Approximate controllability for a system of Schrödinger equations modeling a single trapped ion*

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Résumé

Dans cet article, nous étudions les propriétés de contrôlabilité approchée pour un système d'équations de Schrödinger modélisant un ion piégé. Nous nous limitons à un contrôle d'une forme particulière, correspondant à des restrictions pratiques. Notre approche est fondée sur l'analyse de la contrôlablité d'un système approché de dimension finie, pour lequel il est possible de construire explicitement des contrôles exacts. Nous justifions alors précisément les approximations qui relient le système complet au système approché. Nous en déduisons des résultats de contrôlabilité approchée dans l'espace naturel $(L^2(\mathbb{R}))^2$ mais aussi dans des espaces plus forts correspondants aux domaines de l'opérateur harmonique.

Abstract

In this article, we analyze the approximate controllability properties for a system of Schrödinger equations modeling a single trapped ion. The control we use has a special form, which takes its origin from practical limitations. Our approach is based on the controllability of an approximate finite dimensional system for which one can design explicitly exact controls. We then justify the approximations which link up the complete and approximate systems. This yields approximate controllability results in the natural space $(L^2(\mathbb{R}))^2$ and also in stronger spaces corresponding to the domains of the harmonic operator.

1 Introduction

Notations: Let A be the harmonic oscillator operator on $\mathbb{R}$

$$A = \frac{1}{2}\Big(- \partial_{xx}^2 + x^2\Big). \tag{1.1}$$

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Note that A is a self-adjoint definite positive operator on $L^2(\mathbb{R})$, and we can therefore introduce, for integers $k \in \mathbb{N}$, the spaces $\mathcal{D}(A^{k/2})$, endowed with the norm

$$\|\cdot\|_k = \left\|A^{k/2}\cdot\right\|_{L^2(\mathbb{R})}.$$

In the sequel, we will also consider the product spaces $\mathcal{D}(A^{k/2})^2$, that we endow with the classical product norm

$$\|(\psi_1, \psi_2)\|_{k\times k} = \Big(\, \|\psi_1\|_k^2 + \|\psi_2\|_k^2 \,\Big)^{1/2}, \quad \forall (\psi_1, \psi_2) \in \mathcal{D}(A^{k/2})^2.$$

For a function f, we will denote by f^* its conjugate function. □

In this article, we consider the following system of Schrödinger equations:

$$\begin{cases} i\partial_t \psi_e = \omega A \psi_e + \dfrac{\Omega}{2}\psi_e + (\mathbf{u}+\mathbf{u}^*)\psi_g, & (t,x) \in (0,T)\times\mathbb{R}, \\ i\partial_t \psi_g = \omega A \psi_g - \dfrac{\Omega}{2}\psi_g + (\mathbf{u}+\mathbf{u}^*)\psi_e, & (t,x) \in (0,T)\times\mathbb{R}, \end{cases} \tag{1.2}$$

with initial data

$$\psi_e(0,x) = \psi_e^0(x), \quad \psi_g(0,x) = \psi_g^0(x), \quad x \in \mathbb{R}. \tag{1.3}$$

In (1.2), ω and Ω are real numbers. The function $\mathbf{u} = \mathbf{u}(t,x)$ is the control function, which will be specified later on.

Equation (1.2) models a composite system made of 2-levels of excited and ground states (corresponding respectively to the subscripts e and g). The control $\mathbf{u}$ corresponds to an electro-magnetic wave.

The question we address here consists in describing the action of the control $\mathbf{u}$. To be more precise, we will study the possibility of driving the system from a given initial state to the neighborhood of a given final state.

Due to physical restrictions, we furthermore assume that the control $\mathbf{u}$ has the following specific form:

$$\mathbf{u}(t,x) = u_0 e^{i(\Omega t - \sqrt{2}\eta x)} + u_r e^{i((\Omega-\omega)t - \sqrt{2}\eta x)} + u_b e^{i((\Omega+\omega)t - \sqrt{2}\eta x)}, \tag{1.4}$$

where $(u_0, u_r, u_b) \in \mathbb{C}^3$ and $\eta \in \mathbb{R}_+^*$.

This assumption says that $\mathbf{u}$ is a superposition of three monochromatic waves, one of pulsation Ω (ion electronic transition) and of amplitude $u_0 \in \mathbb{C}$, one of pulsation $\Omega - \omega$ (red shift by a vibration quantum) and of amplitude $u_r \in \mathbb{C}$, and one of pulsation $\Omega + \omega$ (blue shift by a vibration quantum) and of amplitude $u_b \in \mathbb{C}$. We further assume that we can switch on and off these monochromatic waves. In other words, the function $t \mapsto (u_0(t), u_b(t), u_r(t))$ is piecewise constant.

Besides, we assume that only one control is active at each time $t \geq 0$. In other words, there is at most one non-zero component in the vector $(u_0(t), u_r(t), u_b(t))$ in any time $t \in [0,T]$.

The norm of the control $\mathbf{u}$ is of primary importance in applications. We thus furthermore assume that for every time $t \in [0,T]$,

$$\sup\{|u_0(t)|, |u_r(t)|, |u_b(t)|\} \leq K. \tag{1.5}$$

In other words, K represents the size of the controls.

The parameter η is a real positive number, the so-called Lamb-Dicke parameter, which is related to the wavelength of the electro-magnetic wave.

In practice, ω is of order 10^{10}, Ω of order 10^{15}, the control function satisfies $|\mathbf{u}(t,x)| \ll \omega$, or equivalently $K \ll \omega$, and the Lamb-Dicke parameter is of small magnitude $\eta \ll 1$ (see for instance [25]). Therefore, in our analysis, we shall think of Ω and ω as large numbers and of η as a small one. From the physical point of view, $\omega \ll \Omega$. However, this is not needed in our analysis, and we simply require, all along this article, that

$$\omega \leq \frac{2\Omega}{3}, \tag{1.6}$$

which guarantees that ω is the smallest relevant frequency of the free system (1.2). We refer for instance to [25] for more physical motivations.

As a preliminary result, we first study the Cauchy problem for (1.2):

Theorem 1.1. *Let $T > 0$. Assume that $f : (0,T) \times \mathbb{R} \to \mathbb{R}$, and that*

$$f \in L^\infty((0,T); C_b^0(\mathbb{R})). \tag{1.7}$$

Consider the Cauchy problem for

$$\begin{cases} i\partial_t \psi_e = \omega A \psi_e + \dfrac{\Omega}{2}\psi_e + f\psi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \psi_g = \omega A \psi_g - \dfrac{\Omega}{2}\psi_g + f\psi_e, & (t,x) \in (0,T) \times \mathbb{R}, \end{cases} \tag{1.8}$$

with initial data $(\psi_e^0, \psi_g^0) \in L^2(\mathbb{R})^2$ as in (1.3).

Then there exists a unique mild solution (ψ_e, ψ_g) of (1.8) in $C([0,T]; (L^2(\mathbb{R}))^2)$ and, for any time $t \in [0,T]$,

$$\|(\psi_e(t), \psi_g(t))\|_{0\times 0} = \left\|(\psi_e^0, \psi_g^0)\right\|_{0\times 0}. \tag{1.9}$$

Besides, if for some integer $k \in \mathbb{N}$, $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and $f \in L^\infty(0,T; C_b^k(\mathbb{R}))$, then (ψ_e, ψ_g) belongs to $C([0,T]; \mathcal{D}(A^{k/2})^2)$.

To analyze the control properties of system (1.2), we consider the following system, the so-called Law-Eberly equations (see [17]), which is a simplified model of (1.2):

$$\begin{cases} i\partial_t \phi_e = \left(u_0^* + v_r^* \mathbf{a} + v_b^* \mathbf{a}^\dagger\right)\phi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \phi_g = \left(u_0 + v_r \mathbf{a}^\dagger + v_b \mathbf{a}\right)\phi_e, & (t,x) \in (0,T) \times \mathbb{R}, \end{cases} \tag{1.10}$$

where v_r and v_b correspond, respectively, to $-i\eta u_r$ and $-i\eta u_b$, with initial conditions

$$\phi_e(0,x) = \phi_e^0(x), \quad \phi_g(0,x) = \phi_g^0(x), \qquad x \in \mathbb{R}. \tag{1.11}$$

In (1.10), we use the notations $\mathbf{a}$ and $\mathbf{a}^\dagger$ for:

$$\mathbf{a} = \frac{1}{\sqrt{2}}\Big(x + \partial_x\Big), \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}\Big(x - \partial_x\Big). \tag{1.12}$$

These notations are standard in quantum mechanics. The operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ are, respectively, the so-called annihilation and creation operators. Also note that $\mathbf{a}^\dagger$ is the adjoint of $\mathbf{a}$.

System (1.10) indeed corresponds to a simplified model of (1.2) written in the interaction frame, and after several approximations which, to our knowledge, have not been rigorously justified yet. We will justify rigorously these approximations below.

In order to use the controllability properties of system (1.10) for our original system (1.2), we will recall and refine the results in [17] on the controllability of (1.10). Roughly speaking, it is stated in [17] that any finite linear combination of eigenvectors of A can be steered to any finite linear combination of eigenvectors of A.

To state rigorously the results in [17], we introduce the spectrum of A, which simply consists in a sequence (λ_j, Φ_j) of increasing eigenvalues and normalized (in $L^2(\mathbb{R})$) eigenvectors (see [24] and Section 3 below). It is then convenient to introduce, for an integer $M \geq 0$, the finite dimensional subspace spanned by the $M+1$ first eigenvectors of A:

$$V_M = \operatorname{span}\Big\{\Phi_j\ ;\ \ 0\ \leq j \leq M\Big\}.$$

Indeed, following the strategy in [17], we obtain a precise controllability result for system (1.10):

Theorem 1.2 (Based on [17]). *Let $M \geq 0$ be an integer. Given any (ϕ_e^0, ϕ_g^0) and (ϕ_e^1, ϕ_g^1) in V_M^2 of equal $(L^2(\mathbb{R}))^2$ norms, there exist a time $T > 0$ and a piecewise constant function $t \mapsto (u_0(t), v_r(t), v_b(t))$ such that the solution (ϕ_e, ϕ_g) of (1.10) with initial data (ϕ_e^0, ϕ_g^0) satisfies $(\phi_e(T), \phi_g(T)) = \beta(\phi_e^1, \phi_g^1)$, for some complex number β of modulus 1.*

Besides, the following properties hold:

1. *For each time $t \in [0,T]$, $(\phi_e(t), \phi_g(t)) \in V_M^2$.*
2. *At each time $t \in [0,T]$, there is only one nonzero component in the vector $(u_0(t), v_r(t), v_b(t))$, and there is at most $4M+2$ switching times.*
3. *If we impose a priori that the control functions u_0, v_r and v_b satisfy $|u_0| \leq K_0$ and $|v_r|, |v_b| \leq K_1$ for some constants K_0 and K_1, then T can be chosen to be any time satisfying*

$$T \geq \frac{(M+1)\pi}{K_0} + \frac{\pi}{K_1}\sum_{j=1}^{M} \frac{1}{\sqrt{j}}. \tag{1.13}$$

Note that, in Theorem 1.2, the control is exact up to a phase term β. This parameter is actually irrelevant for physical purposes, and thus does not affect the results of Theorem 1.2.

Theorem 1.2 is then used to deduce the following approximate controllability result:

Theorem 1.3 (Approximate Controllability in $(L^2(\mathbb{R}))^2$). *Consider two couples of data (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) of unit $(L^2(\mathbb{R}))^2$ norms.*

For any $\delta > 0$, there exist a constant $\aleph = \aleph(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) > 0$, two parameters $\eta_0 = \eta_0(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) > 0$ and $\rho_0(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1)$, such that for (ω, Ω) as in (1.6) and for

$$0 < \eta \leq \eta_0, \qquad KT = \frac{\aleph}{\eta}, \qquad \frac{\omega\eta}{K} \geq \rho_0, \tag{1.14}$$

for a control function $\mathbf{u}(t,x)$ of the form (1.4), given by a map $t \mapsto (u_0(t), u_r(t), u_b(t))$ of piecewise constant functions,

- *The solution (ψ_e, ψ_g) of (1.2) with initial data (ψ_e^0, ψ_g^0) satisfies, for some complex number β of modulus 1,*

$$\left\|(\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1)\right\|_{0\times 0} \leq \delta. \tag{1.15}$$

- *For every time $t \in [0,T]$, $\mathbf{u}$ satisfies (1.5).*

- *At each time $t \in [0,T]$, there is at most one nonzero component in the vector $(u_0(t), u_r(t), u_b(t))$.*

This result shows approximate controllability for system (1.2). This notion is relevant because, after reaching a neighborhood of the target state (ψ_e^1, ψ_g^1), if we switch off the control, due to estimate (1.9), the solution will stay in this neighborhood.

Note that the condition $KT = \aleph/\eta$ in Theorem 1.3 corresponds to a condition on the $L^1(0,T;L^\infty(\mathbb{R}))$ norm of the control $\mathbf{u}$. Theorem 1.3 can be interpreted in several different ways:

- If we are limited by the size of the controls we can use, we need to control system (1.2) during a time $T = \aleph/(K\eta) = T^*/\eta$, which blows up when $\eta \to 0$. In this case, condition (1.14) imposes that ω shall satisfy $\omega \geq \omega^*/\eta$.

- If we want to obtain an approximate controllability result in a prescribed time T, our method needs large controls to work, and K must be like $K = \aleph/(T\eta) = K^*/\eta$. Thus (1.14) imposes that ω shall satisfy the condition $\omega \geq K^*\rho_0/\eta^2$.

- If (ω, Ω) are constant positive numbers satisfying (1.6), one shall choose K smaller than $K^*\eta$, for a suitable small enough positive constant K^*. In this case, the time T must be larger than $\aleph/(K\eta)$.

- If $K = \eta$, condition (1.14) simply imposes on ω that $\omega \geq \omega_0$, for $\omega_0 = \rho_0$ independent of η and K. In this case, note that the time T has to be large enough and grows as $\aleph/\eta^2$.

Also remark that condition (1.14) can be satisfied only when ω/K is large enough. This corresponds to the condition $|\mathbf{u}| \ll \omega$, which is of physical nature.

One of the interesting features of Theorem 1.3 is that the construction of the approximate control function $\mathbf{u}$ is explicit. This result fully justifies the approximate control problem (1.10).

Besides, similar results can be proved for the stronger norms $\|(\cdot,\cdot)\|_{k\times k}$. Indeed, our proofs can be extended to deal with these norms, again by using Theorem 1.2. This will be done in Theorem 4.6. Note that, as in the L^2 case, this is relevant since, after reaching a $\mathcal{D}(A^{k/2})^2$ neighborhood of the target state, if we switch off the control, the solution will stay in this neighborhood (see Lemma 2.1).

Let us briefly present the context of our work. We refer the interested reader to [15] for a pedagogical introduction to controllabilty theory untill recent developments of the theory.

In the pioneer work [2], the bilinear exact controllability for general abstract systems has been proved to be impossible in natural spaces. As noticed in [26], the analysis in [2] applies to the classical Schrödinger harmonic operator and proves the lack of (even local) exact controllability in the natural space L^2. Though, this does not prove that local exact controllability does not hold when considering higher order norms. Indeed, as proved in [6, 7, 8], it does hold for the harmonic oscillator in a potential well (and thus in a bounded interval) in H^7 neighborhoods of the ground state (or any other stationary solutions). Let us also mention the results in [14], which state the existence of a minimal time of control even when dealing with H^7 neighborhoods of the ground state. We also refer to [20] where the controlled Schrödinger operator is decoupled into a free uncontrolled part and a controllable one, which coincides with the classical harmonic oscillator.

Note that the results in [2] also applies to system (1.2) and proves the lack of local exact controllability in $L^2(\mathbb{R})^2$. To our knowledge, the case of stronger norms has not been studied so far. We will present some comments related to this issue at the end of the article.

It is then natural to consider weaker forms of controllability. For instance, a lot of attention has been devoted to the study of controllability properties for finite-dimensional harmonic oscillator Schrödinger equations and of their numerical computations, essentially based on Lyapunov techniques [27, 9, 21]. In particular, this yields to approximate controllability results in L^2 in infinite time [18, 10, 19], and can be adapted for more regular spaces [22].

Let us also mention the optimal control approach developped in [4, 16] (and [3] for a Hartree Fock model) in the infinite dimensional setting, and analyzed numerically in [5], which provides other techniques for approximate controllability results.

In [1], approximate state to state controllability (which consists in steering the system from an eigenstate to a neighborhood of an eigenstate) is proved in L^2, by using specific features of the controlled systems when the eigenvalues cross each other. The main disadvantage of this technique is that it requires a good knowledge of the eigenvalues. Thus, a large time is needed to ensure a good behavior of the spectrum, using the adiabatic theorem.

In [13], approximate controllability in L^2 for abstract Schrödinger type systems is deduced from approximate controllability results for the Galerkin approximations of the system. This method yields L^2 controllability results, using piecewise constant controls. The setting and method in [13] are close to ours. Roughly speaking, it consists in deriving global controllability results by using the controls of conveniently chosen finite dimensional systems. Though, several differences appear: in [13], spectral conditions, which are not satisfied in our case, are required to avoid resonant cases; there is only one scalar control, whereas we handle several controls; the control is chosen as a piecewise constant function, whereas we are looking for highly oscillatoring controls (recall (1.4)); in [13], no explicit form of the control is given, and no estimate on the control time is available. Moreover, to our knowledge, the results in [13] do not extend to the case of stronger norms.

The outline of the article is the following. In Section 2, we prove Theorem 1.1. In Section 3, we formally present the approximations which link (1.2) to (1.10) and prove Theorem 1.2. In Section 4, we prove Theorem 1.3 and some variants. We finally provide some further comments.

2 On the Cauchy problem

This section aims at proving Theorem 1.1. This part is inspired by the article [4]. We first prove the existence of mild solutions for (1.8), which justifies the computations which will be done in a second step to derive the estimates in Theorem 1.1.

2.1 Mild solutions

Lemma 2.1. *Let us denote by $(S(t))_{t\in\mathbb{R}}$ the free Schrödinger semi-group $e^{-it\omega A}$. Then, for any $T > 0$, for any integer $s \geq 0$, if $\psi^0 \in \mathcal{D}(A^{s/2})$, the function ψ defined for $t \in (0,T)$ by $\psi(t) = S(t)\psi^0$, which is the unique solution of*

$$i\partial_t\psi = \omega A\psi, \quad (t,x) \in (0,T)\times\mathbb{R}, \qquad \psi(0,x) = \psi^0(x), \quad x \in \mathbb{R}, \tag{2.1}$$

belongs to $C([0,T];\mathcal{D}(A^{s/2}))$, and satisfies the following estimates:

$$\|\psi(t)\|_s = \left\|\psi^0\right\|_s, \quad t \in [0,T]. \tag{2.2}$$

Indeed, Lemma 2.1 simply follows from the fact that A is a self-adjoint (unbounded) operator on $L^2(\mathbb{R})$ and thus that $(S(t))_{t\in\mathbb{R}}$ is a semi-group of isometries on $L^2(\mathbb{R})$ and on any $\mathcal{D}(A^{s/2})$ for $s \geq 0$.

Proposition 2.2. *Let k be a nonnegative integer. If $f \in L^\infty((0,T);C_b^k(\mathbb{R}))$ and if the initial data (ψ_e^0,ψ_g^0) belongs to $\mathcal{D}(A^{k/2})^2$, then there exists a unique mild solution $(\psi_e,\psi_g) \in C([0,T];\mathcal{D}(A^{k/2})^2)$ of* (1.8).

Let $\rho > 0$ be such that $\|f\|_{L^\infty((0,T);C_b^k(\mathbb{R}))} \leq \rho$. Then there exists a positive constant $C_{T,\rho}$ such that

$$\|(\psi_e,\psi_g)\|_{C([0,T];\mathcal{D}(A^{k/2})^2)} \leq C_{T,\rho} \left\|(\psi_e^0,\psi_g^0)\right\|_{k\times k}. \tag{2.3}$$

Proof. Let us introduce the space $Y = C([0,T];\mathcal{D}(A^{k/2})^2)$ endowed with the norm

$$\|(\psi_e,\psi_g)\|_Y = \sup_{t\in[0,T]} \left\{ e^{-\lambda t} \|(\psi_e(t),\psi_g(t))\|_{k\times k} \right\},$$

where λ is a positive parameter which will be chosen later on. Remark that Y is obviously a complete space.

The solution of (1.8) is obtained as a mild solution, *i.e.* as a solution of

$$\psi_e(t) = S(t)e^{-i\Omega t/2}\psi_e^0 + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f(s)\psi_g(s)\, ds,$$
$$\psi_g(t) = S(t)e^{i\Omega t/2}\psi_g^0 + i\int_0^t S(t-s)e^{i\Omega(t-s)/2} f(s)\psi_e(s)\, ds.$$

We are thus going to show that this equation has a unique solution in Y, by proving that the operator Ψ defined by

$$\Psi_e(\psi_e,\psi_g)(t) = S(t)e^{-i\Omega t/2}\psi_e^0 + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f(s)\psi_g(s)\, ds,$$
$$\Psi_g(\psi_e,\psi_g)(t) = S(t)e^{i\Omega t/2}\psi_g^0 + i\int_0^t S(t-s)e^{i\Omega(t-s)/2} f(s)\psi_e(s)\, ds,$$

has a unique fixed point in Y.

First, remark that Ψ indeed maps Y into Y, since $(\psi_e^0,\psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and since there exists a constant $c(\rho)$, which only depends on ρ, such that

$$\|f(s)\psi\|_k \leq c(\rho)\|\psi\|_k, \quad \forall s \in [0,T],\ \forall \psi \in \mathcal{D}(A^{k/2}).$$

This, combined with Lemma 2.1, implies that $\Psi : Y \to Y$.

It is then sufficient to prove that Ψ is a strict contraction on Y. Consider then $\psi = (\psi_e,\psi_g)$ and $\phi = (\phi_e,\phi_g)$ in Y. Then

$$\Psi_e(\psi_e,\psi_g)(t) - \Psi_e(\phi_e,\phi_g)(t) = i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f(s)(\psi_g - \phi_g)(s)\, ds,$$

and thus, for all $t \in [0,T]$,

$$
\begin{aligned}
&\|\Psi_e(\psi_e,\psi_g)(t) - \Psi_e(\phi_e,\phi_g)(t)\|_k \\
&\qquad\qquad\qquad \le \int_0^t \left\| S(t-s)e^{-i\Omega(t-s)/2} f(s)(\psi_g - \phi_g)(s) \right\|_k \, ds \\
&\qquad\qquad\qquad \le \int_0^t \|f(s)(\psi_g - \phi_g)(s)\|_k \, ds \\
&\qquad\qquad\qquad \le c(\rho) \int_0^t \|(\psi_g - \phi_g)(s)\|_k \, ds \\
&\qquad\qquad\qquad \le c(\rho) \int_0^t e^{\lambda s} \left(e^{-\lambda s} \|(\psi_g - \phi_g)(s)\|_k \right) ds \\
&\qquad\qquad\qquad \le c(\rho) \int_0^t e^{\lambda s} \|\psi - \phi\|_Y \, ds \\
&\qquad\qquad\qquad \le \frac{c(\rho)}{\lambda} e^{\lambda t} \|\psi - \phi\|_Y .
\end{aligned}
$$

The same can be done for Ψ_g. This yields the following estimate:

$$
\|\Psi(\psi) - \Psi(\phi)\|_Y \le \frac{2c(\rho)}{\lambda} \|\psi - \phi\|_Y .
$$

Then, choosing $\lambda = 4c(\rho)$, the map Ψ is a strict contraction on Y, and therefore has a unique fixed point in Y, which coincides, by construction, with the solution of (1.8) in $C([0,T];\mathcal{D}(A^{k/2})^2)$. □

Remark 2.3. In Proposition 2.2, we do not require f to be real-valued. Though, this assumption, assumed in Theorem 1.1, will be used later on to derive *a priori* estimates for solutions of (1.8).

Proposition 2.4. *If $f \in L^\infty((0,T);C_b(\mathbb{R}))$ is a real-valued function and if the initial data (ψ_e^0,ψ_g^0) belongs to $(L^2(\mathbb{R}))^2$, then the mild solution (ψ_e,ψ_g) of* (1.8) *satisfies:*

$$
\int_{\mathbb{R}} \left(|\psi_e(t)|^2 + |\psi_g(t)|^2 \right) dx = \int_{\mathbb{R}} \left(|\psi_e^0|^2 + |\psi_g^0|^2 \right) dx, \quad t \in [0,T]. \tag{2.4}
$$

Proof. The proof strongly uses the assumption that f is real-valued, and is divided into several steps. We first prove Proposition 2.4 for smooth initial data and potential f. We then develop a standard density argument to extend this result to functions $f \in L^\infty((0,T);C_b(\mathbb{R}))$ and initial data in $L^2(\mathbb{R})^2$.

We first assume that (ψ_e^0,ψ_g^0) belongs to $\mathcal{D}(A)^2$ and that $f \in L^\infty((0,T);C_b^2(\mathbb{R}))$. Note that, in this case, the computations below are justified due to the regularity of the solutions of (1.8) proved in Proposition 2.2. Indeed, since the mild solution (ψ_e,ψ_g) of (1.8) belongs to $C([0,T];\mathcal{D}(A)^2)$, for all $t \in [0,T]$, $A\psi_e(t)$ and $A\psi_g(t)$ belong to $L^2(\mathbb{R})$.

Let us now prove (2.4). Multiplying the first line of (1.8) by ψ_e^*, we get, for $t \in [0, T]$,

$$i \int_{\mathbb{R}} \partial_t \psi_e(t) \psi_e^*(t) \, dx = \frac{\omega}{2} \int_{\mathbb{R}} \Big(|\partial_x \psi_e(t)|^2 + x^2 |\psi_e(t)|^2 \Big) \, dx \\ + \frac{\Omega}{2} \int_{\mathbb{R}} |\psi_e(t)|^2 \, dx + \int_{\mathbb{R}} f(t) \psi_g(t) \psi_e^*(t) \, dx.$$

Taking the imaginary part, we obtain:

$$\frac{1}{2} \frac{d}{dt} \Big(\int_{\mathbb{R}} |\psi_e(t)|^2 \, dx \Big) = \Im \Big(\int_{\mathbb{R}} f(t) \psi_g(t) \psi_e^*(t) \, dx \Big), \quad t \in [0, T].$$

Similarly, multiplying the second line of (1.8) by $\psi_g^*(t)$ and taking the imaginary part yield:

$$\frac{1}{2} \frac{d}{dt} \Big(\int_{\mathbb{R}} |\psi_g(t)|^2 \, dx \Big) = \Im \Big(\int_{\mathbb{R}} f(t) \psi_e(t) \psi_g^*(t) \, dx \Big), \quad t \in [0, T].$$

Therefore, we obtain that, for $t \in [0, T]$,

$$\frac{1}{2} \frac{d}{dt} \Big(\int_{\mathbb{R}} \Big(|\psi_e(t)|^2 + |\psi_g(t)|^2 \Big) \, dx \Big) = \Im \Big(\int_{\mathbb{R}} f(t) [\psi_g(t) \psi_e^*(t) + \psi_e(t) \psi_g^*(t)] \, dx \Big).$$

As f is real-valued, this implies

$$\frac{d}{dt} \Big(\int_{\mathbb{R}} \Big(|\psi_e(t)|^2 + |\psi_g(t)|^2 \Big) \, dx \Big) = 0, \quad \forall t \in [0, T]. \tag{2.5}$$

We now assume that $f \in L^\infty((0, T); C_b(\mathbb{R}))$ and that $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A)^2$.

Choose $\zeta \in C_c^\infty(\mathbb{R})$ such that for all $x \in \mathbb{R}$, $\zeta(x) \geq 0$ and $\int_{\mathbb{R}} \zeta(x) \, dx = 1$. For $\epsilon > 0$, define the regularization function

$$\zeta^\epsilon(x) = \frac{1}{\epsilon} \zeta \Big(\frac{x}{\epsilon} \Big).$$

Now, introduce, for $\epsilon > 0$, the function $f^\epsilon = f \star \zeta_\epsilon$, where the convolution is meant in the space variable. Remark that, with this definition, for each $\epsilon > 0$, f^ϵ is in $L^\infty((0, T); C_b^2(\mathbb{R}))$, and then (2.4) holds for solutions of (1.8) corresponding to f^ϵ with initial data in $\mathcal{D}(A)^2$.

Consider $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A)^2$. Define (ψ_e, ψ_g) as the mild solution of (1.8) with initial data (ψ_e^0, ψ_g^0). For $\epsilon > 0$, introduce the mild solution $(\psi_e^\epsilon, \psi_g^\epsilon) \in C([0, T]; (L^2(\mathbb{R}))^2)$ of (1.8) corresponding to f^ϵ with initial data (ψ_e^0, ψ_g^0). We thus have:

$$\psi_e^\epsilon(t) = S(t)e^{-i\Omega t/2}\psi_e^0 + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f^\epsilon(s)\psi_g^\epsilon(s)\ ds,$$
$$\psi_g^\epsilon(t) = S(t)e^{i\Omega t/2}\psi_g^0 + i\int_0^t S(t-s)e^{i\Omega(t-s)/2} f^\epsilon(s)\psi_e^\epsilon(s)\ ds,$$
$$\psi_e(t) = S(t)e^{-i\Omega t/2}\psi_e^0 + i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f(s)\psi_g(s)\ ds,$$
$$\psi_g(t) = S(t)e^{i\Omega t/2}\psi_g^0 + i\int_0^t S(t-s)e^{i\Omega(t-s)/2} f(s)\psi_e(s)\ ds.$$

In particular,

$$\psi_e^\epsilon(t) - \psi_e(t) = i\int_0^t S(t-s)e^{-i\Omega(t-s)/2} f^\epsilon(s)(\psi_g^\epsilon(s) - \psi_g(s))\ ds$$
$$+ i\int_0^t S(t-s)e^{-i\Omega(t-s)/2}(f^\epsilon(s) - f(s))\psi_g(s)\ ds.$$

We thus obtain

$$\|\psi_e^\epsilon(t) - \psi_e(t)\|_0 \le \|f\|_{L^\infty((0,T)\times\mathbb{R})}\int_0^t \left\|\psi_g^\epsilon(s) - \psi_g(s)\right\|_0\ ds$$
$$+ \int_0^t \|(f^\epsilon(s) - f(s))\psi_g(s)\|_0\ ds$$

Doing the same estimate for $\psi_g^\epsilon(t) - \psi_g(t)$, we obtain that for $t \in [0,T]$,

$$\left\|(\psi_e^\epsilon(t), \psi_g^\epsilon(t)) - (\psi_e(t), \psi_g(t))\right\|_{0\times 0}$$
$$\le \|f\|_{L^\infty((0,T)\times\mathbb{R})}\int_0^t \left\|(\psi_e^\epsilon(s), \psi_g^\epsilon(s)) - (\psi_e(s), \psi_g(s))\right\|_{0\times 0}\ ds$$
$$+ \int_0^T \|(f^\epsilon(s) - f(s))(\psi_e(s), \psi_g(s))\|_{0\times 0}\ ds. \quad (2.6)$$

But, in any time $s \in [0,T]$, $(\psi_e(s), \psi_g(s)) \in (L^2(\mathbb{R}))^2$ and almost everywhere in $s \in [0,T]$, $f(s) \in C_b^0(\mathbb{R})$.

Recall that, for $g \in C_b^0(\mathbb{R})$, the sequence $(g^\epsilon) = (g\star\zeta_\epsilon)_{\epsilon>0}$ strongly converges to g in $C^0(K)$ for any compact K (see [12, Proposition IV.21]). It follows that, if $\psi \in L^2(\mathbb{R})$, the sequence $(g^\epsilon\psi)_{\epsilon>0}$ strongly converges in $L^2(\mathbb{R})$ to $g\psi$.

Hence, almost everywhere in $s \in [0,T]$, $\|(f^\epsilon(s) - f(s))(\psi_e(s), \psi_g(s))\|_{0\times 0}$ converges to zero. We finally use Lebesgue's dominated convergence theorem to prove that

$$\int_0^T \|(f^\epsilon(s) - f(s))(\psi_e(s), \psi_g(s))\|_{0\times 0}\ ds \underset{\epsilon\to 0}{\longrightarrow} 0,$$

since, for $s \in [0,T]$,

$$\|(f^\epsilon(s) - f(s))(\psi_e(s), \psi_g(s))\|_{0\times 0} \leq 2\,\|f\|_{L^\infty((0,T)\times\mathbb{R})}\,\|(\psi_e(s), \psi_g(s))\|_{0\times 0}\,.$$

Applying Grönwall's Lemma to (2.6), we then obtain

$$(\psi_e^\epsilon, \psi_g^\epsilon) \underset{\epsilon\to 0}{\longrightarrow} (\psi_e, \psi_g) \quad \text{in } C([0,T];(L^2(\mathbb{R}))^2).$$

In particular, passing to the limit in ϵ in (2.4), estimate (2.4) holds for mild solutions of (1.8) with initial data in $\mathcal{D}(A)^2$ for $f \in L^\infty(0,T;C_b^0(\mathbb{R}))$.

The same conclusion holds for $(\psi_e^0, \psi_g^0) \in L^2(\mathbb{R})^2$, using the standard density argument of $\mathcal{D}(A)$ in $L^2(\mathbb{R})$ for the L^2 norm. Indeed, for $(\psi_e^0, \psi_g^0) \in (L^2(\mathbb{R}))^2$, consider a sequence $(\psi_{ne}^0, \psi_{ng}^0) \in \mathcal{D}(A)^2$ such that

$$(\psi_{ne}^0, \psi_{ng}^0) \underset{n\to\infty}{\longrightarrow} (\psi_e^0, \psi_g^0) \quad \text{in } (L^2(\mathbb{R}))^2.$$

Then, denoting by (ψ_{ne}, ψ_{ng}) and (ψ_e, ψ_g) the corresponding mild solutions, arguing as above, one can prove that

$$(\psi_{ne}, \psi_{ng}) \underset{n\to\infty}{\longrightarrow} (\psi_e, \psi_g) \quad \text{in } C([0,T];(L^2(\mathbb{R}))^2).$$

This completes the proof, since one can then pass to the limit in (2.4). $\square$

Theorem 1.1 then simply combines the results of Propositions 2.2 and 2.4.

3 Law-Eberly equations

3.1 Preliminaries

This subsection presents several standard results in quantum mechanics.

The operator A introduced in (1.1) is self-adjoint, positive definite, and with compact resolvent. Besides, its spectral decomposition is well-known. Indeed (see [24]), the eigenvalues of A are $\lambda_n = n+1/2$ for $n \in \mathbb{N}$, and the corresponding eigenvectors Φ_n normalized in $L^2(\mathbb{R})$ form an orthonormal basis of $L^2(\mathbb{R})$ (the so-called Hermite functions).

Besides, from the identities

$$A = \mathbf{a}^\dagger\mathbf{a} + \frac{1}{2} = \mathbf{a}\mathbf{a}^\dagger - \frac{1}{2}, \tag{3.1}$$

and the explicit forms of $\mathbf{a}$ and $\mathbf{a}^\dagger$, one can prove that the operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ act on the eigenvectors of A in the following way:

$$\mathbf{a}\Phi_0 = 0, \quad \begin{cases} \mathbf{a}\Phi_{n+1} &= \sqrt{n+1}\;\Phi_n, \\ \mathbf{a}^\dagger\Phi_n &= \sqrt{n+1}\;\Phi_{n+1}, \end{cases} \quad \forall n \in \mathbb{N}. \tag{3.2}$$

3.2 From system (1.2) to (1.10)

Let us now briefly explain the approximations and change of variables which yield from (1.2)-(1.4) to (1.10). This is done in a formal way in a first step.

First, since the Lamb-Dicke parameter η is small, $\mathbf{u}$ is approximated by $\mathbf{u}_{LD}$ given by

$$\mathbf{u}_{LD}(t,x) = \Big(u_0 e^{i\Omega t} + u_r e^{i(\Omega-\omega)t} + u_b e^{i(\Omega+\omega)t}\Big)(1 - i\sqrt{2}\eta x). \tag{3.3}$$

Then we make the change of variables

$$\tilde{\phi}_e(t) = S(-t)e^{i\Omega t/2}\tilde{\psi}_e(t), \qquad \tilde{\phi}_g(t) = S(-t)e^{-i\Omega t/2}\tilde{\psi}_g(t),$$

where $S(t) = \exp(-it\omega A)$ is the free Schrödinger group and $(\tilde{\psi}_e, \tilde{\psi}_g)$ is the solution of (1.8) with $f = \mathbf{u}_{LD} + \mathbf{u}_{LD}^*$.

In these variables, we obtain the following equations:

$$\begin{cases} i\partial_t\tilde{\phi}_e = e^{i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)\tilde{\phi}_g, & (t,x) \in (0,T)\times\mathbb{R}, \\ i\partial_t\tilde{\phi}_g = e^{-i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)\tilde{\phi}_e, & (t,x) \in (0,T)\times\mathbb{R}, \\ \tilde{\phi}_e(0,x) = \psi_e^0(x), \quad \tilde{\phi}_g(0,x) = \psi_g^0(x), & x \in \mathbb{R}. \end{cases} \tag{3.4}$$

In physical language, system (3.4) corresponds to the so-called interaction frame for system (1.2) with the Lamb-Dicke approximation (3.3).

Then we compute the operator $S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)$. Due to the form of $\mathbf{u}_{LD}$, we actually need to compute $\exp(it\omega A)\sqrt{2}\,x\exp(-it\omega A)$, or, equivalently, $\exp(it\omega A)(\mathbf{a}+\mathbf{a}^\dagger)\exp(-it\omega A)$. Using the identities (3.2), one easily proves that

$$\exp(it\omega A)(\mathbf{a}+\mathbf{a}^\dagger)\exp(-it\omega A) = e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger.$$

Thus, with $\mathbf{u}_{LD}$ as in (3.3), we obtain

$$\begin{aligned} &e^{i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t) \\ &= u_0 e^{2i\Omega t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) + u_0^*\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\ &+ u_r e^{i(2\Omega-\omega)t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) + u_r^* e^{i\omega t}\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) \\ &+ u_b e^{i(2\Omega+\omega)t}\Big(1 - i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big) + u_b^* e^{-i\omega t}\Big(1 + i\eta\Big(e^{-i\omega t}\mathbf{a} + e^{i\omega t}\mathbf{a}^\dagger\Big)\Big). \end{aligned} \tag{3.5}$$

The last approximation, the so-called averaging one, consists in neglecting all the (highly) oscillating terms. In our setting, this yields the following approximation:

$$e^{i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t) \simeq u_0^* + i\eta u_r^*\mathbf{a} + i\eta u_b^*\mathbf{a}^\dagger.$$

Thus, doing the same for $e^{-i\Omega t}S(-t)(\mathbf{u}_{LD} + \mathbf{u}_{LD}^*)S(t)$, we obtain the Law-Eberly system (1.10), by setting, as claimed in the introduction, $v_r = -i\eta u_r$ and $v_b = -i\eta u_b$.

3.3 Controllability results for the Law-Eberly equations

The goal of this subsection is to prove Theorem 1.2.

Before entering into the proof, remark that, in [17] or in the proof presented below, only the two controls u_0 and v_r are used.

In [17], the method, roughly speaking, consists in steering the data to the ground state $(0, \Phi_0)$, the time reversibility of (1.10) yielding Theorem 1.2.

At this step, it is essential to remark that system (1.10) is skew-adjoint, and thus that the $(L^2(\mathbb{R}))^2$ norm of solutions of (1.10) (at least those that are sufficiently regular, which is the case here since at each time s, $(\phi_e(s), \phi_g(s)) \in V_M^2$) is constant in time.

Since Theorem 1.2 is fundamental for our results, we give here a brief idea of its proof.

Sketch of the proof. When u_0 is the only active control (we recall that u_0 has to be constant), system (1.10) takes the form

$$i\partial_t \phi_e = u_0^* \phi_g, \quad i\partial_t \phi_g = u_0 \phi_e, \quad (t, x) \in (0, T) \times \mathbb{R}. \tag{3.6}$$

Expand $(\phi_e^0, \phi_g^0) \in V_M^2$ on the basis Φ_j:

$$\phi_e^0 = \sum_{j \leq M} a_j^0 \Phi_j, \quad \phi_g^0 = \sum_{j \leq M} b_j^0 \Phi_j. \tag{3.7}$$

Solving (3.6), we obtain

$$\phi_e(t) = \sum_{j \leq M} a_j(t) \Phi_j, \quad \phi_g(t) = \sum_{j \leq M} b_j(t) \Phi_j, \tag{3.8}$$

where, for $j \in \{0, \cdots, M\}$,

$$a_j(t) = \cos(|u_0|t) a_j^0 - i \sin(|u_0|t) \frac{u_0^*}{|u_0|} b_j^0,$$

$$b_j(t) = \cos(|u_0|t) b_j^0 - i \sin(|u_0|t) \frac{u_0}{|u_0|} a_j^0.$$

In other words, equations (3.6) are constituted by decoupled systems corresponding to the projections onto (Φ_j, Φ_j), and the ratio of populations at the energy level Φ_j oscillates with a frequency $|u_0|$.

When v_r is the only active control, system (1.10) takes the form

$$i\partial_t \phi_e = v_r^* \mathbf{a} \phi_g, \quad i\partial_t \phi_g = v_r \mathbf{a}^\dagger \phi_e, \quad (t, x) \in (0, T) \times \mathbb{R}. \tag{3.9}$$

In this case, writing $(\phi_e^0, \phi_g^0) \in V_M^2$ as in (3.7) with the additional assumption that $a_M^0 = 0$, one can solve explicitly (3.9), and the solution of (3.9), expanded as in (3.8), satisfies:

$$\begin{aligned}
a_j(t) &= \cos(t|v_r|\sqrt{j+1}) a_j^0 - i \sin(t|v_r|\sqrt{j+1}) \frac{v_r^*}{|v_r|} b_{j+1}^0, \ 0 \leq j \leq M-1, \\
b_j(t) &= \cos(t|v_r|\sqrt{j}) b_j^0 - i \sin(t|v_r|\sqrt{j}) \frac{v_r}{|v_r|} a_{j-1}^0, \quad 1 \leq j \leq M, \\
b_0(t) &= b_0^0, \quad a_M(t) = 0.
\end{aligned} \tag{3.10}$$

Again, one checks that the energy levels are associated by several coupled 2×2 systems corresponding to the projections onto (Φ_j, Φ_{j+1}), for which the system oscillates at a frequency $|v_r|\sqrt{j+1}$.

Thus, we only have to design, for a given initial data as in (3.7), a sequence of impulses which yields to the ground state $(0, \Phi_0)$. This is actually easy, since one can, in two impulses, steer functions in V_M^2 to functions in V_{M-1}^2.

Indeed, first, we turn on only u_0, during a time τ_0 such that $a_M(\tau_0) = 0$. This can be done by solving

$$|a_M^0| \cos(|u_0|\tau_0) = |b_M^0| \sin(|u_0|\tau_0), \quad \arg(u_0) = \frac{\pi}{2} + \arg(b_M^0) - \arg(a_M^0).$$

This always has a solution for a time $\tau_0 \leq \pi/(2|u_0|)$. Taking $|u_0| = K_0$, which corresponds to the maximal size of the control function, we can thus solve the equation above in a time $\tau_0 \leq \pi/(2K_0)$.

Once this is done, at time τ_0, we turn off the control u_0 and activate v_r during a time τ_r in such a way that $b_M(\tau_0 + \tau_r) = 0$. This again yields to explicit equations which can be solved within a time $\tau_r \leq \pi/(2|v_r|\sqrt{M})$. Taking $|v_r| = K_1$, which again corresponds to the maximal size of the controls, this can be solved in a time $\tau_1 \leq \pi/(2K_1\sqrt{M})$.

It follows that any couple of functions in V_M^2 can be steered to V_{M-1}^2 in a time less $\pi/(2K_0) + \pi/(2K_1\sqrt{M})$. Iterating this process, and using the reversibility of (1.10), one easily checks Theorem 1.2 and estimate (1.13). $\square$

Remark 3.1. Given $K > 0$, in view of the constraints (1.5), we will set in the sequel $K_0 = K$ and $K_1 = \eta K$. Thus, we will choose a time T such that

$$TK \geq \pi(M+1) + \frac{2\pi}{\eta}\sqrt{M}. \tag{3.11}$$

In particular, for $\eta \leq \eta_M = 1/(2\sqrt{M+1})$, (3.11) holds for

$$TK = \frac{\aleph}{\eta}, \quad \text{with } \aleph = 3\pi\sqrt{M}. \tag{3.12}$$

Remark 3.2. We do not know if the strategy proposed above minimizes the norm of the control in general. In particular, here we did not use the control v_b, which might help to obtain better time estimates. This question is, to our knowledge, widely open.

4 Approximate controllability for (1.2)

This section aims at proving Theorem 1.3. Our proof is divided into two main steps. The first one is devoted to derive precise estimates on the approximation process of Subsection 3.2. The second one proves the result of Theorem 1.3. We then give several extensions of Theorem 1.3, which derive approximate controllability results in the stronger norms $\|(\cdot,\cdot)\|_{k\times k}$.

In the sequel, to simplify notations, we will denote without ambiguity $\int_{\mathbb{R}} g(x)\, dx$ by $\int_{\mathbb{R}} g$.

4.1 Approximate controllability in $(L^2(\mathbb{R}))^2$

Fix an integer $M \geq 1$ and consider initial and target data in V_M^2.

The sizes of the parameters η, ω, Ω, K and T are not fixed *a priori*, and will be chosen later on. Below, we will make the dependence in these parameters as explicit as possible. In particular, C will denote a generic constant which does not depend on any of these parameters.

Fix an initial state

$$\psi_e^0 = \sum_{j \leq M} a_j^0 \Phi_j, \quad \psi_g^0 = \sum_{j \leq M} b_j^0 \Phi_j, \tag{4.1}$$

and a target state

$$\psi_e^1 = \sum_{j \leq M} a_j^1 \Phi_j, \quad \psi_g^1 = \sum_{j \leq M} b_j^1 \Phi_j, \tag{4.2}$$

with same $(L^2)^2$ norm, say 1.

We assume that the control impulses u_0, $v_b = -i\eta u_b$ and $v_r = -i\eta u_r$ satisfy, for some constant $K > 0$,

$$|u_0| \leq K, \qquad |v_b| \leq K\eta, \quad |v_r| \leq K\eta. \tag{4.3}$$

In the sequel, we assume that $\eta \leq \eta_M = 1/(2\sqrt{M+1})$.

Then Theorem 1.2 (see also Remark 3.1) guarantees that, for $\eta \leq \eta_M$, in a time T as in (3.12), for any couples of functions (ϕ_e^0, ϕ_g^0) and (ϕ_e^1, ϕ_g^1) in V_M^2, there exists a control function

$$t \mapsto (u_0(t), v_r(t), v_b(t)), \tag{4.4}$$

which corresponds to a sequence of impulses and satisfies the properties of Theorem 1.2, such that the solution (ϕ_e, ϕ_g) of (1.10) with initial data (ϕ_e^0, ϕ_g^0) satisfies

$$(\phi_e(T), \phi_g(T)) = \beta(\phi_e^1, \phi_g^1), \tag{4.5}$$

where β is a complex number of modulus 1.

We then set

$$(\phi_e^1, \phi_g^1) = \big(S(-T)\exp(i\Omega T/2)\psi_e^1, S(-T)\exp(-i\Omega T/2)\psi_g^1\big), \tag{4.6}$$

which belongs to V_M^2 when (ψ_e^1, ψ_g^1) belongs to V_M^2, and which is of unit $(L^2)^2$ norm. From Theorem 1.2, one can then choose a control function as in (4.4) satisfying (4.3) which steers solutions of (1.10) from $(\phi_e^0, \phi_g^0) = (\psi_e^0, \psi_g^0)$ to (ϕ_e^1, ϕ_g^1) defined in (4.6) in time T up to complex number β of modulus 1 as in (4.5).

Now, set $u_r(t) = iv_r(t)/\eta$ and $u_b(t) = iv_b(t)/\eta$, and define $\mathbf{u}$ as in (1.4). Then consider the solution (ψ_e, ψ_g) of (1.2) with initial data (ψ_e^0, ψ_g^0) as in (4.1). Our goal is to prove that

$$(\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1)$$

is small.

Theorem 4.1 (Approximate Controllability in V_M^2 in $(L^2(\mathbb{R}))^2$ norm)**.** *Let M be a positive integer, and consider two couples of data (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in V_M^2 of unit $(L^2(\mathbb{R}))^2$ norms.*

For $\eta \in (0, \eta_M)$, let K and T be such that $KT = 3\pi\sqrt{M}/\eta$ as in (3.12). Consider the control function $t \mapsto (u_0, v_r, v_b)$ given by Theorem 1.2 under the constraints (4.3), which steers solutions of (1.10) from (ψ_e^0, ψ_g^0) to (ϕ_e^1, ϕ_g^1) (defined by (4.6)) up to a complex number β of modulus 1 as in (4.5).

Set $u_r = iv_r/\eta$ and $u_b = iv_b/\eta$, and consider the control function $\mathbf{u}$ as defined in (1.4).

Then, for any $\delta > 0$, there exist $\eta_0 \in (0, \eta_M)$ and $\rho_0 > 0$ such that for (ω, Ω) as in (1.6), for

$$0 < \eta \leq \eta_0, \qquad KT = \frac{3\pi\sqrt{M}}{\eta}, \qquad \frac{\omega\eta}{K} \geq \rho_0,$$

the solution (ψ_e, ψ_g) of (1.2) with initial data (ψ_e^0, ψ_g^0) and the above defined control $\mathbf{u}$ satisfies (1.15).

Remark 4.2. Note that the controlled trajectory $t \mapsto (\psi_e(t), \psi_g(t))$ does not stay in V_M^2 but, as one can check following the proof, it stays always close (in $(L^2)^2$ norm) to an element of V_M^2.

Proof of Theorem 4.1. Before going into the proof, we shall introduce the functions

$$f = f(t,x) = \mathbf{u} + \mathbf{u}^*, \qquad f_{LD} = f_{LD}(t,x) = \mathbf{u}_{LD} + \mathbf{u}_{LD}^*, \tag{4.7}$$

where $\mathbf{u}_{LD}$ is as in (3.3). To simplify notations, we will often omit the dependence in x of these functions and simply write $f(t)$ and $f_{LD}(t)$.

We also define the operator

$$f_{LE} = f_{LE}(t) = \Big(u_0 - i\eta u_r \mathbf{a}^\dagger - i\eta u_b \mathbf{a}\Big), \tag{4.8}$$

where $\mathbf{a}$ and $\mathbf{a}^\dagger$ are defined by (1.12) and its adjoint operator $f_{LE}^\dagger = f_{LE}^\dagger(t)$.

Let (ψ_e, ψ_g) be the solution of (1.2) with initial data (ψ_e^0, ψ_g^0) for $\mathbf{u}$ being the one given by Theorem 4.1.

Set

$$\xi_e(t) = S(-t)e^{i\Omega t/2}\psi_e(t), \qquad \xi_g(t) = S(-t)e^{-i\Omega t/2}\psi_g(t).$$

Then (ξ_e, ξ_g) is the solution of

$$\begin{cases} i\partial_t \xi_e = e^{i\Omega t} S(-t) f(t) S(t) \xi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \xi_g = e^{-i\Omega t} S(-t) f(t) S(t) \xi_e, & (t,x) \in (0,T) \times \mathbb{R}, \end{cases} \tag{4.9}$$

with initial data $(\xi_e(0), \xi_g(0)) = (\psi_e^0, \psi_g^0)$.

Consider (ϕ_e, ϕ_g) the solution of (1.10) with initial data (ϕ_e^0, ϕ_g^0) for the control function (4.4) computed from Theorem 1.2, which steers (ϕ_e^0, ϕ_g^0) to (ϕ_e^1, ϕ_g^1) as in (4.6).

Note that, from (4.6),

$$\begin{aligned} \left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{0\times 0} &= \left\| (\xi_e(T), \xi_g(T)) - \beta(\phi_e^1, \phi_g^1) \right\|_{0\times 0} \\ &= \left\| (\xi_e(T), \xi_g(T)) - (\phi_e(T), \phi_g(T)) \right\|_{0\times 0}. \end{aligned}$$

We thus directly work on (ξ_e, ξ_g), which we will compare with (ϕ_e, ϕ_g).

Recall that (ϕ_e, ϕ_g) satisfies

$$\begin{cases} i\partial_t \phi_e = f_{LE}(t)^\dagger \phi_g, & (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \phi_g = f_{LE}(t) \phi_e, & (t,x) \in (0,T) \times \mathbb{R}. \end{cases}$$

We therefore introduce the functions

$$\epsilon_e = \epsilon_e(t,x) = \xi_e - \phi_e, \qquad \epsilon_g = \epsilon_g(t,x) = \xi_g - \phi_g, \tag{4.10}$$

which satisfy

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{0\times 0} = \left\| (\epsilon_e(T), \epsilon_g(T)) \right\|_{0\times 0}.$$

Besides, the functions ϵ_e, ϵ_g satisfy the following system:

$$\begin{cases} i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g + e^{i\Omega t} S(-t)(f(t) - f_{LD}(t)) S(t) \phi_g \\ \qquad + \Big(e^{i\Omega t} S(-t) f_{LD}(t) S(t) - f_{LE}^\dagger(t) \Big) \phi_g, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e + e^{-i\Omega t} S(-t)(f(t) - f_{LD}(t)) S(t) \phi_e \\ \qquad + \Big(e^{-i\Omega t} S(-t) f_{LD}(t) S(t) - f_{LE}(t) \Big) \phi_g, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ \epsilon_e(0) = 0, \quad \epsilon_g(0) = 0. \end{cases} \tag{4.11}$$

From Theorem 1.2, the functions ϕ_e and ϕ_g belong to V_M^2 in any time $t \geq 0$. Moreover, from Proposition 2.2, the functions ψ_e and ψ_g, which are solutions of (1.2), are in $\cap_{l>0} \mathcal{D}(A^l)$ in any time $t \geq 0$, and so are ξ_e and ξ_g. Then the functions ϵ_e and ϵ_g inherit the same regularity: this justifies all the computations below.

Define, for $t \in [0,T]$, the following functions

$$\begin{aligned} h_{LDe}(t,x) &= e^{i\Omega t} S(-t)(f(t,x) - f_{LD}(t,x)) S(t) \phi_g(t,x), \\ h_{me}(t,x) &= \int_0^t \Big(e^{i\Omega s} S(-s) f_{LD}(s,x) S(s) - f_{LE}(s)^\dagger \Big) \phi_g(s,x)\, ds, \\ h_{LDg}(t,x) &= e^{-i\Omega t} S(-t)(f(t,x) - f_{LD}(t,x)) S(t) \phi_e(t,x), \\ h_{mg}(t,x) &= \int_0^t \Big(e^{-i\Omega s} S(-s) f_{LD}(s,x) S(s) - f_{LE}(s) \Big) \phi_e(s,x)\, ds. \end{aligned} \tag{4.12}$$

System (4.11) then reads as

$$\begin{cases} i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g + h_{LDe}(t,x) + \partial_t h_{me}(t,x), \\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e + h_{LDg}(t,x) + \partial_t h_{mg}(t,x). \end{cases} \tag{4.13}$$

Below, we give precise estimates for the quantities in (4.12), which will be needed to bound the norm of (ϵ_e, ϵ_g).

Lamb-Dicke approximation. To deal with the Lamb-Dicke approximation, roughly speaking measured by h_{LDe} and h_{LDg} in (4.12), one has essentially to estimate the norm of the multiplication operator $f(t,x) - f_{LD}(t,x)$. Note that, due to the explicit form of $f(t,x) - f_{LD}(t,x)$, one can check that $g_{LD}(t,x) = f(t,x) - f_{LD}(t,x)$ is a smooth function in x, for which there exists a constant C such that for all (t,x),

$$|g_{LD}(t,x)| \leq C\eta^2 K |x|^2, \qquad |\partial_x g_{LD}(t,x)| \leq C\eta^2 K |x|,$$
$$\forall k \geq 2, \quad |\partial_x^k g_{LD}(t,x)| \leq C\eta^k K.$$

It follows that for all $t \geq 0$,

$$\|g_{LD}(t,x)\phi\|_k \leq C(k)\eta^2 K \|\phi\|_{k+2}.$$

Now, remark that $\phi_g(t)$ and $\phi_e(t)$ both belong to V_M for any $t > 0$, and thus are smooth. Furthermore, for any $\phi \in V_M$, we have

$$\|\phi\|_k = \left\| A^{k/2}\phi \right\|_0 \leq (M+1)^{k/2} \|\phi\|_0. \tag{4.14}$$

Also recall that $S(t)$ is a unitary map from V_M to V_M and from $\mathcal{D}(A^{k/2})$ to $\mathcal{D}(A^{k/2})$. It follows that there exists a constant $C_1(k)$, which depends only on k, such that

$$\begin{aligned} \sup_{t\in[0,T]} \|h_{LDe}(t)\|_k &\leq C_1(k)\eta^2 K (M+1)^{(k+2)/2}, \\ \sup_{t\in[0,T]} \|h_{LDg}(t)\|_k &\leq C_1(k)\eta^2 K (M+1)^{(k+2)/2}, \end{aligned} \tag{4.15}$$

since $\|(\phi_e(t), \phi_g(t))\|_{0\times 0} = \|(\phi_e(0), \phi_g(0))\|_{0\times 0} = 1$ for all $t > 0$.

The mean approximation. Let us now focus on the integrals h_{me} and h_{mg} in (4.12). According to (3.5), we define the operator $g_m = g_m(s)$ by

$$\begin{aligned} g_m^\dagger &= g_m^\dagger(s) = e^{i\Omega s} S(-s) f_{LD}(s) S(s) - f_{LE}^\dagger(s) \\ &= u_0 e^{2i\Omega s} - i\eta u_0 \Big(e^{i(2\Omega-\omega)s}\mathbf{a} + e^{i(2\Omega+\omega)s}\mathbf{a}^\dagger \Big) + i\eta u_0^* \Big(e^{-i\omega s}\mathbf{a} + e^{i\omega s}\mathbf{a}^\dagger \Big) \\ &+ u_r e^{i(2\Omega-\omega)s} - i\eta u_r \Big(e^{2i(\Omega-\omega)s}\mathbf{a} + e^{2i\Omega s}\mathbf{a}^\dagger \Big) + u_r^* e^{i\omega s} + i\eta u_r^* e^{2i\omega s}\mathbf{a}^\dagger \\ &+ u_b e^{i(2\Omega+\omega)s} - i\eta u_b \Big(e^{2i\Omega s}\mathbf{a} + e^{2i(\Omega+\omega)s}\mathbf{a}^\dagger \Big) + u_b^* e^{-i\omega s} + i\eta u_b^* e^{-2i\omega s}\mathbf{a}. \end{aligned} \tag{4.16}$$

Recall that for any $t \geq 0$, $(\phi_e(t), \phi_g(t)) \in V_M^2$ and that it is explicitly given by the construction in Theorem 1.2. In particular, they are functions of time which oscillate at a frequency $\sup\{K, K\eta\sqrt{M}\} = K$ at most. To be more precise, there is a sequence of times

$$0 = T_0 \leq T_1 \leq \cdots \leq T_{4M+2} = T,$$

which corresponds to the switching times of the controls, such that the functions $\phi_e(t)$ and $\phi_g(t)$ for $t \in (T_l, T_{l+1})$ are linear combination of complex exponential $\exp(\pm iFt)$ with F smaller than K. For instance, if the only active control in (T_l, T_{l+1}) is v_r and

$$\phi_e(T_l) = \sum_{j \leq M-1} a_j^0 \Phi_j, \qquad \phi_g(T_l) = \sum_{j \leq M} b_j^0 \Phi_j,$$

then $(\phi_e(t + T_l), \phi_g(t + T_l))$, expanded as in (3.8), is explicitly given by (3.10).

Our goal is to estimate

$$\left\| \int_{T_l}^{T_{l+1}} g_m(s)^\dagger \phi_g(s)\ ds \right\|_k .$$

Due to the form of g_m, we first focus on

$$\begin{aligned}\int_{T_l}^{T_{l+1}} e^{i\omega s} \phi_g(s)\ ds &= e^{i\omega T_l} \int_0^{T_{l+1}-T_l} e^{i\omega t} \phi_g(t + T_l)\ dt \\ &= e^{i\omega T_l} \Big[\sum_{1\ \leq j \leq M} \Big(\int_0^{T_{l+1}-T_l} \cos(t|v_r|\sqrt{j}) e^{i\omega t}\ dt \Big) b_j^0 \Phi_j \\ - i \Big(\int_0^{T_{l+1}-T_l} &\sin(t|v_r|\sqrt{j}) e^{i\omega t}\ dt \Big) \frac{v_r}{|v_r|} a_{j-1}^0 \Phi_j \Big] + e^{i\omega T_l} \Big(\int_0^{T_{l+1}-T_l} e^{i\omega t}\ dt \Big) b_0^0\end{aligned}$$

Thus for $\omega > K$ (recall that $|v_r|\sqrt{j} \leq K\eta M \leq K$ since $\eta \leq \eta_M$), explicit computations give

$$\left\| \int_{T_l}^{T_{l+1}} e^{i\omega s} \phi_g(s)\ ds \right\|_k \leq C \frac{1}{(\omega - K)} \left\| (\phi_e(T_l), \phi_g(T_l)) \right\|_{k \times k} .$$

But $(\phi_e(T_l), \phi_g(T_l)) \in V_M^2$, and thus estimate (4.14) holds. Besides, $(\phi_e(T_l), \phi_g(T_l))$ is of the same $L^2(\mathbb{R})^2$ norm as (ϕ_e^0, ϕ_g^0), which we assumed to be 1. Consequently,

$$\left\| \int_{T_l}^{T_{l+1}} e^{i\omega s} \phi_g(s)\ ds \right\|_k \leq \frac{C}{(\omega - K)} (M+1)^{k/2}.$$

Similarly,

$$\begin{aligned}\left\| \int_{T_l}^{T_{l+1}} e^{2i\omega s} \mathbf{a}^\dagger \phi_g(s)\ ds \right\|_k &\leq \left\| \int_{T_l}^{T_{l+1}} e^{i\omega s} \phi_g(s)\ ds \right\|_{k+1} \\ &\leq \frac{C}{(\omega - K)} (M+1)^{(k+1)/2}.\end{aligned}$$

Doing the same computations for the other oscillating terms in (4.16), one can obtain

$$\left\| \int_{T_l}^{T_{l+1}} g_m(s)^\dagger \phi_g(s)\ ds \right\|_k \leq C \frac{K}{(\omega - K)} (M+1)^{(k+1)/2}.$$

(Notice that under condition (1.6), the slowest oscillating terms in (4.16) are indeed the ones oscillating at frequency ω.)

The same estimates can be obtained when u_0 is the only active control. The computations actually are easier, and are left to the reader.

Since there are at most $4M+2$ switching times, it follows that there exists a constant $C_2(k)$, which depends only on k, such that

$$\sup_{t\in[0,T]} \|h_{me}(t)\|_k \leq C_2(k) \frac{K}{(\omega - K)} (M+1)^{(k+3)/2}, \tag{4.17}$$

and, similarly, that,

$$\sup_{t\in[0,T]} \|h_{mg}(t)\|_k \leq C_2(k) \frac{K}{(\omega - K)} (M+1)^{(k+3)/2}. \tag{4.18}$$

Approximate controllablity. Using estimates (4.15) and (4.17)-(4.18) derived above, we will prove that the norm of (ϵ_e, ϵ_g) is small. This will be done by energy techniques. Note that Grönwall's estimates are not sufficient to prove the smallness of the norm of (ϵ_e, ϵ_g) since the leading term can only be bounded by K, while KT blows up when $\eta \to 0$.

Multiplying in (4.13) the first equation by ϵ_e^* and the second by ϵ_g^*, and summing them, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\epsilon_e(t)\|_0^2 + \|\epsilon_g(t)\|_0^2 \Big) = \Im\Big(\int_{\mathbb{R}} h_{LDe}(t)\epsilon_e^*(t) + \int_{\mathbb{R}} \partial_t h_{me}(t)\epsilon_e^*(t) + \int_{\mathbb{R}} h_{LDg}(t)\epsilon_g^*(t) + \int_{\mathbb{R}} \partial_t h_{mg}(t)\epsilon_g^*(t)\Big).$$

Integrating in time, we get

$$\frac{1}{2}\Big(\|\epsilon_e(t)\|_0^2 + \|\epsilon_g(t)\|_0^2 \Big) = \Im\Big(\int_0^t \int_{\mathbb{R}} h_{LDe}(s)\epsilon_e^*(s)\ ds + \int_0^t \int_{\mathbb{R}} \partial_t h_{me}(s)\epsilon_e^*(s)\ ds + \int_0^t \int_{\mathbb{R}} h_{LDg}(s)\epsilon_g^*(s)\ ds + \int_0^t \int_{\mathbb{R}} \partial_t h_{mg}(s)\epsilon_g^*(s)\ ds\Big). \tag{4.19}$$

Set, for $t \geq 0$,

$$F_0(t) = \frac{1}{2}\Big(\|\epsilon_e(t)\|_0^2 + \|\epsilon_g(t)\|_0^2 \Big).$$

For the terms corresponding to the Lamb-Dicke approximation, using (4.15), we check that

$$\Big| \int_0^t \int_{\mathbb{R}} h_{LDe}(s)\epsilon_e^*(s)\ ds + \int_0^t \int_{\mathbb{R}} h_{LDg}(s)\epsilon_g^*(s)\ ds \Big| \leq C\eta^2 K(M+1) \int_0^t \sqrt{F_0(s)}\ ds. \tag{4.20}$$

For the mean value approximations, we need to be more careful, since we cannot guarantee $\partial_t h_{me}$ to be small. Though, an integration by parts yields

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \partial_t h_{me}(s)\epsilon_e^*(s)\ ds &= \int_{\mathbb{R}} h_{me}(t)\epsilon_e^*(t) - \int_0^t \int_{\mathbb{R}} h_{me}(s)\partial_t \epsilon_e^*(s)\ ds \\ &= \int_{\mathbb{R}} h_{me}(t)\epsilon_e^*(t) - i\int_0^t \int_{\mathbb{R}} h_{me}(s)\partial_t h_{me}^*(s)\ ds \\ &\quad -i\int_0^t \int_{\mathbb{R}} h_{me}(s)\Big(e^{-i\Omega s}S(s)f(s)S(-s)\epsilon_g^*(s) + h_{LDe}^*(s)\Big)\ ds.\end{aligned}$$

Remark that $e^{-i\Omega s}S(s)f(s)S(-s)$ is a bounded operator on L^2 with norm less than CK and that $\sup_{t\in[0,T]} \|h_{LDe}(t,x)\|_0 \le C\eta^2 K(M+1)$ (see (4.15)). Thus, taking the imaginary part, we get

$$\begin{aligned}\Big|\Im\Big(\int_0^t \int_{\mathbb{R}} \partial_t h_{me}(s)\epsilon_e^*(s)\ ds\Big)\Big| &\le C\Big(\|h_{me}(t)\|_0 \sqrt{F_0(t)} + \|h_{me}(t)\|_0^2 \\ &\quad + \int_0^t \|h_{me}(s)\|_0 \Big(K\sqrt{F_0(s)} + \eta^2 K(M+1)\Big)\ ds\Big),\end{aligned} \quad (4.21)$$

Similar computations can be done for the term involving h_{mg}.

For convenience, set

$$H_m = \sup_{t\in[0,T]} \Big\{ \|h_{me}(t)\|_0, \|h_{mg}(t)\|_0 \Big\},$$

which, from (4.17)-(4.18), satisfies

$$H_m \le C\frac{K}{(\omega - K)}(M+1)^{3/2}. \quad (4.22)$$

Identity (4.19), combined with estimate (4.20), yields, for $t\in[0,T]$,

$$\begin{aligned}F_0(t) \le C\Big(&\Big(\eta^2 K(M+1) + KH_m\Big)\int_0^t \sqrt{F_0(s)}\ ds + H_m\sqrt{F_0(t)} \\ &+ H_m^2 + T\eta^2 H_m K(M+1)\Big),\end{aligned}$$

and, using $2ab \le a^2+b^2$,

$$F_0(t) \le C\Big(\Big(\eta^2 K(M+1) + KH_m\Big)\int_0^t \sqrt{F_0(s)}\ ds + H_m^2 + T\eta^2 H_m K(M+1)\Big).$$

Lemma 4.3. *Let $F(t)$ be a continuous nonnegative function of time $t\in[0,T]$. Assume that there exist nonnegative numbers α,β such that, for $t\in[0,T]$,*

$$F(t) \le \alpha\int_0^t \sqrt{F(s)}\ ds + \beta. \quad (4.23)$$

Then, for $t\in[0,T]$, F satisfies

$$\sqrt{F(t)} \le \frac{\alpha}{2}t + \sqrt{\beta}.$$

The proof of Lemma 4.3 is left to the reader. The trick consists in setting $G(t) = \alpha \int_0^t \sqrt{F(s)}\, ds + \beta$ and writing (4.23) as

$$\frac{G'(t)}{2\sqrt{G(t)}} \leq \frac{\alpha}{2}.$$

Applied to the function F_0, Lemma 4.3 yields, for $t \in [0, T]$,

$$\sqrt{F_0(t)} \leq C\Big(\Big(\eta^2 K(M+1) + K H_m\Big)T + \Big(H_m^2 + T\eta^2 H_m K(M+1)\Big)^{1/2}\Big). \quad (4.24)$$

Now, set $\delta > 0$. Recall that M is fixed, $KT = 3\pi\sqrt{M}/\eta$ is as in (3.12). We now choose η_0 such that for any $\eta \leq \eta_0$.

$$3\pi C\eta(M+1)^{3/2} \leq \frac{\delta}{2}, \quad (4.25)$$

For $\eta \leq \eta_0$, we shall adjust the other free parameters to obtain

$$C\Big(\frac{3\pi\sqrt{M}}{\eta} H_m + \Big(H_m^2 + 3\pi\eta H_m (M+1)^{3/2}\Big)^{1/2}\Big) \leq \frac{\delta}{2}.$$

This can be done due to the estimate (4.22). Indeed, this last estimate is equivalent to

$$\begin{aligned}\frac{3\pi(M+1)^2}{\eta}&\Big(\frac{K}{\omega - K}\Big)\\ &+ \Big[\Big(\frac{K}{\omega-K}\Big)^2 (M+1)^3 + 3\pi\eta\Big(\frac{K}{\omega-K}\Big)(M+1)^3\Big]^{1/2} \leq \frac{\delta}{2C}, \quad (4.26)\end{aligned}$$

which can be satisfied if we assume $\omega\eta/K$ to be large enough (Recall that η is supposed to be small, and then we can assume without restriction that $\eta_0 < 1$). The proof of Theorem 4.1 is then complete. $\square$

We now focus on Theorem 1.3.

Proof of Theorem 1.3. Roughly speaking, the idea consists in using Theorem 4.1 to control the most significant part of the system. We thus need to estimate the error terms which have been introduced by this technique.

Set $\delta > 0$.

Since $(\psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1)$ belongs to $(L^2(\mathbb{R}))^4$, and since the family (Φ_j) is an orthonormal basis of L^2, there exists an nonegative integer $M > 0$ such that one can find a quadruplet of functions $(\tilde\psi_e^0, \tilde\psi_g^0, \tilde\psi_e^1, \tilde\psi_g^1)$ in V_M^4 with

$$\Big\|(\psi_e^0, \psi_g^0) - (\tilde\psi_e^0, \tilde\psi_g^0)\Big\|_{0\times 0} + \Big\|(\psi_e^1, \psi_g^1) - (\tilde\psi_e^1, \tilde\psi_g^1)\Big\|_{0\times 0} \leq \frac{\delta}{2}.$$

Besides, since the couples (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) both have unit $(L^2(\mathbb{R}))^2$ norm, we can further impose that the couples $(\tilde\psi_e^0, \tilde\psi_g^0)$ and $(\tilde\psi_e^1, \tilde\psi_g^1)$ also have unit norm in $(L^2(\mathbb{R}))^2$.

Set
$$(\epsilon_e^0, \epsilon_g^0, \epsilon_e^1, \epsilon_g^1) = (\psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) - (\tilde{\psi}_e^0, \tilde{\psi}_g^0, \tilde{\psi}_e^1, \tilde{\psi}_g^1).$$
Remark that the solution (ψ_e, ψ_g) of (1.2) with initial data (ψ_e^0, ψ_g^0) coincides with the sum of the solutions $(\tilde{\psi}_e, \tilde{\psi}_g)$ of (1.2) with initial data $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$ and (ϵ_e, ϵ_g) of (1.2) with initial data $(\epsilon_e^0, \epsilon_g^0) \in (L^2(\mathbb{R}))^2$.

Now, $(\tilde{\psi}_e^0, \tilde{\psi}_g^0, \tilde{\psi}_e^1, \tilde{\psi}_g^1)$ are in V_M^4. Using Theorem 4.1, there exist positive constants $\eta_0 > 0$ and $\rho_0 > 0$, such that for $\eta \leq \eta_0$, $KT\eta = \aleph = 3\pi\sqrt{M}$, (ω, Ω) as in (1.6) and $\omega\eta/K \geq \rho_0$, there exists a control function $\mathbf{u}$ as in (1.4) satisfying the constraints (4.3) such that the solution $(\tilde{\psi}_e, \tilde{\psi}_g)$ of (1.2) with initial data $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$ satisfies, for a complex number β of modulus 1,
$$\left\|(\tilde{\psi}_e(T), \tilde{\psi}_g(T)) - \beta(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0} \leq \frac{\delta}{2}.$$
From now on, the parameters K, T, $\mathbf{u}$, η, ω and β are fixed as above.

Now recall that, from Theorem 1.1, the $(L^2(\mathbb{R}))^2$ norm of the solutions of (1.2) is constant. In particular
$$\|(\epsilon_e(T), \epsilon_g(T))\|_{0\times 0} = \left\|(\epsilon_e^0, \epsilon_g^0)\right\|_{0\times 0}.$$

Combining these two estimates, we obtain that
$$\begin{aligned}\left\|(\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1)\right\|_{0\times 0} &\leq \left\|(\epsilon_e^0, \epsilon_g^0)\right\|_{0\times 0} \\ &+ \left\|(\tilde{\psi}_e(T), \tilde{\psi}_g(T)) - \beta(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0} + \left\|(\epsilon_e^1, \epsilon_g^1)\right\|_{0\times 0} \leq \delta,\end{aligned}$$
and the proof is complete. □

Remark 4.4. To sum up, our approach justifies the approximations presented in Section 3 when the parameters satisfy the following orders of magnitude
$$\eta \ll 1, \qquad \frac{\omega\eta}{K} \gg 1.$$
In this case, the time shall be chosen as $T = \aleph/(K\eta)$.

4.2 Approximate controllability in $\mathcal{D}(A^{k/2})$

We notice that the control given by Theorem 1.2 is smooth in space and piecewise constant in time. It follows from Theorem 1.1 that the controlled trajectories are as smooth (in the space variable) as the initial data. It is then natural to look for extensions of the results of the previous section for stronger norms.

Several extensions can be considered, and we indicate below some of them.

Theorem 4.5 (Approximate Controllability in V_M^2 in $\|\cdot\|_{k\times k}$ norm)**.** *Let k be a nonnegative integer. Let M be a positive integer, and consider two couples of data (ψ_e^0, ψ_g^0) and (ψ_e^1, ψ_g^1) in V_M^2 of unit $(L^2(\mathbb{R}))^2$ norms.*

For any $\eta \in (0, \eta_M)$, let K and T be such that $KT = 3\pi\sqrt{M}/\eta$ as in (3.12). *Consider the control function $t \mapsto (u_0, v_r, v_b)$ given by Theorem 1.2 under the constraints* (4.3) *which steers solutions of* (1.10) *from (ψ_e^0, ψ_g^0) to (ϕ_e^1, ϕ_g^1) as in* (4.6).

Set $u_r = iv_r/\eta$ and $u_b = iv_b/\eta$, and consider the control function $\mathbf{u}$ as defined in (1.4).

Then, for any $\delta > 0$, there exist $\eta_k(\delta) \in (0, \eta_M)$ and $\rho_k(\delta) > 0$ such that for (ω, Ω) as in (1.6) *and for*

$$0 < \eta \leq \eta_k, \qquad KT = \frac{3\pi\sqrt{M}}{\eta}, \qquad \frac{\omega\eta}{K} \geq \rho_k,$$

the solution (ψ_e, ψ_g) of (1.2) *with initial data (ψ_e^0, ψ_g^0) and the above defined control $\mathbf{u}$ satisfies, for β a complex number of modulus* 1,

$$\left\|(\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1)\right\|_{k\times k} \leq \delta. \tag{4.27}$$

When considering more general data than the ones in V_M^2, we can prove the following:

Theorem 4.6 (Approximate Controllability in $\|\cdot\|_k$ norm)**.** *Let k be a nonnegative integer. Consider two couples of data $(\psi_e^0, \psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and $(\psi_e^1, \psi_g^1) \in \mathcal{D}(A^{k/2})^2$ of unit $(L^2(\mathbb{R}))^2$ norms.*

For any $\delta > 0$, there exist a constant $\aleph = \aleph(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) > 0$, and two positive parameters $\eta_k = \eta_k(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) > 0$, and $\rho_k(\delta, \psi_e^0, \psi_g^0, \psi_e^1, \psi_g^1) > 0$ such that for (ω, Ω) as in (1.6) *and for*

$$0 < \eta \leq \eta_k, \qquad KT = \frac{\aleph}{\eta}, \qquad \frac{\omega\eta}{K} \geq \rho_k,$$

then for a control function $\mathbf{u}(t, x)$ of the form (1.4)*, given by a map $t \mapsto (u_0(t), u_r(t), u_b(t))$ of piecewise constant functions,*

- *The solution (ψ_e, ψ_g) of* (1.2) *with initial data (ψ_e^0, ψ_g^0) satisfies* (4.27) *for some complex number β of modulus* 1.
- *For all time $t \in [0, T]$, estimates* (1.5) *are satisfied.*
- *At each time $t \in [0, T]$, there is only one nonzero component in the vector $(u_0(t), u_r(t), u_b(t))$.*

These two theorems are a consequence of the following Lemma:

Lemma 4.7. *Assume that (ϵ_e, ϵ_g) satisfies*

$$\begin{cases} i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g + h_{LDe} + \partial_t h_{me}, \\ i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e + h_{LDg} + \partial_t h_{mg}, \\ (\epsilon_e(0), \epsilon_g(0)) = (\epsilon_e^0, \epsilon_g^0), \end{cases} \tag{4.28}$$

where $(\epsilon_e^0, \epsilon_g^0) \in \cap_{k\in\mathbb{N}}\mathcal{D}(A^{k/2})$, $f = \mathbf{u}+\mathbf{u}^*$ *for* $\mathbf{u}$ *being as in* (1.4) *and satisfying the constraints* (1.5)*. Also assume that* $h_{LDe}, h_{LDg}, h_{me}, h_{mg}$ *satisfy* $h_{me}(0) = h_{mg}(0) = 0$ *and that, for all* $k \in \mathbb{N}$*, there exists* a_{LDk}*,* a_{mk} *such that*

$$\sup_{t\in[0,T]} \Big\{ \|h_{LDe}(t)\|_k \,, \|h_{LDg}(t)\|_k \Big\} \le a_{LDk},$$
$$\sup_{t\in[0,T]} \Big\{ \|h_{me}(t)\|_k \,, \|h_{mg}(t)\|_k \Big\} \le a_{mk}.$$

For any $k \in \mathbb{N}$*, define*

$$F_k(t) = \frac{1}{2}\Big(\ \|\epsilon_e(t)\|_k^2 + \|\epsilon_g(t)\|_k^2 \Big). \tag{4.29}$$

Then there exists a constant $C_0(k)$ *such that*

$$\begin{aligned} \sup_{t\in[0,T]} \Big\{\sqrt{F_k(t)}\Big\} \le C_0(k)\Big(K\eta \sup_{t\in[0,T]} \Big\{\sqrt{F_{k-1}(t)}\Big\} + a_{LDk} + Ka_{mk}\Big)T \\ + \sqrt{2F_k(0)} + C_0(k)\Big(Ta_{mk}a_{LDk} + a_{mk}^2\Big)^{1/2}. \end{aligned} \tag{4.30}$$

Proof of Lemma 4.7. Multiplying the first and second lines of (4.28) respectively by $A^k\epsilon_e^*$ and $A^k\epsilon_g^*$, we obtain

$$\begin{aligned} & i\int_{\mathbb{R}} \Big(\partial_t A^{k/2}\epsilon_e(t)\Big)A^{k/2}\epsilon_e^*(t) + i\int_{\mathbb{R}} \Big(\partial_t A^{k/2}\epsilon_g(t)\Big)A^{k/2}\epsilon_g^*(t) \\ & = \int_{\mathbb{R}} \Big(e^{i\Omega t}S(-t)f(t)S(t)\epsilon_g(t)A^k\epsilon_e^*(t) + e^{-i\Omega t}S(-t)f(t)S(t)\epsilon_e(t)A^k\epsilon_g^*(t)\Big) \\ & \qquad + \int_{\mathbb{R}} \Big(h_{LDe}(t)A^k\epsilon_e^*(t) + h_{LDg}(t)A^k\epsilon_g^*(t)\Big) \\ & \qquad + \int_{\mathbb{R}} \Big(\partial_t h_{me}(t)A^k\epsilon_e^*(t) + \partial_t h_{mg}(t)A^k\epsilon_g^*(t)\Big) \\ & = I_f(t) + I_{LD}(t) + I_m(t). \end{aligned} \tag{4.31}$$

Below, we estimate each of these integrals separately, or to be more precise, only their imaginary part, since

$$\frac{d}{dt}F_k(t) = \Im\Big(i\int_{\mathbb{R}} \Big(\partial_t A^{k/2}\epsilon_e(t)\Big)A^{k/2}\epsilon_e^*(t) + i\int_{\mathbb{R}} \Big(\partial_t A^{k/2}\epsilon_g(t)\Big)A^{k/2}\epsilon_g^*(t)\Big).$$

We first consider $I_f(t)$.

In a first step, assume that k is even.

$$\int_{\mathbb{R}} \Big(e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g(t) A^k \epsilon_e^*(t) + e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e(t) A^k \epsilon_g^*(t) \Big)$$
$$= \int_{\mathbb{R}} \Big(e^{i\Omega t} A^{k/2} (f(t) S(t) \epsilon_g(t)) A^{k/2} (S(t)\epsilon_e(t))^* \\ + e^{-i\Omega t} A^{k/2} (f(t) S(t) \epsilon_e(t)) A^{k/2} (S(t)\epsilon_g(t))^* \Big)$$

But, for $\psi \in \mathcal{D}(A^{k/2})$, one easily checks that there exists a constant $C(k)$ which depends on k such that, for any function $g = g(x)$,

$$\left\| A^{k/2}(g(x)\psi) - g(x) A^{k/2}\psi \right\|_0 \\ \leq C(k) \sup \Big\{ \left\| \partial_x g \right\|_{L^\infty(\mathbb{R})}, \cdots, \left\| \partial_x^k g \right\|_{L^\infty(\mathbb{R})} \Big\} \left\| A^{(k-1)/2} \psi \right\|_0 .$$

In our case, recall that $f = \mathbf{u} + \mathbf{u}^*$ where $\mathbf{u}$ is as in (1.4). Then we obtain

$$\left\| A^{k/2}(f(t) S(t) \epsilon_g(t)) - f(t) A^{k/2} S(t) \epsilon_g(t) \right\|_0 \leq C(k) K \eta \sqrt{F_{k-1}(t)},$$
$$\left\| A^{k/2}(f(t) S(t) \epsilon_e(t)) - f(t) A^{k/2} S(t) \epsilon_e(t) \right\|_0 \leq C(k) K \eta \sqrt{F_{k-1}(t)},$$

and thus

$$\Bigg| I_f(t) - \int_{\mathbb{R}} \Big[f(t) \Big(e^{i\Omega t} A^{k/2} (S(t)\epsilon_g(t)) A^{k/2} (S(t)\epsilon_e(t))^* \\ + e^{-i\Omega t} A^{k/2} (S(t)\epsilon_e(t)) A^{k/2} (S(t)\epsilon_g(t))^* \Big) \Big] \Bigg| \leq C(k) K \eta \sqrt{F_{k-1}(t)} \sqrt{F_k(t)}.$$

But

$$f(t) \Big(e^{i\Omega t} A^{k/2} (S(t)\epsilon_g(t)) A^{k/2} (S(t)\epsilon_e(t))^* + e^{-i\Omega t} A^{k/2} (S(t)\epsilon_e(t)) A^{k/2} (S(t)\epsilon_g(t))^* \Big)$$

is real, and then we obtain that for all t,

$$\Im(I_f(t)) \leq C(k) K \eta \sqrt{F_{k-1}(t)} \sqrt{F_k(t)}. \tag{4.32}$$

If $k = 2l+1$ is odd, the situation slightly differs. Write

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g(t) A^k \epsilon_e^*(t) + e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e(t) A^k \epsilon_g^*(t) \Big) \\
&\quad = \int_{\mathbb{R}} \Big(e^{i\Omega t} A^l (f(t) S(t) \epsilon_g(t)) \Big(\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \Big) A^l (S(t) \epsilon_e(t))^* \\
&\qquad\qquad\qquad + e^{-i\Omega t} A^l (f(t) S(t) \epsilon_e(t)) \Big(\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \Big) A^l (S(t) \epsilon_g(t))^* \Big) \\
&\quad = \frac{1}{2} \int_{\mathbb{R}} \Big(e^{i\Omega t} A^l (f(t) S(t) \epsilon_g(t)) A^l (S(t) \epsilon_e(t))^* \\
&\qquad\qquad\qquad + e^{-i\Omega t} A^l (f(t) S(t) \epsilon_e(t)) A^l (S(t) \epsilon_g(t))^* \Big) \\
&\quad + \int_{\mathbb{R}} \Big(e^{i\Omega t} \mathbf{a} A^l (f(t) S(t) \epsilon_g) \mathbf{a} A^l (S(t) \epsilon_e)^* \\
&\qquad\qquad\qquad + e^{-i\Omega t} \mathbf{a} A^l (f(t) S(t) \epsilon_e(t)) \mathbf{a} A^l (S(t) \epsilon_g(t))^* \Big).
\end{aligned}$$

Each term can now be estimated as before, and one can then prove (4.32) as for the case k even.

For the Lamb-Dicke approximation term, we get

$$I_{LD}(t) = \int_{\mathbb{R}} (A^{k/2} h_{LDe}(t))(A^{k/2}\ \epsilon_g^*(t)) + \int_{\mathbb{R}} (A^{k/2} h_{LDg}(t))(A^{k/2}\ \epsilon_e^*(t)),$$

and thus,

$$|I_{LD}(t)| \le a_{LDk} \sqrt{F_k(t)}. \tag{4.33}$$

Finally, we compute $\int_0^t I_m(s)\ ds$. Since $h_{me}(0) = 0$ from the hypotheses,

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}} \partial_t h_{me}(s) A^k \epsilon_e^*(s)\ ds &= \int_0^t \int_{\mathbb{R}} \partial_t A^k h_{me}(s) \epsilon_e^*(s)\ ds \\
&= \int_{\mathbb{R}} A^{k/2} h_{me}(t) A^{k/2} \epsilon_e^*(t) - \int_0^t \int_{\mathbb{R}} A^k h_{me}(s) \partial_t \epsilon_e^*(s)\ ds.
\end{aligned}$$

As for the L^2 case, one can use the equations (4.28) to obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} A^k h_{me}(s) \partial_t \epsilon_e^*(s)\ ds \\
&= i \int_0^t \int_{\mathbb{R}} A^k h_{me}(s) e^{-i\Omega s} (S(-s) f(s) S(s) \epsilon_g(s))^*\ ds \\
&+ i \int_0^t \int_{\mathbb{R}} A^k h_{me}(s) h_{LDe}^*(s)\ ds + i \int_0^t \int_{\mathbb{R}} A^k h_{me}(s) \partial_t h_{me}^*(s)\ ds \\
&= i \int_0^t \int_{\mathbb{R}} e^{-i\Omega s} f(s) \Big(S(s) A^{k/2} h_{me}(s) \Big) \Big(A^{k/2} S(s) \epsilon_g(s) \Big)^*\ ds \\
&+ i \int_0^t \int_{\mathbb{R}} \Big(A^{k/2} h_{me}(s) \Big) \Big(A^{k/2} h_{LDe}^*(s) \Big) ds + i \int_0^t \int_{\mathbb{R}} A^{k/2} h_{me}(s) \partial_t A^{k/2} h_{me}^*(s)\ ds.
\end{aligned}$$

Thus, doing the same computations for the other term in I_m, taking the imaginary parts, we obtain, for $t \in [0,T]$,

$$\begin{aligned} &\Big|\Im\Big(\int_0^t I_m(s)\ ds\Big)\Big| \\ &\quad\le C(k)\Big(a_{mk}\sqrt{F_k(t)} + K a_{mk}\int_0^t \sqrt{F_k(s)}\ ds + T a_{mk} a_{LDk} + a_{mk}^2\Big) \\ &\quad\le C(k)\Big(K a_{mk}\int_0^t \sqrt{F_k(s)}\ ds + T a_{mk} a_{LDk} + a_{mk}^2\Big) + \frac{1}{2}\ F_k(t). \end{aligned} \tag{4.34}$$

Combining (4.32), (4.33) and (4.34) in (4.31) and integrating in time, we obtain

$$\begin{aligned} F_k(t) \le F_k(0) + C(k)\Big(K\eta\int_0^t \sqrt{F_{k-1}(s)}\sqrt{F_k(s)}\ ds + a_{LDk}\int_0^t \sqrt{F_k(s)}\ ds \\ + K a_{mk}\int_0^t \sqrt{F_k(s)}\ ds + T a_{mk} a_{LDk} + a_{mk}^2\Big) + \frac{1}{2}\ F_k(t), \end{aligned}$$

and then

$$\begin{aligned} F_k(t) &\le 2F_k(0) + C(k)\Big(T a_{mk} a_{LDk} + a_{mk}^2\Big) \\ &\quad + C(k)\Big(K\eta\Big\|\sqrt{F_{k-1}}\Big\|_{L^\infty(0,T)} + a_{LDk} + K a_{mk}\Big)\int_0^t \sqrt{F_k(s)}\ ds. \end{aligned} \tag{4.35}$$

The proof of Lemma 4.7 then follows directly from Lemma 4.3. □

Remark 4.8. Note that Lemma 4.7 applies to a wider class of initial data. Indeed, if $(\epsilon_{0e}, \epsilon_{0g})$ is only in $\mathcal{D}(A^{k_0/2})^2$ for some $k_0 \in \mathbb{N}$, similarly as in the proof of Theorem 1.1, we can use the density of $\cap_{k\in\mathbb{N}}\mathcal{D}(A^k)$ in $\mathcal{D}(A^{k_0/2})$ to conclude that estimates (4.30) hold for any solution of (4.13) and any $k \le k_0$.

Proof of Theorem 4.5. Set $M > 0$.

Using the same notations as in the proof of Theorem 4.1, the norm $\|(\cdot,\cdot)\|_{k\times k}$ of (ϵ_e, ϵ_g) solution of (4.13) with initial data $(0,0)$ measures precisely the defect of controllability in $\mathcal{D}(A^{k/2})^2$. Besides, in Theorem 4.1 we proved that, for $\delta > 0$, K, T and η as in (3.12), for η small enough and $\omega\eta/K$ large enough, for (ω, Ω) as in (1.6),

$$\sup_{t\in[0,T]} \|(\epsilon_e(t), \epsilon_g(t))\|_{0\times 0} \le \delta.$$

We now proceed by induction on k. Assume that for $k-1$, for any $\delta > 0$, there exist $\eta_{k-1}(\delta) > 0$ and $\rho_{k-1}(\delta) > 0$ such that for $\eta \le \eta_{k-1}$ and $\omega\eta/K \ge \rho_{k-1}$, for (ω, Ω) as in (1.6) and for K, T and η as in (3.12), taking the control function given by Theorem 1.2 under the constraints (4.3), the solution (ϵ_e, ϵ_g) of (4.13) with initial data $(0,0)$ satisfies

$$\sup_{t\in[0,T]} \|(\epsilon_e(t), \epsilon_g(t))\|_{(k-1)\times(k-1)} \le \delta.$$

We then estimate the norm $\|(\cdot,\cdot)\|_{k\times k}$ of (ϵ_e,ϵ_g) solution of (4.13) with initial data $(0,0)$, and $h_{LDe}, h_{LDg}, h_{me}, h_{mg}$ satisfying (4.15)-(4.17)-(4.18).

Using the notations of Lemma 4.7, we can take (see (4.15)-(4.17)-(4.18))

$$a_{LDk} = C(k)\eta^2 K(M+1)^{(k+2)/2}, \quad a_{mK} = C(k)\frac{K}{(\omega-K)}(M+1)^{(k+3)/2}. \tag{4.36}$$

Besides, from the induction hypothesis, there exist $\eta_{k-1}>0$ and $\rho_{k-1}>0$ such that for $\eta \le \eta_{k-1}$, $TK = 3\pi\sqrt{M}/\eta$ and (ω,Ω) as in (1.6) satisfying $\omega\eta/K \ge \rho_{k-1}$, we get

$$\sup_{t\in[0,T]} \|(\epsilon_e(t),\epsilon_g(t))\|_{(k-1)\times(k-1)} \le \frac{\delta}{9C_0(k)\pi\sqrt{M}} = \frac{\delta}{3C_0(k)K\eta T}.$$

Then Lemma 4.7 yields

$$\sup_{t\in[0,T]} \sqrt{F_k(t)} \le \frac{\delta}{3} + C_0(k)a_{LDk}T + C_0(k)\Big[Ka_{mk}T + \Big(Ta_{mk}a_{LDk} + a_{mk}^2\Big)^{1/2}\Big].$$

Using (4.36), we can then choose $\eta_k(\delta)$ smaller than η_{k-1} such that any η with $\eta \le \eta_k(\delta)$ satisfies

$$C_0(k)a_{LDk}T = 3\pi C_0(k)C_1(k)(M+1)^{(k+3)/2} \times \eta \le \frac{\delta}{3}. \tag{4.37}$$

Now, for $\eta \le \eta_k(\delta)$, we want to adjust the parameters such that the following estimate holds.

$$C_0(k)\Big(Ka_{mk}T + \Big(Ta_{mk}a_{LDk} + a_{mk}^2\Big)^{1/2}\Big) \le \frac{\delta}{3}.$$

This can be done by imposing the condition $\eta\omega/K \ge \rho_k(\delta)$ for a suitable constant $\rho_k(\delta)$ large enough (and greater than $\rho_{k-1}(\delta)$), since this last inequality is equivalent to

$$\begin{aligned} C(k)\Big[&\frac{3\pi}{\eta}\Big(\frac{K}{\omega-K}\Big)(M+1)^{k/2+2} \\ &+ \Big(3\pi\eta(M+1)^{k+3}\Big(\frac{K}{\omega-K}\Big) + (M+1)^3\Big(\frac{K}{\omega-K}\Big)^2\Big)^{1/2}\Big] \le \frac{\delta}{3}. \end{aligned} \tag{4.38}$$

It follows that, if $\eta \le \eta_k(\delta)$ and $\omega\eta/K \ge \rho_k(\delta)$, then

$$\sup_{t\in[0,T]} \sqrt{F_k(t)} \le \delta,$$

and the proof is complete by induction. $\square$

Proof of Theorem 4.6. Take $(\psi_e^0,\psi_g^0) \in \mathcal{D}(A^{k/2})^2$ and $(\psi_e^1,\psi_g^1) \in \mathcal{D}(A^{k/2})^2$ as

$$\begin{cases} \psi_e^0 = \sum_j \alpha_j^0 \Phi_j, \\ \psi_g^0 = \sum_j \beta_j^0 \Phi_j, \end{cases} \qquad \begin{cases} \psi_e^1 = \sum_j \alpha_j^1 \Phi_j, \\ \psi_g^1 = \sum_j \beta_j^1 \Phi_j, \end{cases}$$

with

$$\sum_j \Big(|\alpha_j^0|^2 + |\beta_j^0|^2\Big)(j+1/2)^k < \infty, \qquad \sum_j \Big(|\alpha_j^1|^2 + |\beta_j^1|^2\Big)(j+1/2)^k < \infty.$$

Fix $\tilde{\delta} > 0$ and choose M such that

$$\begin{aligned}
&\sum_{j>M} \Big(|\alpha_j^0|^2 + |\beta_j^0|^2\Big)(j+1/2)^k \le \Big(\frac{\tilde{\delta}}{\big\|(\psi_e^1, \psi_g^1)\big\|_{k\times k}}\Big)^2 \le \tilde{\delta}^2, \\
&\sum_{j>M} \Big(|\alpha_j^1|^2 + |\beta_j^1|^2\Big)(j+1/2)^k \le \Big(\frac{\tilde{\delta}}{\big\|(\psi_e^1, \psi_g^1)\big\|_{k\times k}}\Big)^2 \le \tilde{\delta}^2.
\end{aligned} \tag{4.39}$$

Then define

$$\begin{aligned}
\tilde{\psi}_e^0 &= \sum_{j\le M} \alpha_j^0 \Phi_j, \quad \tilde{\psi}_e^1 = \sum_{j\le M} \alpha_j^1 \Phi_j, \\
\tilde{\psi}_g^0 &= \sum_{j\le M} \beta_j^0 \Phi_j, \quad \tilde{\psi}_g^1 = \sum_{j\le M} \beta_j^1 \Phi_j.
\end{aligned}$$

Considering the couples of functions $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)/\left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0}$ and $(\tilde{\psi}_e^1, \tilde{\psi}_g^1)/\left\|(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0}$, one can apply Theorem 4.5 : For $\eta \le \eta_k(\tilde{\delta})$, $KT = 3\pi\sqrt{M}/\eta$ and $\omega\eta/K \ge \rho_k(\tilde{\delta})$, for (ω, Ω) as in (1.6), there exist a complex number β of modulus 1 and a control $\mathbf{u}$ as in (1.4) satisfying (1.5) such that the solution $(\check{\psi}_e, \check{\psi}_g)$ of (1.2) with initial data $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)/\left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0}$ satisfies

$$\left\|(\check{\psi}_e(T), \check{\psi}_g(T)) - \beta \frac{(\tilde{\psi}_e^1, \tilde{\psi}_g^1)}{\left\|(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0}}\right\|_{k\times k} \le \tilde{\delta}.$$

We fix $\tilde{\delta}$, T, K, η, $\mathbf{u}$, ω and Ω as above.

Remark that $(\check{\psi}_e, \check{\psi}_g) = (\tilde{\psi}_e, \tilde{\psi}_g)/\left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0}$, where $(\tilde{\psi}_e, \tilde{\psi}_g)$ is the solution of (1.2) with initial data $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$. Thus we have from (4.39)

$$\begin{aligned}
\left\|(\tilde{\psi}_e(T), \tilde{\psi}_g(T)) - \beta(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{k\times k} &\le \left\|(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{k\times k} \left|1 - \frac{\left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0}}{\left\|(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0}}\right| \\
&+ \left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0} \left\|\frac{(\tilde{\psi}_e(T), \tilde{\psi}_g(T))}{\left\|(\tilde{\psi}_e^0, \tilde{\psi}_g^0)\right\|_{0\times 0}} - \beta \frac{(\tilde{\psi}_e^1, \tilde{\psi}_g^1)}{\left\|(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{0\times 0}}\right\|_{k\times k} \le 3\tilde{\delta}.
\end{aligned}$$

We then deduce

$$\begin{aligned}
&\left\|(\tilde{\psi}_e(T), \tilde{\psi}_g(T)) - \beta(\tilde{\psi}_e^1, \tilde{\psi}_g^1)\right\|_{k\times k} \le 3\tilde{\delta}, \\
&\left\|(\tilde{\psi}_e(T), \tilde{\psi}_g(T)) - \beta(\psi_e^1, \psi_g^1)\right\|_{k\times k} \le 4\tilde{\delta}.
\end{aligned} \tag{4.40}$$

Let us now consider the error term we introduced by truncating the expansions of (ψ_e^0, ψ_g^0).

Set

$$\begin{cases} (\epsilon_e^0, \epsilon_g^0) = (\psi_e^0, \psi_g^0) - (\tilde{\psi}_e^0, \tilde{\psi}_g^0), \\ (\epsilon_e(t), \epsilon_g(t)) = (e^{i\Omega t/2} S(-t)\psi_e(t), e^{-i\Omega t/2} S(-t)\psi_g(t)) \\ \qquad\qquad\qquad -(e^{i\Omega t/2} S(-t)\tilde{\psi}_e(t), e^{-i\Omega t/2} S(-t)\tilde{\psi}_g(t)), \end{cases} \tag{4.41}$$

where (ψ_e, ψ_g) and $(\tilde{\psi}_e, \tilde{\psi}_g)$ are, respectively, the solutions of (1.2) with initial data (ψ_e^0, ψ_g^0) and $(\tilde{\psi}_e^0, \tilde{\psi}_g^0)$. This choice makes (ϵ_e, ϵ_g) satisfy the equation

$$i\partial_t \epsilon_e = e^{i\Omega t} S(-t) f(t) S(t) \epsilon_g, \qquad i\partial_t \epsilon_g = e^{-i\Omega t} S(-t) f(t) S(t) \epsilon_e. \tag{4.42}$$

with initial data $(\epsilon_e^0, \epsilon_g^0)$.

The intial data $(\epsilon_e^0, \epsilon_g^0)$ have the following expansion

$$\epsilon_e^0 = \sum_{j>M} \alpha_j^0 \Phi_j, \quad \epsilon_g^0 = \sum_{j>M} \beta_j^0 \Phi_j,$$

with

$$\left\|(\epsilon_e^0, \epsilon_g^0)\right\|_{k\times k}^2 = \sum_{j>M} \Big(|\alpha_j^0|^2 + |\beta_j^0|^2\Big)(j+1/2)^k < \tilde{\delta}^2.$$

In particular, with the notations introduced in (4.29), we have

$$\forall l \in \{0, \cdots, k\}, \qquad (M+1)^{k/2-l/2} \sqrt{F_l(0)} \leq \tilde{\delta}.$$

Applying Lemma 4.7, we obtain, for $l \in \{1, \cdots, k\}$,

$$\sup_{t\in[0,T]} \sqrt{F_l(t)} \leq C_0(l) K \eta T \left\|\sqrt{F_{l-1}}\right\|_{L^\infty(0,T)} + \Big(2F_l(0)\Big)^{1/2}, \tag{4.43}$$

At this point, it is crucial that K, η and T are related by $KT\eta = 3\pi\sqrt{M}$. Indeed, we can now conclude by induction on l between 0 and k similarly as in the proof of Theorem 4.5. Let us briefly explain the first step. Note that $\sqrt{F_0(t)}$ is constant and bounded by $\tilde{\delta}/(M+1)^{k/2}$. Then (4.43) writes

$$\sup_{t\in[0,T]} \sqrt{F_1(t)} \leq 2\pi C_0(1) \frac{\tilde{\delta}}{(M+1)^{k/2-1/2}} + \sqrt{2}\frac{\tilde{\delta}}{(M+1)^{k/2-1/2}}.$$

Hence we get

$$\sup_{t\in[0,T]} \sqrt{F_1(t)} \leq \frac{C_1 \tilde{\delta}}{(M+1)^{k/2-1/2}},$$

where C_1 is an explicit constant, which does not depend on the parameters $\eta, \omega, M, K, T, \tilde{\delta}$.

Similarly, by induction on l, we can prove that for all $l \in \{0, \cdots, k\}$,

$$\sup_{t \in [0,T]} \sqrt{F_l(t)} \leq \frac{C_l \tilde{\delta}}{(M+1)^{k/2 - l/2}},$$

where C_l is an explicit constant which does not depend on the parameters $\eta, \omega, M, K, T, \tilde{\delta}$.

For $l = k$, this gives that

$$\sup_{t \in [0,T]} \sqrt{F_k(t)} \leq C_k \tilde{\delta}.$$

Combined with (4.40), we obtain that for $\eta \leq \eta_k$ and $\omega\eta/K \geq \rho_k$, the solution (ψ_e, ψ_g) of (1.2) with initial data (ψ_e^0, ψ_g^0) satisfies

$$\left\| (\psi_e(T), \psi_g(T)) - \beta(\psi_e^1, \psi_g^1) \right\|_{k \times k} \leq (4 + C_k)\tilde{\delta},$$

and the proof is complete by choosing $\tilde{\delta} = \delta / C_k$ for $\delta > 0$. $\square$

Remark 4.9. If we have an estimate on the size M of the truncation, we can have a bound on the constant $\aleph$ in Theorem 4.6. In particular, such bounds can be obtained when the initial and target states both belong to some $\mathcal{D}(A^{k_0/2})^2$ and we are interested in approximate controllability results in the norms $\mathcal{D}(A^{k/2})^2$ for $k < k_0$.

5 Further comments

1. To go on further on the study of the controllability properties of (1.2), one would like to study the local exact controllability of (1.2). As said in the introduction, the resuts in [2] applies and proves the lack of local exact controllability in the natural space $(L^2(\mathbb{R}))^2$ but, similarly as in [6, 7], one could hope local exact controllability results to hold in stronger norms. Though, this question seems widely open.

Actually, one could first consider the local exact controllability properties around the ground state of the simpler model

$$i\partial_t \psi = A\psi + f(t)\eta(x)\psi, \quad (t, x) \in (0, T) \times \mathbb{R}, \tag{5.1}$$

where $\eta = \eta(x)$ is a smooth real-valued function, say for instance in $\mathcal{S}'(\mathbb{R})$ and $f = f(t)$ is the control function, which we assume to be real-valued.

This issue has not been dealt with yet. If one wants to use the same strategy as in [6], it would be convenient to have a function η such that the coefficients

$$\int_{\mathbb{R}} \eta(x)\Phi_0(x)\Phi_j(x)\, dx$$

(recall that the functions Φ_j are the eigenvectors of A) never vanish and decay polynomially, say as $1/j^k$, in j. Indeed, this would allow to prove an exact controllability result for the linearization of (5.1) in the space $\mathcal{D}(A^k)$ around the ground state Φ_0. We did not succeed to find a reasonable function η satisfying this condition.

2. As said in Remark 3.2, we do not know if the strategy of [17] (and recalled in the proof of Theorem 1.2) is sharp. It is very likely that better strategies exist, since ours does not use all the controls. It would be very interesting to optimize our strategy in order to use smaller controls/times, and it thus deserves further work. Perhaps this issue might be addressed by using graph theory to study system (1.10) as presented for instance in [23] (see also [11]).

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