Observability in arbitrary small time for discrete approximations of conservative systems*

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Abstract

The goal of this article is to analyze observability results in arbitrary small time for discrete approximations of conservative systems. In previous works, under the assumption that the continuous conservative system is admissible and exactly observable, observability results for the corresponding discrete approximation schemes have been proved within the class of conveniently filtered solutions using resolvent estimates. However, in several situations and in particular for Schrödinger equations when the Geometric Control Condition is satisfied, the exact observability property holds in any arbitrary small time. We prove that in several cases, namely under a stronger resolvent condition, the time-discrete approximations of conservative systems also enjoy uniform observability properties in arbitrary small time, still within the class of conveniently filtered solutions. In particular, our methodology applies to space semi-discrete and fully discrete approximation schemes of Schrödinger equations for which the Geometric Control Condition is satisfied. Our approach is based on the resolvent characterization of the exact observability property and a constructive argument by Haraux in [14].

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1 Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator with compact resolvent. Let us consider the following abstract system:

$$\dot{z}(t) = Az(t), \quad z(0) = z_0. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, and $z = z(t)$ is the *state* of the system. Note that since A is skew-adjoint, solutions of (1.1) have constant energy: $\forall t \in \mathbb{R}$, $\|z(t)\|_X = \|z_0\|_X$. In particular, (1.1) can be solved for all time $t \in \mathbb{R}$.

Such systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell equations) or in quantum mechanics (Schrödinger equation).

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A), Y)$ be an observation operator and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

In order to give a sense to (1.2), we make the assumption that B is an admissible observation operator in the following sense (see [20]):

Definition 1.1. The operator B is an admissible observation operator for system (1.1)-(1.2) if for every $T > 0$ there exists a constant $K_T > 0$ such that

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2, \qquad \forall z_0 \in \mathcal{D}(A). \tag{1.3}$$

Note that if B is *bounded* on X, i.e. if it can be extended such that $B \in \mathfrak{L}(X, Y)$, then B is obviously an admissible observation operator. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property of the solutions of the evolution equation (1.1).

The exact observability property of system (1.1)-(1.2) can be formulated as follows:

Definition 1.2. System (1.1)-(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|Bz(t)\|_Y^2 \, dt, \qquad \forall z_0 \in \mathcal{D}(A). \tag{1.4}$$

Moreover, (1.1)-(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that observability issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [20]). Indeed, controllability and observability are dual notions, and therefore each statement concerning observability has its counterpart in controllability. In the sequel, we focus on the observability properties of (1.1)-(1.2).

It was proved in [21] that system (1.1)-(1.2) is exactly observable if and only if the following assertion holds:

Condition 1. There exist positive constants $M, m > 0$ such that

$$M^2 \left\|(A - i\omega I)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \quad \forall\, \omega \in \mathbb{R},\ \forall\, z \in \mathcal{D}(A). \tag{1.5}$$

This spectral condition can be viewed as a Hautus-type test, and generalizes the classical Kalman rank condition, see for instance [26]. To be more precise, if Condition 1 holds, then system (1.1)-(1.2) is exactly observable in any time $T > T_0 = \pi M$ (see [21]).

The following stronger resolvent condition is more interesting for our purpose:

Condition 2. There exist a positive constant $m > 0$ and a function $M = M(\omega)$ of $\omega \in \mathbb{R}$, bounded on $\mathbb{R}$ and satisfying

$$\lim_{|\omega| \to \infty} M(\omega) = 0, \tag{1.6}$$

such that for all $\omega \in \mathbb{R}$,

$$M(\omega)^2 \left\|(A - i\omega I)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \qquad \forall\, z \in \mathcal{D}(A). \tag{1.7}$$

This condition appears naturally when considering Schrödinger equations for which the Geometric Control Condition is satisfied (see [21] and Section 4 below).

Theorem 1.3 ([5, 21])**.** *When Condition* 2 *is fulfilled, system* (1.1)-(1.2) *is observable in any time* $T^* > 0$.

The proof of Theorem 1.3 in [21] is based on a decoupling argument of high- and low-frequency components. Given $T^* > 0$, take $M > 0$ such that $\pi M < T^*$, and choose a frequency cut $\Omega = \Omega_0 + 1/M$ where Ω_0 satisfies $\sup_{|\omega| \geq \Omega_0} \{M(\omega)\} \leq M$. Then the frequencies higher than Ω are exactly observable in any time $\tilde{T} \in (\pi M, T^*)$. The low-frequency components then correspond to a finite dimensional observability problem and can be handled in any time $T > 0$. Finally these two partial observability properties are combined together using a compactness argument (or simultaneous exact controllability results as in [26]).

But these methods are not constructive and are then not sufficient to obtain observability results for family of operators satisfying (1.7) uniformly. It is then of particular interest to design a constructive proof of Theorem 1.3 when dealing for instance with discrete approximation schemes of (1.1)-(1.2).

In the sequel, we will then propose a constructive proof of Theorem 1.3, based on an explicit method proposed by Haraux in [14]. This allows us to deal with families of operators satisfying Condition 2 uniformly. We then explain how our method applies to time semi-discretizations of (1.1)-(1.2).

In particular, our method implies that when the Geometric Control Condition is satisfied, time continuous and time semi-discrete Schrödinger equations are exactly observable in arbitrary small time, as we will see in Section 4. In this case, based on the abstract approach developed in [8, 9], we can also deal with space semi-discrete and fully discrete approximation schemes. In particular, we will prove uniform (with respect ti the discretization parameters) observability results in arbitrary small time for discrete approximations of Schrödinger equations satisfying the Geometric Control Condition, within the class of conveniently filtered solutions.

Let us now briefly comment the literature. This article follows the works [10, 8, 9] on observability properties for discrete approximation schemes of abstract conservative systems which, in the continuous setting, are exactly observable. The main underlying idea there is to use spectral criteria such as (1.5) which yield explicit dependence on the parameters for the constants entering in the exact observability property (1.4). Indeed, one can then use the following diagram to prove uniform observability results for discrete approximations of (1.1)-(1.2):

$$\begin{array}{ccc}
\text{Exact observability property} & & \text{Uniform observability} \\
\text{for the continuous system} & & \text{for discretizations of } (1.1)\text{-}(1.2) \\
\Downarrow & & \Uparrow \\
\text{Spectral criterion} & \Longrightarrow & \text{Spectral criterion} \\
\text{for the continuous system} & & \text{for the discrete systems}
\end{array}$$

The spectral criteria used in [10, 8, 9] for the exact observability property are due in particular to [5, 21, 24, 26]. As already noticed in [26], if the operators A and B satisfy estimate (1.7) for a function $M(\omega)$ satisfying $\lim_{|\omega|\to\infty} M(\omega) = M$ (M may be different from 0), then system (1.1)-(1.2) is exactly observable in any time $T > \pi M$.

Let us mention that one has to look for *uniform* observability properties for the discrete approximation schemes of (1.1)-(1.2) to guarantee the convergence of the controls computed in the discrete setting to one of the continuous system (1.1)-(1.2). However, as already noticed in

[11, 12, 13] in the nineties, observability properties for discrete versions of (1.1)-(1.2) do not hold uniformly with respect to the discretization parameters due to spurious high-frequencies. We thus need to restrict ourselves to prove uniform observability properties within the class of conveniently filtered solutions.

There are of course several other techniques to study observability properties for discrete versions of (1.1)-(1.2), such as Ingham's Lemma [16], which use is essentially limited to the 1d cases (see [15, 6, 7]), and discrete multiplier methods as in [22, 23]. For extensive references and the state of the art for the observability properties of discrete approximations of the wave equation, we refer to [27].

The paper is organized as follows.

In Section 2, we give a constructive proof of Theorem 1.3. In Section 3, we explain how this can yield uniform observability results in arbitrary small time for time semi-discrete versions of (1.1)-(1.2). in Section 4, we present an application of these techniques to discrete Schrödinger equations -including the fully discrete case- when the Geometric Control Condition is satisfied. We finally end up with some further comments.

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2 A constructive proof of Theorem 1.3

Before going into the proof, we introduce some notations.

For a function $f \in L^2(\mathbb{R}; X)$ depending on time $t \in \mathbb{R}$, we define its time Fourier transform $\hat{f} \in L^2(\mathbb{R}, X)$ by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t}\, dt. \tag{2.1}$$

The Parseval identity then reads:

$$\int_{\mathbb{R}} \|f(t)\|_X^2 \, dt = \int_{\mathbb{R}} \left\| \hat{f}(\omega) \right\|_X^2 \, d\omega, \quad \forall f \in L^2(\mathbb{R}; X). \tag{2.2}$$

It is convenient to introduce the spectrum of the operator A. Since A is skew-adjoint with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : \ j \in \mathbb{Z}\}$, where $(\mu_j)_{j\in\mathbb{Z}}$ is an increasing sequence of real numbers. Set $(\Phi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{Z}}$:

$$A\Phi_j = i\mu_j \Phi_j. \tag{2.3}$$

Moreover, we define the filtered class

$$\mathcal{C}(s) = \operatorname{span}\{\Phi_j : \text{the corresponding } i\mu_j \ \text{ satisfies } |\mu_j| < s\}, \tag{2.4}$$

and its orthogonal $\mathcal{C}(s)^\perp$ in X, which coincides with

$$\mathrm{span}\{\Phi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \geq s\}.$$

We can now focus on the proof of Theorem 1.3, which is decomposed in two main steps, which will be explained in the next subsections:

1. We prove an observability inequality in arbitrary small time for the high-frequency solutions of (1.1).
2. We use the constructive argument in [14] to obtain an observability inequality for any solutions of (1.1).

2.1 High frequency components

We first prove a high-frequency resolvent estimate:

Lemma 2.1. *For all $M > 0$ there exists a constant Ω such that*

$$M^2 \left\|(A - i\omega I)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \quad \forall\, \omega \in \mathbb{R},\ \forall\, z \in \mathcal{D}(A) \cap \mathcal{C}(\Omega)^\perp. \tag{2.5}$$

Proof of Lemma 2.1. Fix $M > 0$. Then there exists Ω_0 such that

$$\forall \omega \geq \Omega_0, \quad |M(\omega)| \leq M.$$

This implies the following version of (2.5):

$$M^2 \left\|(A - i\omega I)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \quad \forall\, \omega \text{ such that } |\omega| \geq \Omega_0,\ \forall\, z \in \mathcal{D}(A).$$

In particular, this implies (2.5) for all ω such that $|\omega| \geq \Omega_0$. We thus only need to prove (2.5) for ω such that $|\omega| \leq \Omega_0$. This can be done using the following remark: If $\Omega \geq \Omega_0$,

$$\forall \omega \text{ such that } |\omega| \leq \Omega_0,\ \forall z \in \mathcal{D}(A) \cap \mathcal{C}(\Omega)^\perp, \quad \left\|(A - i\omega)z\right\|_X^2 \geq (\Omega - \Omega_0)^2 \left\|z\right\|_X^2.$$

Then, with the choice $\Omega = \Omega_0 + 1/M$, (2.5) holds. $\Diamond$

We now prove the following lemma:

Lemma 2.2. *If (2.5) holds for given M and Ω, the observability inequality (1.4) holds in any time $T > \pi M$ for solutions of (1.1) with initial data lying in $\mathcal{C}(\Omega)^\perp$. Besides the corresponding constant $k_T > 0$ of observability in (1.4) can be chosen as*

$$k_T = \frac{1}{2m^2T^2}(T^2 - \pi^2 M^2).$$

This lemma can actually be found in [5, 21]. We provide the proof for completeness.

Proof of Lemma 2.2. Given $z_0 \in \mathcal{C}(\Omega)^\perp \cap \mathcal{D}(A)$, let $z(t)$ be the corresponding solution of (1.1) and define, for $\chi \in C_0^\infty(\mathbb{R})$,

$$g(t) = \chi(t) z(t), \quad f(t) = g'(t) - Ag(t) = \chi'(t) z(t).$$

Then $\hat{f}(\omega) = (i\omega - A)\hat{g}(\omega)$. Besides, $\hat{g}$ belongs to $L^2(\mathbb{R}; \mathcal{D}(A) \cap \mathcal{C}(\Omega)^\perp)$. We can thus apply the resolvent estimate (2.5) to $\hat{g}(\omega)$:

$$\forall \omega \in \mathbb{R}, \quad \|\hat{g}(\omega)\|_X^2 \leq m^2 \left\| \widehat{Bg}(\omega) \right\|_Y^2 + M^2 \left\| \hat{f}(\omega) \right\|_X^2 .$$

Integrating in ω and using the Parseval identity, we obtain

$$\Big(\int_{\mathbb{R}} \chi(t)^2 \, dt - M^2 \int_{\mathbb{R}} \chi'(t)^2 \, dt \Big) \|z_0\|_X^2 \leq m^2 \int_{\mathbb{R}} \chi(t)^2 \, \|Bz(t)\|_Y^2 \, dt,$$

where we used that the energy of solutions of (1.1), given by $\|z(t)\|_X^2$, is constant.

We then look for a function χ which makes the left hand side positive. This can be achieved by taking $\chi(t) = \sin(\pi t/T)$ in $(0, T)$ and vanishing anywhere else. This is not in $C_0^\infty(\mathbb{R})$ but in $H^1(\mathbb{R})$ with compact support, which is sufficient for the proof developed above.

This gives the desired estimate for any initial data $z_0 \in \mathcal{D}(A) \cap \mathcal{C}(\Omega)^\perp$ and we conclude by density. $\Diamond$

2.2 Haraux's constructive argument

To present the construction precisely, remark that since A has compact resolvent, there is only a finite number of eigenvalues for which $|\mu_j| < \Omega$. For convenience, we introduce the finite sequence $(m_j)_{1 \leq j \leq N}$ of strictly increasing real numbers such that $\{m_j\} = \{\mu_j \text{ such that } |\mu_j| < \Omega\}$. For $j \in \{1, \cdots, N\}$, we then denote by X_j the finite-dimensional vector space spanned by the eigenvectors corresponding to eigenvalues $i\mu_j$ with $\mu_j = m_j$. Note that these notations are not needed when the eigenvalues are simple.

Lemma 2.3 ([14])**.** *Let B be an admissible operator for* (1.1)-(1.2)*. Assume that there exist positive constants $\tilde{k} > 0$ and $\tilde{T}$ such that any solution of* (1.1) *with initial data $z_0 \in \mathcal{C}(\Omega)^\perp$ satisfies*

$$\tilde{k} \, \|z_0\|_X^2 \leq \int_0^{\tilde{T}} \|Bz(t)\|_Y^2 \, dt. \tag{2.6}$$

Also assume that there exists a strictly positive number β such that

$$\forall j \in \{1, \cdots, N\},\ \forall z \in X_j, \quad \|Bz\|_Y \geq \beta \|z\|_X . \tag{2.7}$$

Then the observability inequality (1.4) *holds in any time $T > \tilde{T}$, with a strictly positive observability constant $k_T > 0$ depending explicitly on the parameters β, $T - \tilde{T}$, $\tilde{k}$, the number N of low frequencies and the low frequency gap*

$$\gamma = \inf_{j \in \{0, \cdots, N\}} \{m_{j+1} - m_j\}, \text{ where } m_0 = -\Omega \text{ and } m_{N+1} = +\Omega. \tag{2.8}$$

Note that γ in (2.8) is strictly positive as an infimum of a finite number of strictly positive quantities.

We give the proof of this lemma below, since it will later be generalized to more complex situations. Note that this proof can also be found in [18].

Proof of Lemma 2.3. The argument is inductive. We then just need to describe the first step, the others being similar. We then focus on the observability inequality (1.4) for initial data in $X_N + \mathcal{C}(\Omega)^\perp$.

Set $z_0 \in X_N + \mathcal{C}(\Omega)^\perp$, and expand it as $z_{0,N} + z_{0,hf}$ with $z_{0,N} \in X_N$ and $z_{0,hf} \in \mathcal{C}(\Omega)^\perp$.

Let $z(t)$ be the solution of (1.1) corresponding to the initial data z_0, and define, for $\delta > 0$,

$$v(t) = z(t) - \frac{1}{2\delta} \int_{-\delta}^{\delta} e^{i m_N s} z(t-s)\, ds. \tag{2.9}$$

Writing $z_0 = \sum a_j \Phi_j$, the solution $z(t)$ of (1.1) can be explicitly written as $\sum a_j \Phi_j \exp(i\mu_j t)$. In particular,

$$\begin{aligned} v(t) &= \sum_j a_j \Phi_j \exp(i\mu_j t) \Big(1 - \mathrm{sinc}(\delta(m_N - \mu_j))\Big) \\ &= \sum_{j \text{ with } |\mu_j| \geq \Omega} a_j \Phi_j \exp(i\mu_j t) \Big(1 - \mathrm{sinc}(\delta(m_N - \mu_j))\Big) \end{aligned} \tag{2.10}$$

Note in particular that (2.10) implies that the norms of $v_0 = v(0)$ and z_0 satisfy

$$\|z_{0,hf}\|_X^2 \leq \frac{1}{(1 - \mathrm{sinc}(\delta\gamma))^2} \|v_0\|_X^2 . \tag{2.11}$$

Besides, (2.10) also implies that v is a solution of (1.1) with initial data in $\mathcal{C}(\Omega)^\perp$. Hence, it shall also satisfy the observability inequality (2.6):

$$\|v_0\|_X^2 \leq \frac{1}{\tilde{k}} \int_0^{\tilde{T}} \|Bv(t)\|_Y^2 \, dt. \tag{2.12}$$

We then have to estimate the right hand side of (2.12). From (2.9), we get:

$$\begin{aligned}\int_0^{\tilde{T}} \|Bv(t)\|_Y^2 \ dt \leq 2\int_0^{\tilde{T}} \|Bz(t)\|_Y^2 \ dt \\ +2\int_0^{\tilde{T}} \left\| B\Big(\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{im_N s} z(t-s)\ ds\Big)\right\|_Y^2 \ dt \\ \leq 2\int_0^{\tilde{T}} \|Bz(t)\|_Y^2 \ dt + 2\int_{-\delta}^{\tilde{T}+\delta} \|Bz(t)\|_Y^2 \ dt \\ \leq 4\int_{-\delta}^{\tilde{T}+\delta} \|Bz(t)\|_Y^2 \ dt. \end{aligned} \tag{2.13}$$

Combined with (2.11) and (2.12), this yields

$$\|z_{0,hf}\|_X^2 \leq \frac{4}{\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\int_{-\delta}^{\tilde{T}+\delta} \|Bz(t)\|_Y^2 \ dt. \tag{2.14}$$

We then focus on the component of the solution in X_N. Obviously, denoting by z_N, z_{hf} the solutions of (1.1) with initial data $z_{0,N}, z_{0,hf}$ respectively, applying (2.7) we obtain

$$\begin{aligned}\|z_{0,N}\|_X^2 &\leq \frac{1}{\tilde{T}\beta^2}\int_0^{\tilde{T}} \|Bz_N(t)\|_Y^2 \ dt \\ &\leq \frac{1}{\tilde{T}\beta^2}\Big(2\int_0^{\tilde{T}} \|Bz(t)\|_Y^2 \ dt + 2\int_0^{\tilde{T}} \|Bz_{hf}(t)\|_Y^2 \ dt\Big).\end{aligned}$$

Using the admissibility of B for system (1.1)-(1.2), we obtain

$$\|z_{0,N}\|_X^2 \leq \frac{2}{\tilde{T}\beta^2}\int_0^{\tilde{T}} \|Bz(t)\|_Y^2 \ dt + \frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}\|z_{0,hf}\|_X^2. \tag{2.15}$$

Using (2.14) and the orthogonality of X_N and $\mathcal{C}(\Omega)^\perp$, we conclude

$$\|z_0\|_X^2 \leq \Big[\frac{2}{\tilde{T}\beta^2} + \Big(\frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}+1\Big)\frac{4}{\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\Big]\int_{-\delta}^{\tilde{T}+\delta} \|Bz(t)\|_Y^2 \ dt, \tag{2.16}$$

or, using the conservation of the energy for solutions of (1.1) and the semi-group property,

$$\|z_0\|_X^2 \leq \Big[\frac{2}{\tilde{T}\beta^2} + \Big(\frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}+1\Big)\frac{4}{\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\Big]\int_0^{\tilde{T}+2\delta} \|Bz(t)\|_Y^2 \ dt. \tag{2.17}$$

Since $\delta > 0$ is arbitrary small, we have proved the observability inequality (2.17) in any time $T_N > \tilde{T}$ for any solution of (1.1) with initial data in $X_N + \mathcal{C}(\Omega)^{\perp}$.

The induction argument is then left to the reader. $\Diamond$

2.3 End of the proof of Theorem 1.3

Set $T^* > 0$. Choose $M > 0$ such that $\pi M = T^*/4$. From Lemma 2.1, one can choose Ω such that (2.5) holds for $z \in \mathcal{D}(A) \cap \mathcal{C}(\Omega)^{\perp}$. From Lemma 2.2, this implies that any solution of (1.1) with initial data in $\mathcal{C}(\Omega)^{\perp}$ satisfies (2.6) in time $\tilde{T} = T^*/2$.

Since A has compact resolvent, there is only a finite number of eigenvalues μ_j such that $|\mu_j| < \Omega$ and then the low frequency gap γ defined in (2.8) is strictly positive.

We only have to check that estimate (2.7) indeed holds. This is actually obvious, since for $j \in \{1, \cdots, N\}$ and $z \in X_j$, taking $\omega = m_j$ in (1.7), we obtain:

$$m^2 \|Bz\|_Y^2 \geq \|z\|_X^2 .$$

The proof is then complete by applying Lemma 2.3. $\Box$

3 Applications to time-discrete approximations of (1.1)-(1.2)

This section aims at describing how the previous result can be adapted to time-discrete approximations of systems (1.1)-(1.2) satisfying Condition 2. In particular, we shall prove that in that case, time semi-discrete approximations of (1.1)-(1.2) indeed are exactly observable in arbitrary small time within the class of conveniently filtered solutions, uniformly with respect to the time discretization parameter.

3.1 Time discrete approximations of (1.1)-(1.2)

To simplify the presentation, we will focus on the following natural approximation of (1.1)-(1.2), the so-called midpoint scheme. For $\Delta t > 0$, consider

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\Delta t} = A\Big(\dfrac{z^k + z^{k+1}}{2}\Big), & \text{in } X, \quad k \in \mathbb{Z}, \\ z^0 = z_0 \text{ given.} \end{cases} \tag{3.1}$$

Here, z^k denotes the approximation of the solution z of (1.1) at time $t_k = k\Delta t$. Note that the discrete system (3.1) is conservative, in the sense that $k \mapsto \left\|z^k\right\|_X^2$ is constant.

The output function is now given by the discrete sample

$$y^k = Bz^k. \tag{3.2}$$

The admissibility and observability properties for (3.1)-(3.2) have been studied in [10] using spectral criteria such as Condition 1 for the observability properties. In particular, [10] states the following result:

Theorem 3.1 ([10]). *Assume that the continuous system* (1.1)-(1.2) *is admissible and exactly observable in some time $T > 0$. Then for all $\delta > 0$:*

- *For all $T > 0$, there exists a constant $K_{\delta,T}$ such that, for all $\Delta t > 0$, any solution z^k of* (3.1) *with initial data $z_0 \in \mathcal{C}(\delta/\Delta t)$ satisfies*

$$\Delta t \sum_{k\Delta t \in (0,T)} \left\| Bz^k \right\|_Y^2 \leq K_{\delta,T} \left\| z_0 \right\|_X^2. \tag{3.3}$$

- *There exist a time T_δ and a positive constant $k_\delta > 0$ such that, for all $\Delta t > 0$ small enough, any solution z^k of* (3.1) *with initial data $z_0 \in \mathcal{C}(\delta/\Delta t)$ satisfies*

$$k_\delta \left\| z_0 \right\|_X^2 \leq \Delta t \sum_{k\Delta t \in (0,T_\delta)} \left\| Bz^k \right\|_Y^2. \tag{3.4}$$

Note that the observability property (3.4) requires the time to be large enough. Actually, a precise estimate is given in [10] in terms of the resolvent parameters in (1.5) and the scaling parameter δ, but this is not completely satisfactory since, to our knowledge, even in the continuous setting, we are not able in general to recover the optimal time of controllability from (1.5).

But, as explained in the introduction, Condition 2 is sufficient to prove observability of the continuous system (1.1)-(1.2) in any positive time. We thus ask whether or not it is also possible to prove discrete observability properties for (3.1)-(3.2) in arbitrary small time when Condition 2 is satisfied.

Theorem 3.2. *Assume that Condition* 2 *is satisfied.*

- *If $B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y)$ with $\kappa < 1$. Then for any $\delta > 0$, for any time $T^* > 0$, there exists a positive constant $k_{\delta,T^*} > 0$ such that, for all $\Delta t > 0$ small enough, any solution z^k of* (3.1) *with initial data $z_0 \in \mathcal{C}(\delta/\Delta t)$ satisfies*

$$k_{\delta,T^*} \left\| z_0 \right\|_X^2 \leq \Delta t \sum_{k\Delta t \in (0,T^*)} \left\| Bz^k \right\|_Y^2. \tag{3.5}$$

- *If B simply belongs to $\mathfrak{L}(\mathcal{D}(A), Y)$. Then for any time $T^* > 0$, there exist two positive constants $\delta > 0$ and $k_{\delta,T^*} > 0$ such that, for all $\Delta t > 0$*

small enough, any solution z^k *of* (3.1) *with initial data* $z_0 \in \mathcal{C}(\delta/\Delta t)$ *satisfies* (3.5).

Theorem 3.2 is the exact counterpart in the discrete setting of Theorem 1.3, and we will only indicate the modifications needed in its proof to derive Theorem 3.2.

Proof of Theorem 3.2. As we said, the proof of Theorem 3.2 closely follows the one of Theorem 1.3, and we thus only sketch it briefly.

We first deal with the high-frequency components. Lemma 2.1 still holds, since it is by nature independent of time, whether this time is continuous or not. However, Lemma 2.2 has to be modified and replaced by the following

Lemma 3.3. *Assume that* (2.5) *holds for given* m, M *and* Ω. *For any* $\delta > 0$, *set*

$$T_{M,\delta} = \begin{cases} \pi M\Big(1+\dfrac{\delta^2}{4}\Big), & \text{if } B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y) \text{ with } \kappa < 1, \\ \pi \left[M^2\Big(1+\dfrac{\delta^2}{4}\Big)^2 + m^2 \left\|B\right\|^2_{\mathfrak{L}(\mathcal{D}(A),Y)} \dfrac{\delta^4}{16}\right]^{1/2} & \\ \qquad\qquad\qquad\qquad \text{if } B \in \mathfrak{L}(\mathcal{D}(A), Y). \end{cases} \tag{3.6}$$

Then the observability inequality (3.4) *holds in any time* $T > T_M$ *for some positive constant* $k_T > 0$ *for any solution of* (3.1) *with initial data lying in* $\mathcal{C}(\Omega)^\perp \cap \mathcal{C}(\delta/\Delta t)$. *Besides,* k_T *can be chosen explicitly as a function of* T, m, M *and the norm of* B *in* $\mathfrak{L}(\mathcal{D}(A), Y)$.

Proof of Lemma 3.3. In the case $B \in \mathfrak{L}(\mathcal{D}(A), Y)$, this lemma corresponds exactly to Theorem 1.3 in [10], and might be seen as an extension of Lemma 2.2 to the time discrete case, involving in particular discrete Fourier transforms instead of continuous ones.

In the case $B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y)$ with $\kappa < 1$, the proof of Lemma 3.3 can be adapted immediately from the one of Theorem 1.3 in [10] by modifying estimate (2.19) in [10] using

$$\begin{aligned} \left\| B\Big(\frac{z^{k+1}-z^k}{\Delta t}\Big)\right\|_Y &\leq \|B\|_{\mathcal{L}(\mathcal{D}(A^\kappa),Y)} \left\| A^{1+\kappa}\Big(\frac{z^k+z^{k+1}}{2}\Big)\right\|_X \\ &\leq \Big(\frac{\delta}{\Delta t}\Big)^{1+\kappa} \|B\|_{\mathcal{L}(\mathcal{D}(A^\kappa),Y)} \left\|\frac{z^0+z^1}{2}\right\|_X, \end{aligned}$$

and the following ones accordingly. In particular, with the notations of

[10], a_2 in Lemma 2.4 shall be replaced by

$$a_2 = M^2\left(1+\frac{\delta^2}{4}\right)+\frac{\delta^2(\Delta t)^2}{16}(\beta-1) \\ + m^2\left\|B\right\|^2_{\mathfrak{L}(\mathcal{D}(A^\kappa),Y)}\frac{\delta^{2+2\kappa}(\Delta t)^{2-2\kappa}}{16}\left(1+\frac{1}{\alpha}\right).$$

Details are then left to the reader. $\Diamond$

We then deal with the low-frequency components. This is done as in the continuous case:

Lemma 3.4. *Let B be an admissible operator for* (1.1)*. Assume that there exist positive constants $\delta > 0$, $\tilde{k} > 0$ and $\tilde{T}$ such that, for all $\Delta t > 0$ small enough, any solution of* (3.1) *with initial data $z_0 \in \mathcal{C}(\Omega)^\perp \cap \mathcal{C}(\delta/\Delta t)$ satisfies*

$$\tilde{k}\left\|z_0\right\|^2_X \le \Delta t \sum_{k\Delta t\in(0,\tilde{T})}\left\|Bz^k\right\|^2_Y. \tag{3.7}$$

Also assume that there exists a strictly positive number β such that (2.7) *holds. Then, for any $\Delta t > 0$ small enough, for any solutions of* (3.1) *with initial data $z_0 \in \mathcal{C}(\delta/\Delta t)$, the observability inequality* (3.4) *holds in any time $T > \tilde{T}$, with a strictly positive observability constant $k_T > 0$ depending explicitly on the parameters β, $T-\tilde{T}$, $\tilde{k}$, the number N of low frequencies and the low frequency gap* (2.8)*.*

Proof of Lemma 3.4. The proof of Lemma 3.4 closely follows the one of Lemma 2.3.

Fix $\Delta t > 0$. First remark that solutions of (3.1) write as

$$z^k = \sum_j a_j\Phi_j\exp(\lambda_{j,\Delta t}k\Delta t), \qquad \text{with } \lambda_{j,\Delta t} = \frac{1}{2\Delta t}\tan\Big(\frac{\mu_j\Delta t}{2}\Big).$$

Then define, similarly as in (2.8),

$$m_{j,\Delta t} = \frac{1}{2\Delta t}\tan\Big(\frac{m_j\Delta t}{2}\Big) \text{ and } \gamma_{\Delta t} = \inf_{j\in\{0,\cdots,N\}}\{m_{j+1,\Delta t}-m_{j,\Delta t}\}.$$

For simplicity, choose $\delta_{\Delta t}$ such that $\delta/\Delta t$ is an integer. Introduce, similarly as in (2.9),

$$v^k = z^k - \frac{\Delta t}{2\delta}\sum_{\ell\Delta t\in(-\delta,\delta)}\exp(im_{N,\Delta t}\ell\Delta t)\, z^{k-l}.$$

The proof of Lemma 3.4 then follows line to line the one of Lemma 2.3, replacing all the continuous integrals in time by discrete summations and using the admissibility result (3.3) in the class $\mathcal{C}(\delta/\Delta t)$. This yields, similarly as in (2.17), that any solution of (3.1) with initial data $z_0 \in X_N + \mathcal{C}(\Omega)^\perp \cap \mathcal{C}(\delta/\Delta t)$ satisfies

$$\|z_0\|_X^2 \leq \Big[\frac{2}{\tilde{T}\beta^2}+\Big(\frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}+1\Big)\frac{4}{\tilde{k}(1-\operatorname{sinc}(\gamma_{\Delta t}\delta_{\Delta t}))^2}\Big]\Delta t \sum_{k\Delta t\in(0,\tilde{T}+2\delta_{\Delta t})} \left\|Bz^k\right\|_Y^2 .$$

In particular, when Δt goes to zero, one can choose $(\delta_{\Delta t})$ converging to δ. Besides, when $\Delta t \to 0$, $(\gamma_{\Delta t})$ obviously converges to γ. We thus obtain, for $\Delta t > 0$ small enough, that any solution of (3.1) with initial data $z_0 \in X_N + \mathcal{C}(\Omega)^\perp \cap \mathcal{C}(\delta/\Delta t)$ satisfies

$$\|z_0\|_X^2 \leq \Big[\frac{2}{\tilde{T}\beta^2}+\Big(\frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}+1\Big)\frac{4}{\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\Big]\Delta t \sum_{k\Delta t\in(0,\tilde{T}+2\delta_{\Delta t})} \left\|Bz^k\right\|_Y^2 .$$

The inductive argument then again works and allows to conclude Lemma 3.4. $\Diamond$

We now finish the proof of Theorem 3.2. Set $T^* > 0$.

• If $B \in \mathfrak{L}(\mathcal{D}(A^\kappa), Y)$ with $\kappa < 1$. Set $\delta > 0$. Choose M such that $\pi M(1+\delta^2/4) = T^*/4$.

• If $B \in \mathfrak{L}(\mathcal{D}(A), Y)$. Set $\delta < \delta_0$, where δ_0 is such that

$$\pi m \left\|B\right\|_{\mathfrak{L}(\mathcal{D}(A),Y)} \frac{\delta_0^2}{2} = T^*/8.$$

Choose $M > 0$ such that

$$\pi \left[M^2\Big(1+\frac{\delta^2}{4}\Big)^2 + m^2 \left\|B\right\|^2_{\mathfrak{L}(\mathcal{D}(A),Y)} \frac{\delta^4}{16}\right]^{1/2} = T^*/4.$$

Applying successively Lemmas 2.2 and 3.3, we prove uniform observability properties (3.7) for any solution of (3.1) with initial data $z_0 \in \mathcal{C}(\Omega)^\perp \cap \mathcal{C}(\delta/\Delta t)$ in time $T^*/2$. We then conclude as in the continuous case by Lemma 3.4 and estimate (2.7). $\Box$

Remark 3.5. Note that the approach developed in this section can also be applied for more general time discrete approximation schemes. We refer to [10] for the precise assumptions needed on the time discrete numerical schemes. Roughly speaking, any time discrete scheme which preserves the eigenvectors and for which the energy is constant enters in our setting. This includes, for instance, the fourth order Gauss method, or the Newmark method for the wave equation.

4 Schrödinger equations

In this section, we present an application to the above results to Schrödinger equations. Condition 2 is indeed typically satisfied by Schrödinger equations, and can be guaranteed when the corresponding wave equation is observable.

4.1 The continuous case

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, and ω a subdomain of Ω.

Let us consider the following Schrödinger equation:

$$\begin{cases} i\dot{z} + \Delta z = 0, & \text{in } \Omega \times (0, \infty), \\ z = 0, & \text{on } \partial\Omega \times (0, \infty), \\ z(0) = z_0 \in L^2(\Omega), \end{cases} \tag{4.1}$$

observed through $y(t) = \chi_\omega \, z(t)$, where $\chi_\omega = \chi_\omega(x)$ denotes the characteristic function of the set ω.

We thus consider the following observability property: for $T^* > 0$, find a strictly positive constant k_* such that any solution of (4.1) satisfies

$$k_* \, \|z_0\|^2_{L^2(\Omega)} \leq \int_0^{T^*} \|z(t)\|^2_{L^2(\omega)} \; dt. \tag{4.2}$$

Note that this fits the abstract setting presented above: $X = L^2(\Omega)$, $A = i\Delta$ with Dirichlet boundary conditions, the domain of the operator A is $\mathcal{D}(A) = H^2 \cap H^1_0(\Omega)$ and B simply is the multiplication operator by χ_ω, which is continuous from $L^2(\Omega)$ to $L^2(\omega)$ (and therefore admissible).

For Schrödinger equations, due to the infinite velocity of propagation of rays, there are many cases in which the observability inequality (4.2) holds in any time $T^* > 0$, for instance, when the Geometric Control Condition (GCC) is satisfied in some time T.

The GCC in time T can be, roughly speaking, formulated as follows (see [2] for the precise setting): The subdomain ω of Ω is said to satisfy the GCC in time T if all rays of Geometric Optics that propagate inside the domain Ω at velocity one reach the set ω in time less than T.

Note that this is not a necessary condition. For instance, in [17], it has been proved that when the domain Ω is a square, for any non-empty bounded open subset ω, the observability inequality (4.2) holds for system (4.1). Other geometries have also been dealt with: we refer to the articles [18, 19, 3, 1].

Similarly, Condition 2 is not guaranteed in general: it is indeed not clear that the observability property in arbitrary small time for Schrödinger equation (4.1) implies Condition 2.

But there are several cases in which Condition 2 is satisfied: when it has been proven directly to prove observability in arbitrary small time (see for instance [5]), but this approach has not been fully developed in the literature, or when the Geometric Control Condition is satisfied.

Theorem 4.1 ([21]). *Assume that the Geometric Control Condition holds. Then Condition* 2 *is satisfied for system* (4.1) *observed through* $y(t) = \chi_\omega \, z(t)$.

Before going into the proof, we recall that the Geometric Control Condition in time $T > 0$ is equivalent [4] to the exact observability property in time T of the corresponding wave equation

$$\begin{cases} \ddot{u} - \Delta u = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \partial\Omega \times (0, \infty), \\ (u, \dot{u})(0) = (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \end{cases} \tag{4.3}$$

observed by $y(t) = \chi_\omega \dot{u}(t)$. In this case, the observability inequality reads as the existence of a strictly positive constant $c_T > 0$ such that solutions of (4.3) satisfy

$$c_T \left\| (u_0, u_1) \right\|^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq \int_0^T \left\| \dot{u}(t) \right\|^2_{L^2(\omega)} \, dt. \tag{4.4}$$

It is then convenient to introduce an abstract setting, which generalizes Theorem 4.1.

Theorem 4.2 ([21]). *Let* A_0 *be a positive definite operator on* X*, and let* B *be a continuous operator from* $\mathcal{D}(A_0^{1/2})$ *to* Y*. Assume that the wave like equation*

$$\ddot{u} + A_0 u = 0, \quad t \geq 0, \qquad (u(0), \dot{u}(0)) = (u_0, u_1) \in \mathcal{D}(A_0^{1/2}) \times X \tag{4.5}$$

observed through

$$y(t) = B\dot{u}(t), \tag{4.6}$$

is admissible and exactly observable, meaning that there exist a time T *and positive constants* $c_T, K_T > 0$ *such that solutions of* (4.5) *satisfy*

$$c_T \left\| (u_0, u_1) \right\|^2_{\mathcal{D}(A_0^{1/2}) \times X} \leq \int_0^T \left\| B\dot{u}(t) \right\|^2_Y \, dt \leq K_T \left\| (u_0, u_1) \right\|^2_{\mathcal{D}(A_0^{1/2}) \times X}. \tag{4.7}$$

Then the operators $A = -iA_0$ *and* B *satisfy Condition* 2.

In particular, the Schrödinger like equation

$$i\dot{z} = A_0 z, \quad t \geq 0, \qquad z(0) = z_0 \in X, \tag{4.8}$$

observed by

$$y(t) = Bz(t) \tag{4.9}$$

satisfies Condition 2 *and is therefore observable in arbitrary small time: for all* $T^* > 0$*, there exists a positive constant* $k_* > 0$ *such that any solution of* (4.8) *satisfies*

$$k_* \|z_0\|_X^2 \leq \int_0^{T^*} \|Bz(t)\|_Y^2 \, dt \tag{4.10}$$

Proof. Assume that (4.5)-(4.6) is exactly observable. Remark that, setting $\mathfrak{X} = \mathcal{D}(A_0^{1/2}) \times X$, and

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ -A_0 & 0 \end{pmatrix}, \quad \mathcal{B} = (\ 0\ ,\ B), \tag{4.11}$$

equation (4.5) fits the abstract setting given above. In particular, the domain of $\mathcal{A}$ simply is $\mathcal{D}(A_0) \times \mathcal{D}(A_0^{1/2})$ and then the conditions $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ and $\mathcal{B} \in \mathfrak{L}(\mathcal{D}(\mathcal{A}), Y)$ are equivalent.

The admissibility and observability properties (4.7) then imply (see [21]) Condition 1: There exist positive constants $M, m > 0$ such that

$$M^2 \left\| (\mathcal{A} - i\omega I) \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mathfrak{X}}^2 + m^2 \left\| \mathcal{B} \begin{pmatrix} u \\ v \end{pmatrix} \right\|_Y^2 \geq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mathfrak{X}}^2, \quad \forall\, \omega \in \mathbb{R},\ \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \tag{4.12}$$

In particular, for all $\omega \in \mathbb{R}$ and $u \in \mathcal{D}(A_0)$, taking $v = i\omega u$ yields

$$\begin{aligned} M^2 \left\|(A_0 - \omega^2 I)u\right\|_X^2 + m^2\omega^2 \|Bu\|_Y^2 &\geq \left\|A_0^{1/2} u\right\|_X^2 + \omega^2 \|u\|_X^2 \\ &\geq \omega^2 \|u\|_X^2 . \end{aligned}$$

Hence

$$\frac{M^2}{\omega^2} \left\|(A_0 - \omega^2 I)u\right\|_X^2 + m^2 \|Bu\|_Y^2 \geq \|u\|_X^2\,, \ \forall \omega \in \mathbb{R}, \forall u \in \mathcal{D}(A_0),$$

or, equivalently,

$$\frac{M^2}{\omega} \|(A_0 - \omega I)u\|_X^2 + m^2 \|Bu\|_Y^2 \geq \|u\|_X^2\,, \ \forall \omega \in \mathbb{R}_+, \forall u \in \mathcal{D}(A_0). \tag{4.13}$$

Of course, this estimate does not hold for $\omega < 0$ and is not interesting for small values of ω. But this actually corresponds to the easy case.

Indeed, if $\omega < \lambda_1(A_0)$, where $\lambda_1(A_0)$ is the first eigenvalue of A_0 (which is strictly positive since A_0 is positive definite),

$$\|(A_0 - \omega I)u\|_X^2 \geq (\lambda_1(A_0) - \omega)^2 \|u\|_X^2, \quad \forall u \in \mathcal{D}(A).$$

Combined with (4.13), by taking

$$M(\omega) = \begin{cases} \dfrac{M}{\sqrt{\omega}} & \text{for } \omega > \dfrac{\lambda_1(A_0)}{2}, \\ \dfrac{1}{\lambda_1(A_0) - \omega} & \text{for } \omega \leq \dfrac{\lambda_1(A_0)}{2}, \end{cases}$$

we then obtain

$$M(\omega)^2 \|(A_0 - \omega I)u\|_X^2 + m^2 \|Bu\|_Y^2 \geq \|u\|_X^2, \ \forall \omega \in \mathbb{R}, \forall u \in \mathcal{D}(A_0). \tag{4.14}$$

This completes the proof of Theorem 4.2 and, as a particular instance of it, of Theorem 4.1. □

The interest of this approach is that it also applies to space semi-discrete, as well as fully discrete approximation schemes of (4.8) (and in particular to (4.1)).

4.2 Space semi-discrete approximation schemes

Let us now introduce the finite element method for (4.8).

Let $(V_h)_{h>0}$ be a sequence of vector spaces of finite dimension n_h which embed into X via a linear injective map $\pi_h : V_h \to X$. For each $h > 0$, the inner product $< \cdot, \cdot >_X$ in X induces a structure of Hilbert space for V_h endowed with the scalar product $< \cdot, \cdot >_h = < \pi_h \cdot, \pi_h \cdot >_X$.

We assume that, for each $h > 0$, the vector space $\pi_h(V_h)$ is a subspace of $\mathcal{D}(A_0^{1/2})$. We thus define the linear operator $A_{0h} : V_h \to V_h$ by

$$< A_{0h}\phi_h, \psi_h >_h = < A_0^{1/2}\pi_h\phi_h, A_0^{1/2}\pi_h\psi_h >_X, \quad \forall(\phi_h, \psi_h) \in V_h^2. \tag{4.15}$$

The operator A_{0h} defined in (4.15) obviously is self-adjoint and positive definite. If we introduce the adjoint π_h^* of π_h, definition (4.15) reads as:

$$A_{0h} = \pi_h^* A_0 \pi_h. \tag{4.16}$$

This operator A_{0h} corresponds to the finite element discretization of the operator A_0. We thus consider the following space semi-discretization of (4.8):

$$i\dot{z}_h = A_{0h}z_h, \quad t \in \mathbb{R}, \qquad z_h(0) = z_{0h} \in V_h. \tag{4.17}$$

In this context, for all $h > 0$, the observation then naturally becomes

$$y_h(t) = B\pi_h z_h = B_h z_h. \tag{4.18}$$

Note that we shall impose $B \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$ for this definition to make sense.

We now make precise the assumptions we have, usually, on π_h, and which will be needed in our analysis. One easily checks that $\pi_h^* \pi_h = Id_h$. The injection π_h describes the finite element approximation we have chosen. In particular, the vector space $\pi_h(V_h)$ approximates, in the sense given hereafter, the space $\mathcal{D}(A_0^{1/2})$: There exist $\theta > 0$ and $C_0 > 0$, such that for all $h > 0$,

$$\begin{cases} \left\| A_0^{1/2}(\pi_h \pi_h^* - I)\phi \right\|_X \leq C_0 \left\| A_0^{1/2} \phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0^{1/2}), \\ \left\| A_0^{1/2}(\pi_h \pi_h^* - I)\phi \right\|_X \leq C_0 h^\theta \left\| A_0 \phi \right\|_X, & \forall \phi \in \mathcal{D}(A_0). \end{cases} \tag{4.19}$$

When considering finite element discretizations of the Schrödinger equation (4.1), which, as we said, corresponds to take A_0 as the Laplace operator with Dirichlet boundary conditions, estimates (4.19) are satisfied [25] for $\theta = 1$ when using P1 finite elements on a regular mesh (in the sense of finite elements).

We will not discuss convergence results for the numerical approximation schemes presented here, which are classical under assumption (4.19), and which can be found for instance in the textbook [25].

In [8, 9], we proved uniform observability properties for (4.17)-(4.18) in classes of conveniently filtered initial data. In the sequel, our goal is to obtain uniform observability properties for (4.17) similar to (4.10), but in arbitrary small time, still for conveniently filtered initial data.

Therefore, we shall introduce the filtered classes of data. For all $h > 0$, since A_{0h} is a self-adjoint positive definite operator, the spectrum of A_{0h} is given by a sequence of positive eigenvalues

$$0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{n_h}^h, \tag{4.20}$$

and normalized (in V_h) eigenvectors $(\Phi_j^h)_{1 \leq j \leq n_h}$. For any $s > 0$, we can now define, for any $h > 0$, the filtered space

$$\mathcal{C}_h(s) = \operatorname{span}\left\{ \Phi_j^h \text{ with the corresponding eigenvalue satisfies } |\lambda_j^h| \leq s \right\}.$$

We have then proved in Theorem 1.3 in [8]:

Theorem 4.3. *Let A_0 be a self-adjoint positive definite operator with compact resolvent, and $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$, with $\kappa < 1/2$. Assume that the maps $(\pi_h)_{h>0}$ satisfy property* (4.19). *Set*

$$\sigma = \theta \min\left\{ 2(1 - 2\kappa), \frac{2}{3} \right\}. \tag{4.21}$$

Assume that system (4.8)-(4.9) *is admissible and exactly observable. Then there exist* $\varepsilon > 0$*, a time* T^* *and positive constants* $k_*, K_* > 0$ *such that, for any* $h > 0$*, any solution of* (4.17) *with initial data*

$$z_{0h} \in \mathcal{C}_h(\varepsilon/h^\sigma) \tag{4.22}$$

satisfies

$$k_* \left\| z_{0h} \right\|_h^2 \leq \int_0^{T^*} \left\| B_h z_h(t) \right\|_Y^2 \, dt \leq K_* \left\| z_{0h} \right\|_h^2 \tag{4.23}$$

In this result, based on spectral criteria for the admissibility and admissibility of Schrödinger operators, the time of observability T^* cannot be made as small as desired.

When the Geometric Control Condition is satisfied, the following can be proved as a by product on our analysis of the abstract wave like equation (4.5) in [9] and the methods in [21].

Theorem 4.4. *Let* A_0 *be a positive definite unbounded operator with compact resolvent and* $B \in \mathfrak{L}(\mathcal{D}(A_0^\kappa), Y)$*, with* $\kappa < 1/2$*. Assume that the approximations* $(\pi_h)_{h>0}$ *satisfy property* (4.19)*. Set*

$$\varsigma = \theta \min\{2(1-2\kappa), 1\}. \tag{4.24}$$

Assume that system (4.5)-(4.6) *is admissible and exactly observable. Then there exist* $\varepsilon > 0$*, a time* T^* *and positive constants* $k_*, K_* > 0$ *such that, for any* $h > 0$*, any solution of* (4.17) *with initial data in*

$$z_{0h} \in \mathcal{C}_h(\varepsilon/h^\varsigma) \tag{4.25}$$

satisfies (4.23)*.*

Theorem 4.4 indeed improves Theorem 4.3 since $\varsigma \geq \sigma$. This is expected since the assumptions of admissibility and observability for the abstract wave system (4.5)-(4.6) are stronger than the admissibility and observability of Schrödinger equations (4.8)-(4.9).

However, Theorem 4.4 requires the time of observability to be large enough. We shall prove below that the time T can actually be chosen arbitrary small.

Theorem 4.5. *Under the assumptions of Theorem 4.4, assume that system* (4.5)-(4.6) *is admissible and exactly observable. Then there exists* $\varepsilon > 0$ *such that for all* $T^* > 0$*, there exist positive constants* $k_*, K_* > 0$ *such that, for any* $h > 0$*, any solution of* (4.17) *with initial data in* (4.25) *satisfies* (4.23)*.*

Proof. The admissibility result in (4.23) follows from the one in Theorem 4.4 since, when the admissibility inequality holds for some time $T > 0$, it holds for any time. We shall thus not deal further with that question.

Assume that system (4.5)-(4.6) is admissible and exactly observable. Then we can use Theorem 1.1 in [9], which states that, under the assumptions of Theorem 4.4, the space semi-discrete wave systems

$$\ddot{u}_h + A_{0h} u_h = 0, \quad t \geq 0, \qquad y_h(t) = B_h \dot{u}_h,$$

are

- uniformly (with respect to $h > 0$) admissible for any initial data $(u_{0h}, u_{1h}) \in \mathcal{C}_h(\eta\, h^{-\varsigma})^2$, whatever $\eta > 0$ is.
- uniformly (with respect to $h > 0$) observable in some time $T > 0$ for initial data $(u_{0h}, u_{1h}) \in \mathcal{C}_h(\varepsilon\, h^{-\varsigma})^2$, providing ε is small enough.

These uniform admissibility and observability properties imply, as proved in [21], that the resolvent condition (4.12) for the operators A_{0h} and B_h hold uniformly with respect to h for data $z_h \in \mathcal{C}_h(\varepsilon/h^\varsigma)$. The proof of Theorem 4.2 then gives that, uniformly with respect to $h > 0$, we can find positive constants $M,\ m > 0$ such that

$$\frac{M^2}{\omega} \left\| (A_{0h} - \omega I_h) u_h \right\|_h^2 + m^2 \left\| B_h u_h \right\|_Y^2 \geq \left\| u_h \right\|_h^2, \qquad \forall \omega \in \mathbb{R}_+, \forall u_h \in \mathcal{C}_h(\varepsilon/h^\varsigma).$$

To conclude that Condition 2 is uniformly satisfied, following the proof of Theorem 4.2, we only need to check that the first eigenvalue λ_1^h corresponding to the operator A_{0h} stays away from 0. But, writing the Rayleigh coefficient which characterizes λ_1^h and $\lambda_1(A_0)$, one instantaneously checks that $\lambda_1^h \geq \lambda_1(A_0) > 0$ for all $h > 0$.

In other words, we have proved that there exists a bounded positive function $M = M(\omega)$ satisfying $\lim_{|\omega| \to \infty} M(\omega) = 0$ and a positive constant $m > 0$ such that for all $h > 0$

$$M(\omega)^2 \left\| (A_{0h} - \omega I_h) u_h \right\|_h^2 + m^2 \left\| B_h u_h \right\|_Y^2 \geq \left\| u_h \right\|_h^2, \qquad \forall \omega \in \mathbb{R}, \forall u_h \in \mathcal{C}_h(\varepsilon/h^\varsigma). \tag{4.26}$$

Now, we use our constructive proof of Theorem 1.3 to deduce uniform observability properties in any time T^*. However, though this might seem at first a direct consequence of Theorem 1.3, one needs to be cautious.

Following the proof of Theorem 1.3, we see that the high-frequency components can be dealt with uniformly without modification. In particular, for all $\tilde{T} > 0$, there exists Ω such that, for all $h > 0$, any solution of (4.17) with initial data $z_{0h} \in \mathcal{C}_h(\Omega)^\perp \cap \mathcal{C}_h(\varepsilon/h^\varsigma)$ satisfies

$$\tilde{k} \left\| z_{0h} \right\|_h^2 \leq \int_0^{\tilde{T}} \left\| B_h z_h(t) \right\|_Y^2 \, dt, \tag{4.27}$$

for some positive constant $\tilde{k} > 0$ independent of $h > 0$.

Besides the systems (4.17)-(4.18) are uniformly admissible because of Theorem 4.4.

But the low-frequency components require an estimate on the low-frequency gap for each $h > 0$. The constant Ω being independent of $h > 0$ and setting $(m_j^h)_{j \in \{1, \cdots, N_h\}}$ for the increasing sequence of the values taking by the eigenvalues of A_{0h} which are smaller than Ω, we shall estimate

$$\gamma_h = \inf_{j \in \{0, \cdots, N_h\}} \{m_{j+1}^h - m_j^h\} \text{ where } m_0^h = -\Omega \text{ and } m_{N_h+1}^h = \Omega. \tag{4.28}$$

Note in particular that N_h might depend on h. However, since all these correspond to the discrete spectrum of A_{0h} it shall converge to the spectrum of A_0.

Case 1: Each eigenvalue of the spectrum of A_0 is simple. Then the convergence of the discrete spectrum of A_{0h} in the band of eigenvalues smaller than the constant Ω is guaranteed [25]. In particular, N_h is constant for $h > 0$ small enough and the sequence (γ_h) then simply converges to γ.

Case 2: The general case. When the spectrum of A_0 is not simple, this is harder since a multiple eigenvalue of the continuous operator may yield to different but close eigenvalues, making γ_h dangerously small for our argument. The idea then is to refine Haraux' argument, and to think directly at this convergence property of the spectrum.

For each positive $\alpha > 0$ smaller than $\gamma/4$ (γ being the continuous low frequency gap defined in (2.8)), there exists $h_\alpha > 0$ such that for $h \in (0, h_\alpha)$, the spectrum of the operator A_{0h} satisfies

$$\{\lambda_\ell^h \text{ such that } \lambda_\ell^h < \Omega\} \subset \bigcup_{j \in \{1, \cdots, N\}} [m_j - \alpha, m_j + \alpha]. \tag{4.29}$$

Define then the sets $X_j^{h,\alpha} = \operatorname{span}\{\Phi_\ell^h \text{ such that } |\lambda_\ell^h - m_j| \leq \alpha\}$. Since the discrete operators satisfy (4.26), further assuming that α is smaller than $1/(2 \sup M(\omega))$, choosing for instance $\beta = 1/(2m)$, we obtain

$$\forall j \in \{1, \cdots, N\}, \ \forall z \in X_j^{h,\alpha}, \quad \left\| B_h z_h \right\|_Y \geq \beta \left\| z \right\|_h. \tag{4.30}$$

Once we have seen (4.29)-(4.30), the inductive argument developed in Lemma 2.3 works as before, except some small error terms. Let us present it briefly below on the first step.

To write it properly, we shall introduce the orthogonal projections $\mathbb{P}_N^{h,\alpha}$ on $X_N^{h,\alpha}$ and $\mathbb{P}_{hf}^h$ on $\mathcal{C}_h(\Omega)^\perp$, respectively.

Set then $z_{0h} \in X_N^{h,\alpha} + \mathcal{C}_h(\Omega)^\perp \cap \mathcal{C}_h(\varepsilon/h^\varsigma)$, and decompose it into $z_{0h,N} = \mathbb{P}_N^{h,\alpha} z_{0h}$ and $z_{0h,hf} = \mathbb{P}_{hf}^h z_{0h}$. Let $z_h(t)$ be the solution of (4.17) with intial data z_{0h} and, for $\delta > 0$, define v_h as

$$v_h(t) = z_h(t) - \frac{1}{2\delta}\int_{-\delta}^{\delta} e^{im_N s} z_h(t-s)\, ds.$$

Expanding z_{0h} on the basis Φ_j^h, similarly as in (2.11), we obtain

$$\begin{cases} \left\|\mathbb{P}_{hf}^h z_{0h}\right\|_h^2 \leq \dfrac{1}{(1-\mathrm{sinc}(\delta\gamma))^2}\left\|v_h(0)\right\|_X^2. \\ \left\|\mathbb{P}_N^{h,\alpha} v_h(0)\right\|_h^2 \leq (1-\mathrm{sinc}(\alpha\delta))^2 \left\|\mathbb{P}_N^{h,\alpha} z_{0h}\right\|_h^2. \end{cases} \tag{4.31}$$

Besides, v_h is a solution of (4.17), which implies that $\mathbb{P}_{hf}^h v_h$ also is. But $\mathbb{P}_{hf}^h v_h$ lies in $\mathcal{C}_h(\Omega)^\perp \cap \mathcal{C}_h(\varepsilon/h^\varsigma)$, and then one can use (4.27):

$$\begin{aligned} \left\|\mathbb{P}_{hf}^h v_h(0)\right\|_X^2 &\leq \frac{1}{\tilde{k}}\int_0^{\tilde{T}} \left\|B_h \mathbb{P}_{hf}^h v_h(t)\right\|_Y^2\, dt \\ &\leq \frac{2}{\tilde{k}}\int_0^{\tilde{T}} \left\|B_h v_h(t)\right\|_Y^2\, dt + \frac{2K_{\tilde{T}}}{\tilde{k}}\left\|\mathbb{P}_N^{h,\alpha} v_h(0)\right\|_h^2 \\ &\leq \frac{2}{\tilde{k}}\int_0^{\tilde{T}} \left\|B_h v_h(t)\right\|_Y^2\, dt + \frac{2K_{\tilde{T}}}{\tilde{k}}(1-\mathrm{sinc}(\alpha\delta))^2\left\|\mathbb{P}_N^{h,\alpha} z_h(0)\right\|_h^2 \end{aligned} \tag{4.32}$$

Using the same estimates as in (2.13), combined with (2.11), we get

$$\begin{aligned} \left\|\mathbb{P}_{hf}^h z_{0h}\right\|_h^2 \leq \frac{4}{\tilde{k}(1-\mathrm{sinc}(\gamma\delta))^2}\int_{-\delta}^{\tilde{T}+\delta} \left\|B_h z_h(t)\right\|_Y^2\, dt \\ + \frac{2K_{\tilde{T}}}{\tilde{k}}\left(\frac{1-\mathrm{sinc}(\alpha\delta)}{1-\mathrm{sinc}(\gamma\delta)}\right)^2 \left\|\mathbb{P}_N^{h,\alpha} z_h(0)\right\|_h^2. \end{aligned} \tag{4.33}$$

We then focus on the component of the solution in $X_N^{h,\alpha}$. Arguing as in (2.15) and using (4.30), we obtain

$$\left\|\mathbb{P}_N^{h,\alpha} z_{0h}\right\|_h^2 \leq \frac{2}{\tilde{T}\beta^2}\int_0^{\tilde{T}} \left\|B_h z_h(t)\right\|_Y^2\, dt + \frac{2K_{\tilde{T}}}{\tilde{T}\beta^2}\left\|\mathbb{P}_{hf}^h z_{0h}\right\|_h^2. \tag{4.34}$$

Equations (4.33) and (4.34), together, give

$$\left\|\mathbb{P}_N^{h,\alpha} z_{0h}\right\|_h^2 \left(1 - \frac{4K_{\tilde{T}}^2}{\tilde{T}\beta^2\tilde{k}}\left(\frac{1-\operatorname{sinc}(\alpha\delta)}{1-\operatorname{sinc}(\gamma\delta)}\right)^2\right) \leq \left(\frac{2}{\tilde{T}\beta^2} + \frac{8K_{\tilde{T}}}{\tilde{T}\beta^2\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\right)\int_{-\delta}^{\tilde{T}+\delta} \|B_h z_h(t)\|_Y^2 \; dt \tag{4.35}$$

In particular, if one can guarantee the left hand-side to be positive, which can be done simply by choosing $\alpha > 0$ small enough and

$$(1-\operatorname{sinc}(\alpha\delta))^2 \leq \frac{\tilde{T}\beta^2\tilde{k}}{16K_{\tilde{T}}^2}(1-\operatorname{sinc}(\gamma\delta))^2, \tag{4.36}$$

we deduce

$$\left\|\mathbb{P}_N^{h,\alpha} z_{0h}\right\|_h^2 \leq \left(\frac{4}{\tilde{T}\beta^2} + \frac{16K_{\tilde{T}}}{\tilde{T}\beta^2\tilde{k}(1-\operatorname{sinc}(\gamma\delta))^2}\right)\int_{-\delta}^{\tilde{T}+\delta} \|B_h z_h(t)\|_Y^2 \; dt$$

From (4.33), we obtain an estimate for $\left\|\mathbb{P}_{hf}^h z_{0h}\right\|_h^2$. Using the orthogonality of $X_N^{h,\alpha}$ and $\mathcal{C}_h(\Omega)^\perp \cap \mathcal{C}_h(\varepsilon/h^\varsigma)$, this proves that the observability inequality holds in any time $T > \tilde{T}$, uniformly with respect to $h \in (0, h_\alpha)$, for solutions of (4.17) with initial data in $X_N^{h,\alpha} + \mathcal{C}_h(\Omega)^\perp \cap \mathcal{C}_h(\varepsilon/h^\varsigma)$.

Note that (4.36) does not depend on $h > 0$. Thus, once α is chosen according to (4.36), the above proof stands for any $h \in (0, h_\alpha)$.

This concludes the inductive argument, and this slightly generalized Haraux's technique can be applied to conclude the proof of Theorem 4.5. □

4.3 Fully discrete approximation schemes

We can also prove observability properties for fully discrete approximations of (4.8)-(4.9), uniformly with both discretization parameters $\Delta t > 0$ and $h > 0$, in arbitrary small time.

To be more precise, we consider, for $h > 0$ and $\Delta t > 0$, the following system:

$$\begin{cases} i\left(\dfrac{z_h^{k+1} - z_h^k}{\Delta t}\right) = A_{0h}\Big(\dfrac{z_h^k + z_h^{k+1}}{2}\Big), & \text{in } V_h, \quad k \in \mathbb{Z}, \\ z_h^0 = z_{0h}, \end{cases} \tag{4.37}$$

observed by

$$y_h^k = B_h z_h^k. \tag{4.38}$$

For these systems, admissibility and observability results have been derived in [8] using [10] in the class $\mathcal{C}_h(\delta/\Delta t) \cap \mathcal{C}_h(\varepsilon h^{-\sigma})$, with σ as in (4.21), but the observability results in [8] need the time T to be large enough. Later in [9], these admissibility and observability results have been improved by using the Geometric Control Condition, yielding the filtering class $\mathcal{C}_h(\delta/\Delta t) \cap \mathcal{C}_h(\varepsilon/h^{\varsigma})$ with ς as in (4.24), but the observability time is again required to be large enough.

However, using [9] and the techniques developed above, we can prove that the discrete systems (4.37)-(4.38) actually are observable in arbitrary small time.

Theorem 4.6. *Under the assumptions of Theorem 4.4. Assume that system* (4.5)-(4.6) *is admissible and exactly observable. Then, for any time $T^* > 0$, for any $\delta > 0$, there exist two positive constants $\varepsilon > 0$ and $k_{\delta,T^*} > 0$ such that, for all $h, \Delta t > 0$ small enough, any solution of* (4.37) *with initial data*

$$z_{0h} \in \mathcal{C}_h\Big(\inf \Big\{ \frac{\delta}{\Delta t}, \frac{\varepsilon}{h^{\varsigma}} \Big\} \Big),$$

where ς is given by (4.24)*, satisfies*

$$k_{\delta,T^*} \left\| z_{0h} \right\|_h^2 \leq \Delta t \sum_{k\Delta t \in (0,T^*)} \left\| B_h z_h^k \right\|_Y^2 . \tag{4.39}$$

The proof of Theorem 4.6, which can be adapted easily from the previous ones, is left to the reader. The keynote is the convergence of the low components of the spectrum and the fact that all the above proofs are explicit and shortcut any compacity argument.

5 Further comments

This work is based on the resolvent estimate given Condition 2. Under Condition 2, observability properties hold in arbitrary small time. However, there might be systems fitting the abstract setting (1.1)-(1.2) which are observable in arbitrary small time but for which Condition 2 does not hold. In this sense, we did not completely solve the problem.

This is actually part of a more general question: can we read on the operators A and B and their spectral properties the critical time of observability ? To our knowledge, this is still not clear if the resolvent estimates keep precisely track of this information, which is of primary importance in applications, for instance when dealing with waves. It would then be interesting to try to design an efficient spectral characterization of the time of observability.

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