(4)t^{\frac12}\int_0^t (t-s)^{-\frac34}s^{-\frac34}\mu_0 b_n\,\mathrm{d}s \\ &\leq 4C_0\mu_0 b_n. \end{aligned}$$

We can then chose $b_{n+1}=b_0+4C_0\mu_0 b_n$ and thus (recall that $4C_0\mu_0<1$ for our choice of λ_0 since Step 3)

$$\forall n\in\mathbb{N},\quad b_n\leq b_0\left(\frac{1}{1-4C_0\mu_0}\right).$$

In particular, passing to the limit $n\to\infty$, we obtain

$$\sup_{t>0}\|V^a(t)-V^b(t)\|_{\mathcal{L}^2}\leq C\|V_0^a-V_0^b\|_{\mathcal{L}^2}.$$

Thus, our above construction yields a map $V_0\mapsto V$ continuous on the ball of $\mathcal{L}^2$ of radius λ_0 to $L^\infty((0,\infty);\mathcal{L}^2)$, which coincides with the map $V_0\mapsto V_w$ for initial data in $H^1(\mathbb{R}^2)$, where V_w denotes the weak solution of (1.11)–(1.18). Since both maps are continuous (see "The existence part" in the proof of Proposition 2.5 in [22, Section 6] for the continuity of the map $V_0\mapsto V_w$ from $\mathcal{L}^2$ to $L^\infty((0,\infty);\mathcal{L}^2)$), they coincide on the ball of $\mathcal{L}^2$ of radius λ_0. This implies that the solution V constructed in Step 3, as the limit of the sequence Y_n, actually is the unique weak global solution of (1.11)–(1.18).

Step 6: Estimates on the $\mathcal{L}^p$ norm of V for $2\leq p<\infty$. The goal of this step is to show the following Lemma:

Lemma 4.3. *Let λ_0 the constant of Lemma 4.2. For all $p\in[2,\infty)$, there exists a constant $H(p,\lambda_0)$ such that for all $V_0\in\mathcal{L}^2$ satisfying* (4.1)*, the solution V of* (1.11)*–*(1.18) *satisfies:*

$$\sup_{t>0}\{t^{\frac12-\frac1p}\|V(t)\|_{\mathcal{L}^p}\}\leq H(p,\lambda_0). \tag{4.15}$$

Proof. For $p\leq 8$ we obtain (4.15) by interpolation of the estimates of Lemma 4.2. Assume now $p\in[8,\infty)$:

$$\begin{aligned} t^{\frac12-\frac1p}\|V(t)\|_{\mathcal{L}^p} &\leq K_1(p,2)\|V_0\|_{\mathcal{L}^2}+t^{\frac12-\frac1p}K_4(p,4)\int_0^t (t-s)^{-\frac34+\frac1p}\|V(s)\|_{\mathcal{L}^8}^2\,\mathrm{d}s \\ &\quad +t^{\frac12-\frac1p}K_4(p,4)\int_0^t (t-s)^{-\frac34+\frac1p}|\ell_V(s)|\|V(s)\|_{\mathcal{L}^4}\,\mathrm{d}s \\ &\leq K_1(p,2)\|V_0\|_{\mathcal{L}^2}+t^{\frac12-\frac1p}K_4(p,4)\int_0^t (t-s)^{-\frac34+\frac1p}s^{-\frac34}\left(\mu_0^2+\mu_0^2\right)\,\mathrm{d}s \\ &\leq K_1(p,2)\lambda_0+2K_4(p,4)\mu_0^2\int_0^1 (1-\tau)^{-\frac34+\frac1p}\tau^{-\frac34}\,\mathrm{d}\tau, \end{aligned}$$

which gives the desired estimates (4.15) and concludes the proof of Lemma 4.3. ☐

The proof of Proposition 4.1 is then completed. ☐

Remark 4.4 (Remark on the smallness condition)*.* The smallness condition on $\|V_0\|_{\mathcal{L}^2}$ is not surprising, and such an assumption appears in a lot of articles when global well-posedness is required (see e.g. [14]). In dimension 2, several works ([21, 27, 13] in the full plane and [1] in fixed exterior domains) show that the L^2-norm tends to zero when $t\to 0$ for initial data in L^2. Of course, this allows in such situations to get a global result for any initial data in L^2 by proving only a local result for initial data having small L^2-norm. Unfortunately, concerning the case of a moving disk in a 2D viscous fluid, despite the energy estimate satisfied by the solutions of (1.1)–(1.8) which immediately guarantees the global decay of the L^2-norm of the solution, it is still not clear that the L^2-norm of all solutions with initial data in L^2 go to zero as $t \to \infty$. This appears to be a challenging question.

4.2. **The case of an initial data $\mathcal{L}^q$ for $q \in (1,2)$.** The goal of this section is to prove Theorem 1.3.

Proof of Theorem 1.3. The proof is based on the construction done in Proposition 4.1.

Step 1. Decay estimates for $p = 2$ and $p = 4$. Let us consider again the sequence Y_n constructed in (4.3), for which we already know the decay estimates of Lemma 4.2 and, by interpolation,

$$\sup_{t>0}\{t^{1/4}\|Y_n(t)\|_{\mathcal{L}^4}\} \leq \mu_0. \tag{4.16}$$

We then prove the following lemma:

Lemma 4.5. *There exists $\lambda_0(q)$ small enough such that for any $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ satisfying the smallness condition (4.1) with $\lambda_0 \leq \lambda_0(q)$, there exist constants $H(2,q,V_0), H(4,q,V_0)$ for which the sequence Y_n constructed in (4.3) satisfies*

$$\sup_{t>0}\{t^{1/q-1/2}\|Y_n(t)\|_{\mathcal{L}^2}\} \leq H(2,q,V_0), \qquad \sup_{t>0}\{t^{1/q-1/4}\|Y_n(t)\|_{\mathcal{L}^4}\} \leq H(4,q,V_0). \tag{4.17}$$

Consequently, we have

$$\sup_{t>0}\{t^{1/q-1/2}\|V(t)\|_{\mathcal{L}^2}\} \leq H(2,q,V_0), \qquad \sup_{t>0}\{t^{1/q-1/4}\|V(t)\|_{\mathcal{L}^4}\} \leq H(4,q,V_0). \tag{4.18}$$

Proof. We are looking for a sequence H_n such that for all n,

$$\sup_{t>0}\{t^{1/q-1/2}\|Y_n(t)\|_{\mathcal{L}^2}\} \leq H_n, \qquad \sup_{t>0}\{t^{1/q-1/4}\|Y_n(t)\|_{\mathcal{L}^4}\} \leq H_n.$$

Of course, Theorem 1.1 implies that H_0 can be taken as $H_0 = (K_1(2,q) + K_1(4,q))\|V_0\|_{\mathcal{L}^q}$.

For $n \in \mathbb{N}$, using Theorem 1.1 and Corollary 3.10,

$$t^{1/q-1/2}\|Y_{n+1}(t)\|_{\mathcal{L}^2} \leq K_1(2,q)\|V_0\|_{\mathcal{L}^q} + t^{1/q-1/2}\int_0^t K_4(2,2)(t-s)^{-1/2}\left(\|Y_n(s)\|^2_{\mathcal{L}^4} + |\ell_n(s)|\|Y_n(s)\|_{\mathcal{L}^2}\right) \mathrm{d}s.$$

Using the decay estimates of Lemma 4.2 and (4.16),

$$\begin{aligned}
\|Y_n(s)\|^2_{\mathcal{L}^4} &\leq (\mu_0 s^{-1/4})(s^{1/4-1/q}H_n) = \mu_0 H_n s^{-1/q},\\
|\ell_n(s)|\|Y_n(s)\|_{\mathcal{L}^2} &\leq (\mu_0 s^{-1/2})(s^{1/2-1/q}H_n) = \mu_0 H_n s^{-1/q},
\end{aligned}$$

we immediately deduce that

$$\sup_{t>0}\{t^{1/q-1/2}\|Y_{n+1}(t)\|_{\mathcal{L}^2}\} \leq K_1(2,q)\|V_0\|_{\mathcal{L}^q} + 2K_4(2,2)\left(\int_0^1 (1-\tau)^{-1/2}\tau^{-1/q}\,\mathrm{d}\tau\right)\mu_0 H_n.$$

Similar computations yield

$$\sup_{t>0}\{t^{1/q-1/4}\|Y_{n+1}(t)\|_{\mathcal{L}^4}\} \leq K_1(4,q)\|V_0\|_{\mathcal{L}^q} + 2K_4(4,2)\left(\int_0^1 (1-\tau)^{-3/4}\tau^{-1/q}\,\mathrm{d}\tau\right)\mu_0 H_n.$$

One can thus take

$$H_{n+1} := H_0 + \widehat{C}_0(q)\mu_0 H_n,$$

with

$$\widehat{C}_0(q) := 2\left(K_4(2,2)\int_0^1 (1-\tau)^{-1/2}\tau^{-1/q}\,\mathrm{d}\tau + K_4(4,2)\int_0^1 (1-\tau)^{-3/4}\tau^{-1/q}\,\mathrm{d}\tau\right)$$

Choosing $\lambda_0(q) > 0$ such that the corresponding μ_0 given by Lemma 4.2 satisfies

$$\widehat{C}_0(q)\mu_0 \leq 1/2, \tag{4.19}$$

we thus immediately obtain that for all n, $H_{n+1} \leq H_0 + H_n/2$, yielding $H_n \leq 2H_0$ for all $n \in \mathbb{N}$. □

Step 2. The solution V satisfies the decay estimates (1.41)**.** The proof of this result follows the proof of Lemma 4.3. For $p \in [2,4]$, (1.41) can be deduced by interpolation with (4.18). For $p \in [4,\infty)$, we write

$$\begin{aligned} t^{\frac{1}{q}-\frac{1}{p}}\|V(t)\|_{\mathcal{L}^p} &\leq K_1(p,q)\|V_0\|_{\mathcal{L}^q} + t^{\frac{1}{q}-\frac{1}{p}}K_4(p,2)\int_0^t (t-s)^{-1+\frac{1}{p}}\|V(s)\|^2_{\mathcal{L}^4}\,\mathrm{d}s \\ &\quad + t^{\frac{1}{q}-\frac{1}{p}}K_4(p,2)\int_0^t (t-s)^{-1+\frac{1}{p}}|\ell_V(s)|\|V(s)\|_{\mathcal{L}^2}\,\mathrm{d}s \\ &\leq K_1(p,q)\|V_0\|_{\mathcal{L}^q} + t^{\frac{1}{q}-\frac{1}{p}}K_4(p,2)\int_0^t (t-s)^{-1+\frac{1}{p}}s^{-\frac{1}{q}}\,\mathrm{d}s\,(H(4,q,V_0)+H(2,q,V_0))\,\mu_0 \\ &\leq K_1(p,q)\|V_0\|_{\mathcal{L}^q} + K_4(p,2)\,(H(4,q,V_0)+H(2,q,V_0))\,\mu_0\int_0^1 (1-\tau)^{-1+\frac{1}{p}}\tau^{-\frac{1}{q}}\,\mathrm{d}\tau. \end{aligned}$$

Step 3. The decay estimate on $\ell_V(t)$. The proof of (1.42) is very similar to the above one and is based on Corollary 3.11: for $p > 2$ such that $1/p - 1/q > -1/2$,

$$\begin{aligned} t^{1/q}|\ell_V(t)| &\leq K_1(\infty,q)\|V_0\|_{\mathcal{L}^q} + t^{1/q}\int_0^t K_\ell(p)(t-s)^{-1/2-1/p}\left(\|V(s)\|^2_{\mathcal{L}^{2p}} + |\ell_V(s)|\|V(s)\|_{\mathcal{L}^p}\right)\mathrm{d}s \\ &\leq K_1(\infty,q)\|V_0\|_{\mathcal{L}^q} \\ &\quad + K_\ell(p)(H(2p,q,V_0)H(2p,2,V_0) + H(p,q,V_0)\mu_0)\int_0^1 (1-\tau)^{-1/2-1/p}\tau^{1/p-1/2-1/q}\,\mathrm{d}\tau. \end{aligned}$$

Step 4. On the map $q \mapsto \lambda_0(q)$. We remark that, by construction $q \mapsto \lambda_0(q)$ is an increasing function. Indeed, condition (4.19) indicates that our proof of Theorem 1.3 requires the result of Lemma 4.2 with $\mu_0 = \mu_0(q) > 0$, where $\mu_0(q)$ is an increasing function of $q \in (1,2]$. Since the explicit formula (4.7) and (4.9) indicates that $\lambda_0 \mapsto \mu_0$ is a continuous increasing function, the map $q \mapsto \lambda_0(q)$ is an increasing function of $q \in (1,2]$. We also note here that $\lambda_0(q) \to 0$ when $q \to 1$, since $\widehat{C}_0(q) \to \infty$. This concludes the proof of Theorem 1.3. □

4.3. **Proximity with the linearized semi-group.** In this last subsection, we compare the asymptotic structure of solutions to the Navier Stokes and Stokes equations and prove Theorem 1.4.

Proof of Theorem 1.4. Let V_0 satisfy the assumptions of our proposition.

As $\tilde q \mapsto \lambda_0(\tilde q)$ is an increasing function, we note that $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(\tilde q)$ for all $\tilde q \in [q,2]$ so that $V(t)$ satisfies the decay estimates of (1.41)-(1.42) for arbitrary $\tilde q \in [q,2]$ and $p \in [2,\infty)$.

According to estimate (3.33) with $p \in [2,\infty)$ and $q = 2$, for all $t > 0$,

$$\begin{aligned} \|V(t) - S(t)V_0\|_{\mathcal{L}^p} &\leq \int_0^t \|S(t-s)\mathbb{P}\mathrm{div}\,((V(s)-\ell_V(s))\otimes V(s))\|_{\mathcal{L}^p}\,\mathrm{d}s \\ &\leq K_4(p,2)\int_0^t (t-s)^{-1+1/p}\left(\|V(s)\|^2_{\mathcal{L}^4} + |\ell_V(s)|\|V(s)\|_{\mathcal{L}^2}\right)\mathrm{d}s\,. \end{aligned} \tag{4.20}$$

But, using (1.41) with $p = 4$ and $\tilde q \in [q,2]$ and with $p = 2$ and $\tilde q$, we get:

$$\sup_{s>0}\{s^{1/\tilde q-1/4}\|V(s)\|_{\mathcal{L}^4}\} \leq H(4,\tilde q,V_0), \quad \sup_{s>0}\{s^{1/\tilde q-1/2}\|V(s)\|_{\mathcal{L}^2}\} \leq H(2,\tilde q,V_0).$$

and using (1.42), we obtain:

$$\sup_{s>0}\{s^{1/\tilde q}|\ell_V(s)|\} \leq H_\ell(\tilde q,V_0).$$

Hence, for all $s > 0$ and $\tilde q \in [q,2]$, we have:

$$\|V(s)\|^2_{\mathcal{L}^4} + |\ell_V(s)|\|V(s)\|_{\mathcal{L}^2} \leq H(4,\tilde q,V_0)^2 s^{1/2-2/\tilde q} + H(2,\tilde q,V_0)H_\ell(\tilde q,V_0)s^{1/2-2/\tilde q}. \tag{4.21}$$

The case $q \in (4/3, 2]$**: proof of** (1.45)**.** In that case, combining (4.20) and (4.21), taking $\tilde{q} = q$, we immediately obtain:

$$\sup_{t>0} t^{-1/p-1/2+2/q} \|V(t) - S(t)V_0\|_{\mathcal{L}^p} \leq C(p, q, V_0) \int_0^1 (1-\tau)^{-1+1/p} \tau^{1/2-2/q} \, \mathrm{d}\tau.$$

where we used the fact that $1/2 - 2/q > -1$ for $q > 4/3$.

The case $q \in (1, 4/3)$**: proof of** (1.43)**.** Here, we write

$$\begin{aligned}
&\int_0^t (t-s)^{-1+1/p} \left(\|V(s)\|_{\mathcal{L}^4}^2 + |\ell_V(s)| \|V(s)\|_{L^2(\mathcal{F}_0)} \right) \mathrm{d}s \\
&= \underbrace{\int_0^{t/2} (t-s)^{-1+1/p} \left(\|V(s)\|_{\mathcal{L}^4}^2 + |\ell_V(s)| \|V(s)\|_{\mathcal{L}^2} \right) \mathrm{d}s}_{I_1(t)} + \underbrace{\int_{t/2}^t (t-s)^{-1+1/p} \left(\|V(s)\|_{\mathcal{L}^4}^2 + |\ell_V(s)| \|V(s)\|_{\mathcal{L}^2} \right) \mathrm{d}s}_{I_2(t)}.
\end{aligned}$$

Using (4.21) with $\tilde{q} = 2$ for $s \in (0, 1)$ and $\tilde{q} = q$ for $s \in (1, t/2)$ (recall $t > 2$), and using $t - s \geq t/2$ for $s \leq t/2$,

$$I_1(t) \leq C(p, q, V_0) t^{-1+1/p} \left(\int_0^1 s^{-1/2} \, \mathrm{d}s + \int_1^{t/2} s^{1/2-2/q} \, \mathrm{d}s \right) \leq C(p, q, V_0) t^{-1+1/p},$$

where we used that $1/2 - 2/q < -1$ so that $\int_1^\infty s^{1/2-2/q} \, \mathrm{d}s < \infty$.

Using (4.21) with $\tilde{q} = 4/3$ for $s \in (t/2, t)$, we obtain

$$I_2(t) \leq C(p, q, V_0) \int_{t/2}^t (t-s)^{-1+1/p} s^{-1} \, \mathrm{d}s = C(p, q, V_0) t^{-1+1/p} \int_{1/2}^1 (1-\tau)^{-1+1/p} \tau^{-1} \, \mathrm{d}\tau.$$

The case $q = 4/3$**: proof of** (1.44)**.** This case follows similarly as the previous one, except that estimating I_1 yields

$$I_1(t) \leq C(p, q, V_0) t^{-1+1/p} \left(\int_0^1 s^{-1/2} \, \mathrm{d}s + \int_1^{t/2} s^{-1} \, \mathrm{d}s \right) \leq C(p, q, V_0) t^{-1+1/p} (1 + \log(t)).$$

We now concentrate on the estimate (1.46)–(1.48) on $\ell_V(t) - \ell_{S(t)V_0}$. In order to do that, again, we split the integral in two parts:

$$\begin{aligned}
|\ell_V(t) - \ell_{S(t)v_0}| \quad \leq \quad & K_\ell(2) \int_0^{t/2} (t-s)^{-1} \Big(\|V(s)\|_{\mathcal{L}^4}^2 + |\ell_V(s)| \|V(s)\|_{\mathcal{L}^2} \Big) \mathrm{d}s \\
& + K_\ell(p) \int_{t/2}^t (t-s)^{-(\frac{1}{2}+\frac{1}{p})} \Big(\|V(s)\|_{\mathcal{L}^{2p}}^2 + |\ell_V(s)| \|V(s)\|_{\mathcal{L}^p} \Big) \mathrm{d}s =: J_1(t) + J_2(t).
\end{aligned}$$

where K_ℓ is the constant of Corollary 3.11 and $p \in (2, \infty)$. The estimate of J_1 can be done as previously by using (4.21):

$$J_1(t) \leq \begin{cases} C(p, q, V_0) t^{1/2-2/q} & \text{if } q \in (4/3, 2], \\ C(p, q, V_0) t^{-1} (1 + \log(t)) & \text{if } q = 4/3, \\ C(p, q, V_0) t^{-1} & \text{if } q \in (1, 4/3) \end{cases}$$

For J_2, remark that similarly as in (4.21) we can obtain for all $\tilde{q} \in [q, 2]$, $s > 0$,

$$\|V(s)\|_{\mathcal{L}^{2p}}^2 + |\ell_V(s)| \|V(s)\|_{\mathcal{L}^p} \leq \Big(H(2p, \tilde{q}, V_0) \Big)^2 s^{1/p-2/\tilde{q}} + H(p, \tilde{q}, V_0) H_\ell(\tilde{q}, V_0) s^{1/p-2/\tilde{q}} \Big),$$

so that:

$$J_2(t) \leq C(p, \tilde{q}, V_0) t^{1/2-2/\tilde{q}}.$$

This ends the proof by choosing $\tilde{q} = q$ for $q \in (4/3, 2]$ and $\tilde{q} = 4/3$ if $q \in (1, 4/3]$. $\square$

5. Further comments

We list below several comments.

Concerning optimality of Theorem 1.1. When considering the decay estimates of Theorem 1.1, it is natural to ask if the results are sharp, in particular regarding the decay of the gradient estimates when $p > 2$ and $t > 1$, since all other decay estimates correspond to the classical ones for the heat semigroup on $\mathbb{R}^2$. However, in our proof, the decay estimate (1.35) differs from the one corresponding to the heat semigroup on $\mathbb{R}^2$ for all the modes. Each time, this slower decay rate for $t > 1$ and $p > 2$ arises due the presence of the boundary. Let us point out that P. Maremonti and V. A. Solonnikov prove in [17] that, when considering the Stokes equations in an exterior domain of $\mathbb{R}^3$ with homogeneous Dirichlet boundary conditions, estimate (3.3), which is the counterpart of (1.35), is sharp. It is then likely that estimates of Theorem 1.1 are sharp as well.

Straightforward extensions of theorems 1.4 and 1.1.

• Using the density of $\mathcal{L}^1 \cap \mathcal{L}^2$ in $\mathcal{L}^q \cap \mathcal{L}^2$ and the decay estimates of Theorem 1.1, one easily get, for all $q \in (1, 2]$, for all $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ satisfying $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(5/4)$, the unique associated solution $V(t)$ to (1.11)–(1.18) satisfies:

$$\lim_{t\to\infty} t^{1/q-1/p}\|V(t)\|_{\mathcal{L}^p} = 0. \tag{5.1}$$

Indeed, for $V_0 \in \mathcal{L}^q \cap \mathcal{L}^2$ satisfying $\|V_0\|_{\mathcal{L}^2} \leq \lambda_0(5/4)$, by Theorem 1.4, $t^{1/q-1/p}\|V(t) - S(t)V_0\|_{\mathcal{L}^p}$ goes to zero as $t \to \infty$ (recalling that $\lambda_0(q) > \lambda_0(5/4)$ if $q > 5/4$, see the end of Introduction). We then take $\varepsilon > 0$ and $\tilde{V}_0 \in \mathcal{L}^1 \cap \mathcal{L}^2$ satisfying $\|V_0 - \tilde{V}_0\|_{\mathcal{L}^q} \leq \varepsilon$. According to Theorem 1.1, $t^{1/q-1/p}\|S(t)V_0 - S(t)\tilde{V}_0\|_{\mathcal{L}^p} \leq K_1(p, q)\nu^{1/p-1/q}\epsilon$. But Theorem 1.1 also implies $\lim_{t\to\infty} t^{1/q-1/p}\|S(t)\tilde{V}_0\| = 0$ since $\tilde{V}_0$ belongs to $\mathcal{L}^{\tilde{q}}$ for some $\tilde{q} \in (1, q)$. Hence

$$\limsup_{t\to\infty} t^{1/q-1/p}\|V(t)\| \leq C\varepsilon.$$

Since ε was arbitrary, this implies (5.1).

• The proofs of Theorems 1.3–1.4 are only based on the $L^p - L^q$ estimates for the Stokes problem given in Theorem 1.1. As such estimates are already known in the case of a fixed exterior domain (see [5, 6, 17]), we claim that Theorem 1.4 holds true also in this case. Hence, the computations herein extend the results in [10, 12] to the case of finite energy initial data.

• In order to obtain the decay estimates of Theorem 1.1, our approach is strongly based on the fact that the rigid body is an homogenous disk. Indeed, in polar coordinates we decompose in Fourier series. A case which can be easily treated by our analysis is when *the disk is non-homogenous, and the center of mass corresponds to the center of the disk* (e.g. for a density ρ with radial symmetry). In this case, the equations (1.11)-(1.18) and (1.22)-(1.29) are the same, where:

$$m = \int_{B_0} \rho(x)\,\mathrm{d}x; \quad \mathcal{J} = \int_{B_0} \rho(x)|x|^2\,\mathrm{d}x.$$

To our knowledge, the case of a more general shape or more general density is completely open. A similar problem, also open, would be to derive decay estimates in the case of two rigid disks.

Open problems.

• Despite Theorem 1.4, a complete description of the first term in the asymptotic behavior as $t \to \infty$ of the solutions V of (1.11)–(1.18) is still missing. Indeed, Theorems 1.2 and 1.4 cannot be combined since Theorem 1.2 requires the initial data to be $\mathcal{L}^1$ and in that case, Theorem 1.4 only yields that the $\mathcal{L}^p$-norm of the difference between the solution of the complete non-linear system (1.11)–(1.18) and the linear one, given by $S(t)V_0$, decays as $Ct^{1/p-1}$, which is precisely the order of magnitude of the $\mathcal{L}^p$-norm of the solution of the linear Stokes equation when $V_0 \in \mathcal{L}^1$. At this level, let us emphasize that one of the main conceptual difficulties of this problem is that the invariant seems to be the $\mathcal{L}^1$-norm of the solution of (1.11)-(1.18), despite the fact that the linear semigroup does not seem to be well-posed in $\mathcal{L}^1$. To justify this statement, we emphasize that the asymptotic given by Theorem 1.2

does not belong to $\mathcal{L}^1$. Showing that the non-linear term decreases faster than $t^{-1+1/p}$ is the major issue which prevents us from extracting the asymptotic first order.

If we look carefully at the proof of Theorem 1.4, we note that the difficulty comes from the fact that we do not manage to prove that $\|S(t)\mathbb{P}\mathrm{div}\ F\|_{\mathcal{L}^p}$ decays faster for $F \in \mathcal{L}^q$ with $q < 2$ (see (3.34)). In particular, in the case of the Navier-Stokes equations in $\mathbb{R}^2$, using heat kernel estimates, which are better than estimates (3.34) when $q < 2$ and $t > 1$, A. Carpio in [3] shows that the non-linear term is smaller than the Stokes solution for large time. But as we have noted above, the restriction in (3.34) seems to be unavoidable.

Nevertheless, other methods relying on the use of suitable scaling invariance and similarity variables have been used for providing leading order terms in [3, 19, 11]. To keep the unity of this paper we postpone these approaches to a future work.

• Another open problem is to remove the smallness condition in Theorem 1.4, as it is done for Navier-Stokes in the full plane [21, 27] and in fixed exterior domains [1]. Indeed, such result would be expected in view of the energy dissipation law (1.19) which indicates the decay of the $\mathcal{L}^2$-norm of the solutions of (1.1)–(1.8).

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Appendix A. Proof of Proposition 2.2

Assume $V_0 \in \mathcal{L}^2 \cap \mathcal{C}_c^\infty(\mathbb{R}^2)$ Hille-Yosida's theorem implies there exists a unique solution $V \in \mathcal{C}([0,\infty);\mathcal{L}^2)$ to (1.22)–(1.29). Furthermore, the unknowns (ℓ_V, ω_V) and the pressure p that are constructed starting from V have, with V, the following regularity (see [22, Corollary 4.3]):

$$v \in \mathcal{C}([0,\infty);[H^1(\mathcal{F}_0)]^2) \cap L^2((0,\infty);[H^2(\mathcal{F}_0)]^2)\,, \qquad \nabla p \in L^2((0,\infty);L^2(\mathcal{F}_0))\,,$$
$$\ell_V \in H^1(0,\infty)\,, \qquad \omega_V \in H^1(0,\infty)\,.$$

We note also that further smoothing properties of the semigroup (see [2, Theorem 7.7]) imply

$$(v,p) \in [\mathcal{C}^\infty((0,\infty)\times\overline{\mathcal{F}}_0)]^3, \quad \nabla p \in \mathcal{C}((0,\infty);L^2(\mathcal{F}_0))\,.$$

Consequently, we introduce the decomposition (W,Φ,Ψ,V_R) of V in spherical harmonics and a corresponding decomposition of the pressure p:

$$p_1(t,r) = \frac{1}{\pi}\int_0^{2\pi} p(t,r,\theta)\cos\theta\,\mathrm{d}\theta\,, \tag{A.1}$$

$$q_1(t,r) = \frac{1}{\pi}\int_0^{2\pi} p(t,r,\theta)\sin\theta\,\mathrm{d}\theta\,, \tag{A.2}$$

$$p_R(t,r) = p(t,r,\theta) - p_1(t,r)\cos\theta - q_1(t,r)\sin\theta\,. \tag{A.3}$$

Note that, like v_R, the remainder term p_R satisfies:

$$\int_0^{2\pi} p_R(t,r,\theta)\cos(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} p_R(t,r,\theta)\sin(\theta)\,\mathrm{d}\theta = 0. \tag{A.4}$$

Applying the continuity of the spherical-harmonic decomposition together with the continuity of V yields:

$$W \in \mathcal{C}([0,\infty);L^2((0,\infty),r\mathrm{d}r)) \qquad (\partial_r\Psi,\Psi/r) \in \mathcal{C}([0,\infty);L^2((0,\infty),r\mathrm{d}r))$$
$$V_R \in \mathcal{C}([0,\infty);L^2_\sigma(\mathcal{F}_0)) \qquad (\partial_r\Phi,\Phi/r) \in \mathcal{C}([0,\infty);L^2((0,\infty),r\mathrm{d}r)),$$

together with

$$\partial_r p_1 \in \mathcal{C}((0,\infty); L^2((1,\infty), r\mathrm{d}r))\,, \quad \partial_r q_1 \in \mathcal{C}((0,\infty); L^2((1,\infty), r\mathrm{d}r)).$$

Referring to the formulas (2.8)-(2.10) and (A.1)-(A.2) we also obtain at once the smoothness of (W, V_R, Φ, Ψ), of (p_1, q_1) and of (V_R, p_R). It now remains to compute the systems satisfied by these unknowns.

We recall that the spherical-harmonic decomposition of V reads $V = V_r e_r + V_\theta e_\theta$

$$V_r = \frac{\Psi}{r}\sin\theta - \frac{\Phi}{r}\cos\theta + V_R\cdot e_r\,, \quad V_\theta = W\min(1,r) + \partial_r\Psi\cos\theta + \partial_r\Phi\sin\theta + V_R\cdot e_\theta \tag{A.5}$$

and the velocity-field on the disk is given as follows in radial coordinates:

$$(\ell_V + \omega_V x^\perp)_r = \ell_{V,1}\cos(\theta) + \ell_{V,2}\sin(\theta)\,, \quad (\ell_V + \omega_V x^\perp)_\theta = \omega r - \ell_{V,1}\sin(\theta) + \ell_{V,2}\cos(\theta).$$

Identifying V and the velocity-field of the disk on ∂B_0 (*i.e.* for $r = 1$), we obtain the following boundary conditions:

$$\begin{array}{rclllll} w(t,1) &=& \omega(t), & \forall\, t\geq 0, & & & \\ \psi(t,1) &=& \ell_{V,2}(t)\,, & & \partial_r\psi(t,1) &=& \ell_{V,2}(t)\,, \quad \forall\, t\geq 0, \\ \varphi_1(t,1) &=& -\ell_{V,1}(t)\,, & & \partial_r\varphi_1(t,1) &=& -\ell_{V,1}(t), \quad \forall\, t\geq 0\,. \\ v_R(t,x) &=& 0\,, & \forall\, x\in\partial B_0\,, & \forall\, t\geq 0. & & \end{array}$$

In the fluid domain, we remark that, introducing χ such that $\partial_r\chi = w$ and $\chi(0) = 0$, the spherical-harmonic decomposition reads:

$$v = \nabla^\perp\left(\chi + \psi\cos(\theta) + \varphi\sin(\theta)\right) + v_R$$

so that:

$$\partial_t v - \nu\Delta v = \nabla^\perp[\partial_t - \nu\Delta]\left(\chi + \psi\cos(\theta) + \varphi\sin(\theta)\right) + \partial_t v_R - \nu\Delta v_R\,,$$

where, in polar coordinates:

$$\Delta\psi(t,r,\theta) = \frac{1}{r}\partial_r[r\partial_r\psi](t,r,\theta) + \frac{\partial_{\theta\theta}\psi(t,r,\theta)}{r^2}\,.$$

We also recall that, the gradient operator reads:

$$\nabla q = \partial_r q e_r + \frac{\partial_\theta q}{r}e_\theta\,.$$

Finally, we remark that orthogonality conditions such as (2.2) or (A.4) transmit to time and space derivative. Hence, replacing ψ and p by their values in the two last formulas, identifying then the different frequencies: constant, $\cos\theta$, $\sin\theta$, and remainders, we get:

$$\partial_t w - \nu\left(\partial_{rr}w + \frac{\partial_r w}{r} - \frac{w}{r^2}\right) = 0\,, \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty);$$

$$\partial_t\psi - \nu\left(\partial_{rr}\psi + \frac{\partial_r\psi}{r} - \frac{\psi}{r^2}\right) = -r\partial_r q_1\,, \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty);$$

$$\partial_t\partial_r\psi - \nu\partial_r\left(\partial_{rr}\psi + \frac{\partial_r\psi}{r} - \frac{\psi}{r^2}\right) = -\frac{q_1}{r}\,, \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty);$$

$$\partial_t\varphi - \nu\left(\partial_{rr}\varphi + \frac{\partial_r\varphi}{r} - \frac{\varphi}{r^2}\right) = r\partial_r p_1\,, \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{A.6}$$

$$\partial_t\partial_r\varphi - \nu\partial_r\left(\partial_{rr}\varphi + \frac{\partial_r\varphi}{r} - \frac{\varphi}{r^2}\right) = \frac{p_1}{r}\,, \qquad \text{for } (t,r)\in(0,\infty)\times(1,\infty); \tag{A.7}$$

$$\partial_t v_R - \nu\Delta v_R + \nabla p_R = 0\,, \qquad t\geq 0,\ x\in\mathcal{F}_0\,. \tag{A.8}$$

To end up the proof of Proposition 2.2, we write now (1.26)-(1.27). First we recall that, on B_0 there holds $n = -e_r$ (the normal n is here computed outward the fluid domain) and $r = 1$, so that:

$$
\begin{aligned}
-2D(v)n &= \partial_r v + \nabla v_r - [\nabla e_r]^\top v\,, \\
&= (2\partial_r v_r)e_r + (\partial_r v_\theta + \partial_\theta v_r - v_\theta)e_\theta\,, \\
-\Sigma n &= (-p + 2\nu\partial_r v_r)e_r + \nu(\partial_r v_\theta + \partial_\theta v_r - v_\theta)e_\theta.
\end{aligned}
$$

Hence, computing for instance $\ell'_{V,2} = \ell'_V \cdot e_2$ we have:

$$
\begin{aligned}
m\ell'_{V,2} &= \int_0^{2\pi} (2\nu\partial_r v_r - p)\sin\theta\,\mathrm{d}\theta + \nu\int_0^{2\pi} (\partial_r v_\theta + \partial_\theta v_r - v_\theta)\cos\theta\,\mathrm{d}\theta\,, \\
&= \int_0^{2\pi} (2\nu\partial_r v_r + \nu v_r - p)\sin\theta\,\mathrm{d}\theta + \nu\int_0^{2\pi} (\partial_r v_\theta + \partial_\theta v_r)\cos\theta\,\mathrm{d}\theta\,.
\end{aligned}
$$

In these last integrals, we then compute $\partial_r v_r$ and $\partial_r v_\theta$ with respect to w, ψ and φ, and v_R thanks to (A.5). Recalling the orthogonality conditions (2.2) and (A.4), we get:

$$
\begin{aligned}
m\ell'_{V,2} &= \pi\left[\nu\left(2\partial_r\left[\frac{\psi}{r}\right] + \partial_{rr}\psi + \psi - \partial_r\psi\right) - q_1\right]\,, \\
&= \pi\left[\nu\left(\partial_{rr}\psi + \frac{\partial_r\psi}{r} - \frac{\psi}{r^2}\right) - q_1\right]\,.
\end{aligned}
$$

The computations of $\mathcal{J}\omega'_V$ and $m\ell'_{V,1}$ are similar.

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