

Local exact controllability for the 1-d compressible Navier-Stokes equations*

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Joint work with Olivier Glass, Sergio Guerrero and Jean-Pierre Puel

1 Introduction

In this talk, I will present a recent result obtained in [6] with O. Glass, S. Guerrero and J.-P. Puel on the local exact controllability of the 1-d compressible Navier-Stokes equations. The goal of these notes is to give an informal presentation of this article and we refer the reader to it for extensive details.

Setting. We consider the compressible Navier-Stokes equations in a bounded domain $(0, L)$:

$$\begin{cases} \partial_t \rho_S + \partial_x(\rho_S u_S) = 0 & \text{in } (0, T) \times (0, L), \\ \rho_S(\partial_t u_S + u_S \partial_x u_S) - \partial_{xx} u_S + \partial_x p(\rho_S) = 0 & \text{in } (0, T) \times (0, L). \end{cases} \quad (1.1)$$

Here ρ_S is the density and u_S the velocity. The pressure is given by the following law:

$$p(\rho_S) = c_p \rho_S^\gamma, \quad (1.2)$$

where $c_p > 0$ and $\gamma \geq 1$. Recall that this is the classical law when considering isentropic flows, in which case $\gamma = 1.4$ for perfect gases, or isothermal flows ($\gamma = 1$). At this point, let us also remark that actually the only assumption we are going to use is that p is a smooth increasing function of the density.

The initial data are also given:

$$(\rho_S, u_S)|_{t=0} = (\rho_0, u_0) \text{ in } (0, L). \quad (1.3)$$

Note that the boundary conditions are not given explicitly. They actually are the controls and will not be written explicitly. If one really wants to make them explicit, they could for instance be expressed as follows:

- Dirichlet boundary conditions for u_S : $u_S(t, 0) = v_0(t)$ and $u_S(t, L) = v_L(t)$;
- conditions on the density when the flux enters the domain, i.e.: $\rho_S(t, 0) = w_0(t)$ if $u_S(t, 0) > 0$ and $\rho_S(t, L) = w_L(t)$ if $u_S(t, L) < 0$.

We are going to present a local exact controllability result around a constant equilibrium state $(\bar{\rho}, \bar{u})$ with $\bar{\rho} > 0$. To be more precise, we are going to study if there exist a norm $\|\cdot\|_*$ and a constant $\kappa > 0$ such that if $\|(\rho_0, u_0) - (\bar{\rho}, \bar{u})\|_* \leq \kappa$ there exists a trajectory (ρ_S, u_S) solution of (1.1)–(1.3) satisfying

$$(\rho_S, u_S)(T) = (\bar{\rho}, \bar{u}) \text{ in } (0, L). \quad (1.4)$$

Before going further, let us emphasize that the time T is a critical parameter in such problem and in particular because of the hyperbolic nature of the equation of the density. Actually, due to the structure of the equation of the density, if the velocity stays in a neighborhood of the target velocity \bar{u} , the time

$$T_{\bar{u}} = \frac{L}{|\bar{u}|} \quad (1.5)$$

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is expected to be critical for the control properties of the equation of the density in (1.1).

According to this fact, it is already natural to conjecture that the linearized compressible Navier-Stokes equations around the constant state $(\bar{\rho}, \bar{u}) = (\bar{\rho}, 0)$ are not controllable. This is indeed the case since then the linearized equations reduce to

$$\begin{cases} \partial_t \rho + \bar{\rho} \partial_x u = 0 & \text{in } (0, T) \times (0, L), \\ \bar{\rho} \partial_t u - \partial_{xx} u + p'(\bar{\rho}) \partial_x \rho = 0 & \text{in } (0, T) \times (0, L), \end{cases} \quad (1.6)$$

which imply the following equation on u :

$$\bar{\rho} \partial_{tt} u - \partial_{xx} \partial_t u - p'(\bar{\rho}) \bar{\rho} \partial_{xx} u = 0 \text{ in } (0, T) \times (0, L). \quad (1.7)$$

This equation is a wave equation with structural damping. It turns out that the controllability properties of equation (1.7) have already been studied thoroughly in [19] (see also [2]). Using the spectral decomposition of the solutions, the article [19] shows that the equation (1.7) indeed is not exactly controllable. (They also show that this is approximately controllable, but this weaker notion of controllability will not be discussed further here.)

The situation therefore seems to indicate that positive controllability results for linearized compressible Navier-Stokes equation around constant states can be obtained only for non-trivial target velocities $\bar{u} \neq 0$, as already indicated by the critical time (1.5). One could then try to show that the corresponding linear system is exactly controllable provided the time T is larger than $T_{\bar{u}}$ defined in (1.5). Actually, this would lead to the system

$$\begin{cases} \partial_t \rho + \bar{u} \partial_x \rho + \bar{\rho} \partial_x u = 0 & \text{in } (0, T) \times (0, L), \\ \bar{\rho} (\partial_t u + \bar{u} \partial_x u) - \partial_{xx} u + p'(\bar{\rho}) \partial_x \rho = 0 & \text{in } (0, T) \times (0, L). \end{cases} \quad (1.8)$$

Again, this implies the following equation on u :

$$\bar{\rho} (\partial_{tt} u + (|\bar{u}|^2 - p'(\bar{\rho})) \partial_{xx} u + 2\bar{u} \partial_t \partial_x u) - \partial_t \partial_{xxx} u - \bar{u} \partial_{xxx} u = 0 \text{ in } (0, T) \times (0, L).$$

And this equation corresponds to the one studied in [16] when $|\bar{u}|^2 = p'(\bar{\rho})$, $\bar{u} = -1$ and $L = 2\pi$, in which case it is exactly controllable in time $T > 2\pi$. We shall not follow here the approach of [16], which is based on a spectral decomposition of the solutions and which is therefore not robust with respect to perturbations.

Main result. We are then in position to state our main result.

Theorem 1.1 ([6]). *Let $\bar{u} \in \mathbb{R}^*$ and $\bar{\rho} \in \mathbb{R}_+^*$. Let $T > 0$ satisfy*

$$T > T_{\bar{u}} = \frac{L}{|\bar{u}|}. \quad (1.9)$$

Then there exists $\kappa > 0$ such that, for any $u_0 \in H^3(0, L)$ and $\rho_0 \in H^3(0, L)$ such that

$$\|u_0 - \bar{u}\|_{H^3(0, L)} + \|\rho_0 - \bar{\rho}\|_{H^3(0, L)} < \kappa, \quad (1.10)$$

there exists a solution (ρ_S, u_S) of (1.1)–(1.3) satisfying (1.4). Besides, the controlled trajectory satisfies $\rho_S \in H^1((0, T) \times (0, L))$ and $u_S \in H^1((0, T); L^2(0, L)) \cap L^2((0, T); H^2(0, L))$.

Let us give some comments on this result:

- We have proved a local exact controllability result around constant equilibria $(\bar{\rho}, \bar{u})$ when $\bar{\rho} > 0$, which is a natural assumption, and when $\bar{u} \neq 0$, which comes from the hyperbolic nature of the equation of the density, see the above discussion.
- Our result is very weak, since it states local exact controllability *only around constant equilibria*. It is very likely that one can prove local exact controllability around smooth enough (to be made precise) trajectories $(\bar{\rho}, \bar{u})$ of (1.1) under an additional geometric condition. Roughly speaking, we expect such result to be true provided the flow corresponding to the target velocity runs across the whole domain.
- The neighborhood of the initial data is an $H^3 \times H^3$ neighborhood. It is very likely that this can be relaxed to $H^1 \times H^1$, but our strategy of proof does not work with such low regularity assumptions.
- Our approach is mainly linear and we should emphasize that conditions $\bar{u} \neq 0$ and $T > T_{\bar{u}}$ are natural

when staying in a neighborhood of the target $(\bar{\rho}, \bar{u})$. But other strategies could apply and use the nonlinearities, see for instance the return method of J.-M. Coron [3, 4].

References. Theorem 1.1 seems to be the first result on the controllability of the compressible Navier-Stokes equations (1.1) except for the recent result by E. Amosova [1], which states the local exact controllability of compressible viscous fluids in dimension 1 considered in Lagrangian coordinates and when the initial density exactly coincides with the density part of the target.

There are however several results concerning the controllability properties of the incompressible Navier-Stokes and Euler equations.

For incompressible Navier-Stokes equations, the first result seems to be a local exact controllability result around smooth trajectories obtained by A. Fursikov and O. Imanuvilov in [8] for boundary conditions on the normal velocity and the curl. It has then been extended by Imanuvilov in [13] to the case of Dirichlet boundary conditions. Later on in [7], the regularity on the target trajectory was lowered. Since then, other strategies have been proposed, and in particular the one in [11] which consists in adding a fictitious control in the divergence and then do a lifting to absorb it.

In the context of incompressible Euler equations, J.-M. Coron in [3] obtained the global exact controllability in dimension 2 using the return method. Later in [9], O. Glass extended this result to the 3-dimensional setting.

Based on these results, J.-M. Coron and A. Fursikov show in [5] the global exact controllability of the incompressible Navier-Stokes equations provided the control is exerted on the whole boundary. Except for this situation, the global exact controllability of Navier-Stokes equations still is an open problem. Partial answers to that question are available in [12] and [14].

Some controllability results for compressible Euler equations are also available. In [15], a local controllability result has been obtained for classical solutions. In the context of weak entropy solutions, we refer to the result of O. Glass in [10]. Finally, let us also mention the approximate controllability result by H. Nersisyan [18] where the control is a force lying in a finite-dimensional space.

For the rest of the talk, we assume that

$$\bar{u} > 0.$$

This can be done without loss of generality by using the change of coordinates $x \rightarrow L - x$.

2 Main strategy

The strategy for the proof of Theorem 1.1 relies on a fixed point argument.

First, as we will explain later in Section 3, it is convenient to look for a trajectory going from 0 to 0.

This can be done as follows. We define (ρ_{in}, u_{in}) such that $(\bar{\rho} + \rho_{in}, \bar{u} + u_{in})$ is a solution of (1.1) on $(0, T) \times \mathbb{R}$ with initial data (ρ_0, u_0) (this supposes that the initial data have been extended on \mathbb{R} smoothly, which can of course be done easily). We then look for $(\rho, u) = (\rho_S - \bar{\rho} - \Lambda\rho_{in}, u_S - \bar{u} - \Lambda u_{in})$ where $\Lambda : [0, T] \rightarrow [0, 1]$ is a suitable smooth cut-off function taking value 1 close to $t = 0$, say on $[0, T_0]$ for some small $T_0 > 0$, and vanishing after the time $2T_0$.

We are then looking for a solution (ρ, u) of

$$\partial_t \rho + (\bar{u} + u + \Lambda u_{in}) \partial_x \rho + \bar{p} \partial_x u + \bar{p}'(\bar{\rho}) \rho = f(\rho, u) \text{ in } [0, T] \times (0, L), \quad (2.1)$$

$$(\bar{p} + \Lambda \rho_{in})(\partial_t u + \bar{u} \partial_x u) - \partial_{xx} u = g(\rho, u) \text{ in } [0, T] \times (0, L), \quad (2.2)$$

where $f(\rho, u)$ and $g(\rho, u)$ are given as follows:

$$f(\rho, u) = -\Lambda' \rho_{in} + (\Lambda - \Lambda^2) \partial_x (\rho_{in} u_{in}) - \Lambda \partial_x (\rho_{in} u) - \Lambda \rho \partial_x u_{in} - \rho \partial_x u + \bar{p}'(\bar{\rho}) \rho. \quad (2.3)$$

and

$$\begin{aligned} g(\rho, u) = & -(\bar{p} + \Lambda \rho_{in}) \Lambda' u_{in} - (p'(\bar{p} + \Lambda \rho_{in}) - p'(\bar{p} + \rho_{in})) \Lambda \partial_x \rho_{in} \\ & + \rho_{in} \partial_t u_{in} (\Lambda - \Lambda^2) + \rho_{in} \bar{u} \partial_x u_{in} (\Lambda - \Lambda^2) + \bar{p} u_{in} \partial_x u_{in} (\Lambda - \Lambda^2) + \rho_{in} u_{in} \partial_x u_{in} (\Lambda - \Lambda^3) \\ & - \Lambda (\bar{p} + \Lambda \rho_{in}) \partial_x (u u_{in}) - (\bar{p} + \Lambda \rho_{in}) u \partial_x u \\ & - \rho (\partial_t (\Lambda u_{in} + u) + (\bar{u} + \Lambda u_{in} + u) \partial_x (\Lambda u_{in} + u)) \\ & - (p'(\bar{p} + \Lambda \rho_{in} + \rho) - p'(\bar{p} + \Lambda \rho_{in})) \partial_x (\Lambda \rho_{in} + \rho) - p'(\bar{p} + \Lambda \rho_{in}) \partial_x \rho, \end{aligned} \quad (2.4)$$

satisfying

$$\rho(0, \cdot) = \rho(T, \cdot) = 0 \text{ and } u(0, \cdot) = u(T, \cdot) = 0. \quad (2.5)$$

We thus introduce a function $F : (\hat{\rho}, \hat{u}) \mapsto (\rho, u)$ defined as follows:

- u solves the control problem

$$(\bar{p} + \Lambda \rho_{in})(\partial_t u + \bar{u} \partial_x u) - \partial_{xx} u = g(\hat{\rho}, \hat{u}) \text{ in } [0, T] \times (0, L), \quad (2.6)$$

and

$$u(0, \cdot) = u(T, \cdot) = 0 \quad \text{in } (0, L). \quad (2.7)$$

- ρ solves the equation

$$\partial_t \rho + (\bar{u} + u + \Lambda u_{in}) \partial_x \rho + \bar{p} \partial_x u = f(\hat{\rho}, \hat{u}) \text{ in } [0, T] \times (0, L), \quad (2.8)$$

and satisfies

$$\rho(0, \cdot) = \rho(T, \cdot) = 0 \quad \text{in } (0, L). \quad (2.9)$$

Before going further and explaining how these two control problems are solved, let us comment this fixed point approach.

- the source terms f and g both contains linear terms. Here, we do not consider the terms coming from the initial data (i.e. the terms involving components having a subscript “in”) as linear ones since they will eventually be arbitrarily small. The linear term in g is the one coming from the pressure. The linear term in f is $\bar{p}'(\bar{p})\rho$ and is in both sides of the equations. This only is a fictitious term which helps when deriving estimates on ρ , see Section 4.
- the equation of ρ in (2.8) is linear in ρ but non-linear in the couple (ρ, u) . Indeed, our arguments strongly use the fact that the function ρ travels along the characteristics of $\bar{u} + u + \Lambda u_{in}$.
- we have to make precise what is the fixed point space, but this would be done later.
- we will need to estimate the terms coming from the initial data in f and g . This will be done using [17] which yields a bound on (ρ_{in}, u_{in}) in $L^\infty(W^{2,\infty}) \cap W^{1,\infty}(W^{1,\infty}) \times L^\infty(W^{2,\infty}) \cap W^{1,\infty}(L^\infty)$ in terms of the $H^3 \times H^3$ -norm of $(\rho_0 - \bar{p}, u_0 - \bar{u})$. This bound will be called R_{in} in the sequel and is assumed to be small. Roughly speaking, R_{in} is proportional to κ in (1.10).

The talk is organized as follows. In Section 3, we explain how to solve the control problem in u in (2.6)–(2.7). In Section 4, we solve the control problem (2.8)–(2.9) for the density. We then briefly explain in Section 5 how to conclude. We finally end up with some further comments.

3 Controlling u

3.1 Controlling parabolic equations: the classical approach

Before going further, let us briefly explain the strategy of A. Fursikov and O. Imanuvilov (see for instance [8]) to control a parabolic equation.

The control problem. Let $g \in L^2((0, T) \times (0, L))$ and consider the following distributed control problem with a control supported in $(a, b) \subset (0, L)$: Find a function $v \in L^2((0, T) \times (a, b))$ such that the solution u of

$$\partial_t u - \partial_{xx} u = g + v \mathbf{1}_{(a,b)} \text{ in } (0, T) \times (0, L), \quad u(t, 0) = u(t, L) = 0. \quad (3.1)$$

with initial data

$$u(0, \cdot) = 0 \text{ in } (0, L) \quad (3.2)$$

satisfies

$$u(T, \cdot) = 0 \text{ in } (0, L). \quad (3.3)$$

A dual problem. In order to solve the control problem (3.1)–(3.2)–(3.3), the idea is to introduce its weak formulation: v is an admissible control function if and only if for all smooth function z satisfying $z(t, 0) = z(t, L) = 0$,

$$\int_0^T \int_0^L gz + \int_0^T \int_a^b vz + \int_0^T \int_0^L u(\partial_t z + \partial_{xx} z) = 0. \quad (3.4)$$

This equivalent weak formulation of the control problem suggests to focus on the observability properties of the adjoint equation. This is precisely the goal of Carleman estimates.

A Carleman estimate for the adjoint equation. Based on the ideas above, A. Fursikov and O. Imanuvilov in [8] prove a Carleman estimate for the adjoint system. This requires the introduction of several notations.

Let us first set $\psi \in C^\infty([0, L]; \mathbb{R}_+)$ such that

$$3 \leq \min_{[0, L]} \psi \leq \max_{[0, L]} \psi \leq 4, \quad \max_{[b, L]} \psi' < 0 \quad \text{and} \quad \min_{[0, a]} \psi' > 0. \quad (3.5)$$

We then define the weight functions $\varphi(t, x)$ and $\xi(t, x)$, depending on a positive parameter $\lambda \geq 1$, as follows:

$$\varphi(t, x) = \frac{1}{t(T-t)} \left(e^{5\lambda} - e^{\lambda\psi(x)} \right), \quad (t, x) \in (0, T) \times [0, L], \quad (3.6)$$

$$\xi(t, x) = \frac{1}{t(T-t)} e^{\lambda\psi(x)}, \quad (t, x) \in (0, T) \times [0, L]. \quad (3.7)$$

Using these notations, A. Fursikov and O. Imanuvilov in [8] prove the following Carleman estimate:

Theorem 3.1. *There exist constants $C > 0$, $s_0 \geq 1$ and $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$ and $s \geq s_0$, for all smooth function z satisfying $z(t, 0) = z(t, L) = 0$,*

$$\begin{aligned} s^3 \lambda^4 \iint_{(0, T) \times (0, L)} \xi^3 e^{-2s\varphi} |z|^2 + s \lambda^2 \iint_{(0, T) \times (0, L)} \xi e^{-2s\varphi} |\partial_x z|^2 \\ + \frac{1}{s} \iint_{(0, T) \times (0, L)} \frac{1}{\xi} e^{-2s\varphi} (|\partial_{xx} z|^2 + |\partial_t z|^2) \\ \leq C \iint_{(0, T) \times (0, L)} e^{-2s\varphi} |\partial_t z + \partial_{xx} z|^2 + C s^3 \lambda^4 \iint_{(0, T) \times (a, b)} \xi^3 e^{-2s\varphi} |z|^2. \end{aligned} \quad (3.8)$$

Remark that this Carleman estimate is robust with respect to perturbations due to the freedom on the parameters s and λ which can be made arbitrarily large. In particular, one easily checks that the Carleman estimate (3.8) still holds when replacing the right-hand side of (3.8) by

$$C \iint_{(0, T) \times (0, L)} e^{-2s\varphi} |\partial_t z + \partial_{xx} z + a(t, x) \partial_x z + b(t, x) z|^2 + C s^3 \lambda^4 \iint_{(0, T) \times (a, b)} \xi^3 e^{-2s\varphi} |z|^2,$$

provided a and b belong to $L^\infty((0, T) \times (0, L))$ and $s \geq s_1(\|a\|_{L^\infty}, \|b\|_{L^\infty}) \geq s_0$.

A Carleman based duality for solving the control problem. According to Theorem 3.1 and the weak formulation (3.4) of the control problem (3.1)–(3.2)–(3.3), it is natural to introduce the functional J defined by

$$J(z) = \frac{s^3 \lambda^4}{2} \iint_{(0, T) \times (a, b)} \xi^3 e^{-2s\varphi} |z|^2 + \frac{1}{2} \iint_{(0, T) \times (0, L)} e^{-2s\varphi} |\partial_t z + \partial_{xx} z|^2 + \int_0^T \int_0^L gz, \quad (3.9)$$

and the norm $\|\cdot\|_{obs}$ defined by

$$\|z\|_{obs}^2 = s^3 \lambda^4 \iint_{(0, T) \times (a, b)} \xi^3 e^{-2s\varphi} |z|^2 + \iint_{(0, T) \times (0, L)} e^{-2s\varphi} |\partial_t z + \partial_{xx} z|^2$$

for smooth function z with boundary conditions $z(t, 0) = z(t, L) = 0$. Note that this defines a norm according to (3.8) and one has an explicit quantification of weighted $L^2(L^2)$, $H^1(L^2)$ and $L^2(H^2)$ norms of z in terms of $\|z\|_{obs}$.

Of course, the set of functions z in $C^2([0, T] \times [0, L])$ satisfying homogeneous Dirichlet boundary conditions is not complete with respect to that norm, and we shall introduce the completion X_{obs} of the set of functions in $C^2([0, T] \times [0, L])$ satisfying homogeneous Dirichlet boundary conditions with respect to $\|\cdot\|_{obs}$.

According to (3.8), the functional J in (3.9) can then be extended continuously on X_{obs} provided

$$\int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |g|^2 dxdt < \infty. \quad (3.10)$$

In this case, the functional J is continuous, strictly convex and coercive on X_{obs} . Therefore it admits a unique minimizer $Z \in X_{obs}$ and, writing down the corresponding Euler-Lagrange equation, setting

$$U = (\partial_t Z + \partial_{xx} Z) e^{-2s\varphi}, \quad V = s^3 \lambda^4 \xi^3 \mathbf{1}_{(a,b)} Z e^{-2s\varphi}, \quad (3.11)$$

(U, V) solves the weak formulation (3.4) of the control problem (3.1)–(3.2)–(3.3). By uniqueness of the solutions of the heat equation in the sense of transposition, (U, V) solves the control problem (3.1)–(3.2)–(3.3).

Besides, since $J(Z) \leq J(0) = 0$, the Carleman estimate (3.8) yields that

$$s^3 \lambda^4 \|Z\|_{obs}^2 \leq C \int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |f|^2 dxdt.$$

Using the identities in (3.11), this implies

$$s^3 \lambda^4 \int_0^T \int_0^L e^{2s\varphi} |U|^2 + \int_0^T \int_a^b \xi^{-3} e^{2s\varphi} |V|^2 \leq C \int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |f|^2.$$

Then, using weighted energy estimates on the solution U of (3.1), we can furthermore get weighted estimates on higher norms of U , namely:

$$\begin{aligned} s^3 \lambda^4 \int_0^T \int_0^L e^{2s\varphi} |U|^2 + s \lambda^2 \int_0^T \int_0^L \xi^{-2} e^{2s\varphi} |\partial_x U|^2 \\ + \frac{1}{s} \int_0^T \int_0^L \xi^{-4} e^{2s\varphi} (|\partial_t U|^2 + |\partial_{xx} U|^2) + \int_0^T \int_a^b \xi^{-3} e^{2s\varphi} |V|^2 \\ \leq C \int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |g|^2. \end{aligned} \quad (3.12)$$

Comments. The above strategy is very useful when dealing with semilinear parabolic type equations. In this case indeed, the strategy consists of doing a fixed point argument, as in our case, which consists of a map $\hat{u} \mapsto u$ in which u solves a control problem corresponding to a source term which is the semi-linear term for \hat{u} . This is for instance the strategy used for controlling Burgers equations in 1-d, see [8].

In this case, the main difficulty is to show that the fixed point argument works, and in particular that the map $\hat{u} \mapsto u$ defined that way maps a ball into itself. In order to do this, according to the estimates in (3.12), the natural balls have the following form:

$$\begin{aligned} s^3 \lambda^4 \int_0^T \int_0^L e^{2s\varphi} |U|^2 + s \lambda^2 \int_0^T \int_0^L \xi^{-2} e^{2s\varphi} |\partial_x U|^2 \\ + \frac{1}{s} \int_0^T \int_0^L \xi^{-4} e^{2s\varphi} (|\partial_t U|^2 + |\partial_{xx} U|^2) \leq R^2. \end{aligned} \quad (3.13)$$

Of course, the difficulty then is to get suitable estimates on the source term g in the corresponding weighted space, i.e. on

$$\int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |g|^2.$$

But, due to the form of the weight function $\exp(s\varphi)$, which blows up exponentially as $t \rightarrow 0$ and $t \rightarrow T$, even the finiteness of this integral is not at all obvious. Basically, this works fine when the function g

contains only quadratic terms in u , but it is more subtle when g contains linear terms, in which case the powers in s and λ may be of particular relevance.

Moreover, the fact that this weight function blows up as $t \rightarrow 0$ makes difficult to handle initial data directly by letting in the same time the parameters s and λ free, thus bringing a difficulty to handle semilinearities which contains linear terms. It is then nicer to modify the control problem by introducing an additional source term taking into account the initial data so that the problem reduces to go from 0 to 0. This is precisely what we have done in Section 2 with the introduction of the variables (ρ_{in}, u_{in}) .

In our control problem (2.6)–(2.7), the source term $g(\hat{\rho}, \hat{u})$ contains the linear term $\partial_x \hat{\rho}$. We should therefore be very careful to keep the dependence in the parameters s and λ and to get a suitable estimate on

$$\int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |\partial_x \rho|^2.$$

But ρ is a solution of an hyperbolic equation which travels along the characteristics of $\bar{u} + u + \Lambda u_{in}$, which essentially are the ones of \bar{u} . We will therefore construct a weight function which follows the characteristics $t \mapsto x_0 + \bar{u}t$.

3.2 Revisiting the approach of A. Fursikov and O. Imanuvilov

The idea is to modify the original approach of A. Fursikov and O. Imanuvilov to solve the control problem (2.6)–(2.7) by taking into account the hyperbolic nature of the equation of the density.

Extension of the domain for the control problem. Our original control problem is given by (2.6)–(2.7). We first extend the spatial domain to $(-4\bar{u}T, L)$ in order to reformulate the problem with a distributed control localized in $(-4\bar{u}T, -\bar{u}T)$.

The new control problem is then as follows: given a source term $g \in L^2((0, T) \times (0, L))$, find a control function $v \in L^2((0, T) \times (-4\bar{u}T, -\bar{u}T))$ such that the solution u of

$$(\bar{\rho} + \Lambda \rho_{in})(\partial_t u + \bar{u} \partial_x u) - \partial_{xx} u = g \mathbf{1}_{(0, L)} + v \mathbf{1}_{(-4\bar{u}T, -\bar{u}T)} \text{ in } [0, T] \times (-4\bar{u}T, L), \quad (3.14)$$

with the additional boundary conditions

$$u(t, -4\bar{u}T) = u(t, L) = 0 \text{ in } (0, T) \quad (3.15)$$

satisfies

$$u(0, \cdot) = u(T, \cdot) = 0 \text{ in } (-4\bar{u}T, L). \quad (3.16)$$

A new Carleman estimate for the adjoint equation. In order to solve the above control problem, we are going to choose a function $(t, x) \mapsto \psi(x - \bar{u}t)$ which “follows” the flow. To be more precise, we set $\psi \in C^\infty(\mathbb{R}; \mathbb{R}_+)$ such that

$$3 \leq \min_{[-5\bar{u}T, L]} \psi \leq \max_{[-5\bar{u}T, L]} \psi \leq 4, \quad \max_{[-3\bar{u}T, L]} \psi' < 0 \quad \text{and} \quad \min_{[-5\bar{u}T, -4\bar{u}T]} \psi' > 0. \quad (3.17)$$

Then, let $\theta = \theta(t) \in C^2([0, T]; \mathbb{R}_+)$ be defined by

$$\theta(t) = \begin{cases} t & \text{in } [0, 2T_0] \\ 1 & \text{in } [3T_0, T - 3T_0] \\ T - t & \text{in } [T - 2T_0, T], \end{cases} \quad (3.18)$$

and being such that θ is increasing on $[0, 3T_0]$ and decreasing on $[T - 3T_0, T]$.

We then define $\varphi(t, x)$ and $\xi(t, x)$ as follows:

$$\varphi(t, x) = \frac{1}{\theta(t)} \left(e^{5\lambda} - e^{\lambda\psi(x - \bar{u}t)} \right), \quad (t, x) \in (0, T) \times \mathbb{R} \quad (\lambda > 0), \quad (3.19)$$

$$\xi(t, x) = \frac{1}{\theta(t)} e^{\lambda\psi(x - \bar{u}t)}, \quad (t, x) \in (0, T) \times \mathbb{R} \quad (\lambda > 0). \quad (3.20)$$

Note that the critical points of the function ψ in (3.17) belong to $(-4\bar{u}T, -3\bar{u}T)$ and then, for all $t \in [0, T]$, the critical points of the function $x \mapsto \psi(x - \bar{u}t)$ belong to $(-4\bar{u}T, -2\bar{u}T)$. This is of particular relevance to obtain the following Carleman estimate, whose proof closely follows the one of Theorem 3.1:

Theorem 3.2. Assume that $\Lambda\rho_{in}$ is bounded in $W^{1,\infty}((0,T) \times (-4\bar{u}T, L))$ and that $\inf_{(-4\bar{u}T, L)}(\bar{\rho} + \Lambda\rho_{in}) > 0$. Let ψ, θ, φ and ξ be defined as in (3.17)–(3.18)–(3.19)–(3.20).

There exist $s_0 \geq 1, \lambda_0 \geq 1$ and $C > 0$ such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$, for all smooth function $z : [0, T] \times [-4\bar{u}T, L] \rightarrow \mathbb{R}$ satisfying $z(t, -4\bar{u}T) = z(t, L) = 0$,

$$\begin{aligned} & s^3 \lambda^4 \iint_{(0,T) \times (-4\bar{u}T, L)} \xi^3 e^{-2s\varphi} |z|^2 + s\lambda^2 \iint_{(0,T) \times (-4\bar{u}T, L)} \xi e^{-2s\varphi} |\partial_x z|^2 \\ & \quad + \frac{1}{s} \iint_{(0,T) \times (-4\bar{u}T, L)} \frac{1}{\xi} e^{-2s\varphi} (|\partial_{xx} z|^2 + |\partial_t z|^2) \\ & \leq C \iint_{(0,T) \times (-4\bar{u}T, L)} e^{-2s\varphi} |\partial_t((\bar{\rho} + \Lambda\rho_{in})z) + \partial_t((\bar{\rho} + \Lambda\rho_{in})z) + \partial_{xx} z|^2 \\ & \quad + Cs^3 \lambda^4 \iint_{(0,T) \times (-4\bar{u}T, -\bar{u}T)} \xi^3 e^{-2s\varphi} |z|^2. \end{aligned} \quad (3.21)$$

Solving the control problem. Based on the Carleman estimate (3.21) and following the strategy given in Subsection 3.1, provided $g = \hat{g} := g(\hat{\rho}, \hat{u})$ satisfies

$$\int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |\hat{g}|^2 < \infty, \quad (3.22)$$

we get a solution (U, V) of the control problem (3.14)–(3.15)–(3.16) such that

$$\begin{aligned} & s^3 \lambda^4 \int_0^T \int_{-4\bar{u}T}^L e^{2s\varphi} |U|^2 + s\lambda^2 \int_0^T \int_{-4\bar{u}T}^L \xi^{-2} e^{2s\varphi} |\partial_x U|^2 \\ & \quad + \frac{1}{s} \int_0^T \int_{-4\bar{u}T}^L \xi^{-4} e^{2s\varphi} (|\partial_t U|^2 + |\partial_{xx} U|^2) + \int_0^T \int_{-4\bar{u}T}^{\bar{u}T} \xi^{-3} e^{2s\varphi} |V|^2 \\ & \leq C \int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |\hat{g}|^2. \end{aligned} \quad (3.23)$$

Besides, easy multiplier identities yield estimates on the traces of the controlled solution U at $x = 0$ and $x = L$:

$$\begin{aligned} & s^2 \lambda^3 \int_0^T \xi^{-1} e^{2s\varphi(t,0)} |U(t,0)|^2 dt + \lambda \int_0^T \xi^{-3} e^{2s\varphi(t,0)} |\partial_x U(t,0)|^2 dt \\ & \quad + \lambda \int_0^T \xi^{-3} e^{2s\varphi(t,L)} |\partial_x U(t,L)|^2 dt \leq C \int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |\hat{g}|^2. \end{aligned} \quad (3.24)$$

We are now in position to introduce the convex set in which u will be looked for:

$$Y_{s,\lambda,R_u} = \left\{ u \text{ such that } u(t, L) = 0, t \in (0, T), \right. \\ \left. \begin{array}{ll} e^{s\varphi} u \in L^2((0, T) \times (0, L)) & \text{with } \|s^{3/2} \lambda^2 e^{s\varphi} u\|_{L^2((0,T) \times (0,L))} \leq R_u, \\ \xi^{-1} e^{s\varphi} \partial_x u \in L^2((0, T) \times (0, L)) & \text{with } \|s^{1/2} \lambda \xi^{-1} e^{s\varphi} \partial_x u\|_{L^2((0,T) \times (0,L))} \leq R_u, \\ \xi^{-2} e^{s\varphi} \partial_{xx} u \in L^2((0, T) \times (0, L)) & \text{with } \|s^{-1/2} \xi^{-2} e^{s\varphi} \partial_{xx} u\|_{L^2((0,T) \times (0,L))} \leq R_u, \\ \xi^{-2} e^{s\varphi} \partial_t u \in L^2((0, T) \times (0, L)) & \text{with } \|s^{-1/2} \xi^{-2} e^{s\varphi} \partial_t u\|_{L^2((0,T) \times (0,L))} \leq R_u. \end{array} \right\}, \quad (3.25)$$

where φ and ξ are the weight functions defined in (3.19)–(3.20). Here, $R_u > 0$ is the size of the ball. It will eventually be small enough.

4 Controlling the density

4.1 Constructing ρ

Here we explain a natural way to solve the control problem in (2.8)–(2.9).

In order to do that, we assume that u has been constructed as in Section 3 and belongs to some Y_{s,λ,R_u} . Note that $R_u > 0$ will be chosen to be small at the end, so that we can think of u as being small. This in particular implies that the flow

$$\partial_t X(t, \tau, a) = \bar{u} + u(t, X(t, \tau, a)) + \Lambda u_{in}(t, X(t, \tau, a)), \quad X(\tau, \tau, a) = a \quad (4.1)$$

is close to $a + (t - \tau)\bar{u}$.

It follows that $t \mapsto X(t, 0, 0)$ runs across the whole domain $[0, L]$ before the time T since $T > T_{\bar{u}} = L/\bar{u}$. Based on this fact we are going to introduce a forward ρ , denoted by ρ_f , which takes into account the source term and the initial condition $\rho_f(0, \cdot) = 0$ in $[0, L]$, and a backward ρ , denoted by ρ_b , which imposes the goal $\rho_b(T, \cdot) = 0$ in $[0, L]$.

To be more precise, setting $\hat{f} = f(\hat{\rho}, \hat{u})$, we define ρ_f as the solution of

$$\begin{cases} \partial_t \rho_f + (\bar{u} + u + \Lambda u_{in}) \partial_x \rho_f + \bar{\rho} \partial_x u + \bar{\rho} p'(\bar{\rho}) \rho_f = \hat{f} & \text{in } [0, T] \times (0, L), \\ \rho_f(0, x) = 0 & \text{in } (0, L), \\ \rho_f(t, 0) = 0 & \text{in } (0, T), \end{cases} \quad (4.2)$$

and ρ_b as the solution of

$$\begin{cases} \partial_t \rho_b + (\bar{u} + u + \Lambda u_{in}) \partial_x \rho_b + \bar{\rho} \partial_x u + \bar{\rho} p'(\bar{\rho}) \rho_b = \hat{f} & \text{in } [0, T] \times (0, L), \\ \rho_b(T, x) = 0 & \text{in } (0, L), \\ \rho_b(t, L) = 0 & \text{in } (0, T). \end{cases} \quad (4.3)$$

We then choose $T_0 > 0$ small enough such that $(T - 8T_0)\bar{u} = L$ and a small interval $(t_1, t_2) \subset [3T_0, 5T_0]$ such that the trajectories $t \mapsto X(t, t_1, 0)$ and $t \mapsto X(t, t_2, 0)$ reach the boundary $x = L$ in times smaller than $T - 3T_0$, see Figure 1. Note that the choice of (t_1, t_2) can be done independently of R_u small enough since R_u also bounds the L^∞ -norm of u by classical Sobolev's embedding.

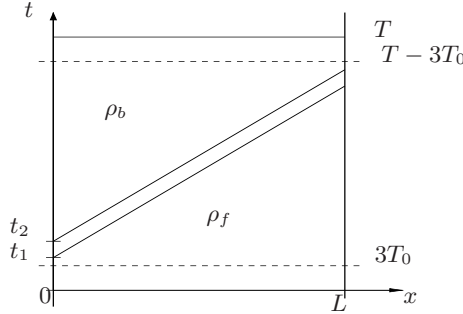


Figure 1: The geometric configuration. Here, the lines represent the characteristics associate to \bar{u} starting from $t = t_1, x = 0$ and from $t = t_2, x = 0$.

We then introduce η the solution of

$$\begin{cases} \partial_t \eta + (\bar{u} + u + \Lambda u_{in}) \partial_x \eta = 0 & \text{in } [0, T] \times [0, L], \\ \eta(0, \cdot) = 0 & \text{in } [0, L], \\ \eta(t, 0) = \tilde{\eta}(t) & \text{in } [0, T], \end{cases} \quad (4.4)$$

where $\tilde{\eta}(t)$ is a smooth function satisfying

$$\tilde{\eta}(t) = \begin{cases} 0 & \text{if } t < t_1, \\ 1 & \text{if } t > t_2, \end{cases}$$

and we glue ρ_f and ρ_b along the characteristics using the function η :

$$\rho(t, x) = \rho_f(t, x)(1 - \eta(t, x)) + \rho_b(t, x)\eta(t, x). \quad (4.5)$$

By construction, this ρ solves the control problem (2.8)–(2.9).

4.2 Getting estimates

The main difficulty then is to get suitable estimates for ρ . Recall that the source term g in the equation of u contains a term of the form $\partial_x \rho$. According to Section 3, we should show that

$$\int_0^T \int_0^L \xi^{-3} e^{2s\varphi} |\partial_x \rho|^2 < \infty$$

and give an explicit bound on this quantity.

This suggests to consider the equation of $\partial_x \rho$:

$$\partial_t \partial_x \rho + (\bar{u} + u + \Lambda u_{in}) \partial_x (\partial_x \rho) + \partial_x (u + \Lambda u_{in}) \partial_x \rho + \bar{\rho} \partial_{xx} u + \bar{\rho} p'(\bar{\rho}) \partial_x \rho = \partial_x \hat{f}.$$

Here, assuming that the source term and boundary term will not create any difficulties, we may get bounds on $\partial_x \rho$ in terms of $\partial_{xx} u$. But this latter term is too singular to run the fixed point argument since it has the same strength as the principal part of the parabolic operator. This statement can be made more precise by checking that an estimate in $\|\xi^{-2} e^{2s\varphi} \partial_{xx} u\|_{L^2}$ given by Section 3 is not sufficient to yield a suitable estimate on $\|\xi^{-3/2} e^{2s\varphi} \partial_x \rho\|_{L^2}$ to run the fixed point argument.

It is therefore natural to try to get rid of this term $\partial_{xx} u$. Remarking that this term also appears in the equation of u , we then combine these two equations and introduce the new variables

$$\mu_f(t, x) = u + \frac{1}{\bar{\rho}^2} \partial_x \rho_f, \quad \mu_b(t, x) = u + \frac{1}{\bar{\rho}^2} \partial_x \rho_b, \quad (4.6)$$

which both solve

$$\partial_t \mu + (\bar{u} + u + \Lambda u_{in}) \partial_x \mu + k \mu = h, \quad (4.7)$$

where the functions h and k can be computed explicitly:

$$\begin{aligned} \bar{\rho} h &:= \frac{1}{\bar{\rho}} \partial_x \hat{f} + \hat{g} - \Lambda \rho_{in} (\partial_t u + \bar{u} \partial_x u) + \bar{\rho} \partial_x [u(u + \Lambda u_{in})] + p'(\bar{\rho}) \bar{\rho}^2 u, \\ k &:= p'(\bar{\rho}) \bar{\rho} + \partial_x (u + \Lambda u_{in}). \end{aligned} \quad (4.8)$$

Of course, to determine μ_f and μ_b completely, we shall also make precise the boundary conditions in space-time:

$$\mu_f(0, x) = 0 \text{ for } x \in (0, L), \quad \mu_b(T, x) = 0 \text{ for } x \in (0, L), \quad (4.9)$$

$$\mu_f(t, 0) = m_f(t) := u(t, 0) + \frac{1}{\bar{\rho}^2} \left(\frac{1}{\bar{u} + u(t, 0) + \Lambda u_{in}(t, 0)} \right) \left(\hat{f}(t, 0) - \bar{\rho} \partial_x u(t, 0) \right), \quad (4.10)$$

$$\mu_b(t, L) = m_b(t) := \frac{1}{\bar{\rho}^2} \left(\frac{1}{\bar{u} + \Lambda u_{in}(t, L)} \right) \left(\hat{f}(t, L) - \bar{\rho} \partial_x u(t, L) \right), \quad (4.11)$$

The equation (4.7) is nicer than the one of $\partial_x \rho$ since the source term h is much less singular than $\partial_{xx} u$. Let us also point out that the term $\bar{\rho} p'(\bar{\rho}) \hat{\rho}$ in $\hat{f} = f(\hat{\rho}, \hat{u})$ (recall (2.3)) is the one needed so that $\partial_x \hat{f} / \bar{\rho} + \hat{g}$ does not contain any linear term in $\partial_x \hat{\rho}$. This explains our perhaps surprising choice of writing the equation (2.1) with the addition in both sides of the equation of the linear term $\bar{\rho} p'(\bar{\rho}) \hat{\rho}$.

But one should still be careful when doing the estimates since the boundary conditions contain terms of the form $\partial_x u(t, 0)$ and $\partial_x u(t, L)$.

Also note that the boundary conditions $\rho_f(t, 0) = \rho_b(t, L) = 0$ allow to recover ρ_f, ρ_b directly from the knowledge of μ_f, μ_b :

$$\rho_f(t, x) := \bar{\rho}^2 \int_0^x (\mu_f - u)(t, y) dy, \quad \rho_b(t, x) := -\bar{\rho}^2 \int_x^L (\mu_b - u)(t, y) dy. \quad (4.12)$$

The main difficulty then reduces to get estimates on μ_f and μ_b .

We now have to get estimates on the solution of a transport equation with a source term, see (4.7). We thus explicitly write the solution μ_f (of course, the same can be done with μ_b):

- if $x \geq X(t, 0, 0)$:

$$\mu_f(t, x) = \int_0^t h(\tilde{\tau}, X(\tilde{\tau}, t, x)) e^{-\int_0^{\tilde{\tau}} k(\tau, X(\tau, t, x)) d\tau} d\tilde{\tau}. \quad (4.13)$$

- if $x \leq X(t, 0, 0)$: in this case, there exists a unique $t^*(t, x) \in [0, t]$ such that $X(t, t^*(t, x), 0) = x$ and

$$\begin{aligned} \mu_f(t, x) &= m_f(t^*(t, x)) e^{-\int_{t^*(t, x)}^t k(\tau, X(\tau, t, x)) d\tau} \\ &\quad + \int_{t^*(t, x)}^t h(\tilde{\tau}, X(\tilde{\tau}, t, x)) e^{-\int_{t^*(t, x)}^{\tilde{\tau}} k(\tau, X(\tau, t, x)) d\tau} d\tilde{\tau}. \end{aligned} \quad (4.14)$$

We then use the following facts:

- the flow X corresponding to the velocity field $\bar{u} + u + \Lambda u_{in}$ is close to the one corresponding to \bar{u} :

$$|X(t, \tau, x) - (x + (t - \tau)\bar{u})| \leq C|t - \tau| \|u + \Lambda u_{in}\|_{L^\infty},$$

- the Carleman weight φ in (3.19) decreases along the characteristics of the flow corresponding to \bar{u} while $t < T - 3T_0$: for all $0 < \tau < t < T - 3T_0$ and $x \in [0, L - \bar{u}(t - \tau)]$,

$$\varphi(\tau, x) \geq \varphi(t, x + (t - \tau)\bar{u}).$$

(This estimate is needed to estimate μ_f .)

- the Carleman weight φ in (3.19) increases along the characteristics of the flow corresponding to \bar{u} while $t > 3T_0$: for all $3T_0 < \tau < t < T$ and $x \in [0, L - \bar{u}(t - \tau)]$,

$$\varphi(\tau, x) \leq \varphi(t, x + (t - \tau)\bar{u}).$$

(This estimate is needed to estimate μ_b .)

Note that the two last points should both be true on $(3T_0, T - 3T_0)$. This imposes that $\theta(t)$ in the Carleman weight function, defined in (3.18), should be constant on $(3T_0, T - 3T_0)$.

Other technical estimates (especially one on the Jacobian of the change of variable $x \mapsto X(t, \tau, x)$) are also needed that we do not report here but are precisely done in [6].

Accordingly, based on the explicit formula (4.13)–(4.14), we can show that

$$\begin{aligned} \sup_{[0, T-3T_0]} \int_0^L |\mu_f(t, x)|^2 \frac{e^{2s\varphi(t, x)}}{\xi^3(t, x)} dx &\leq C e^{C_0 s \lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} |m_f(\tau)|^2 \frac{e^{2s\varphi(\tau, 0)}}{\xi^3(\tau, 0)} d\tau \\ &\quad + C e^{C_0 s \lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |h(\tau, y)|^2 \frac{e^{2s\varphi(\tau, y)}}{\xi^3(\tau, y)} d\tau dy. \end{aligned} \quad (4.15)$$

And such estimates allow to obtain bounds on $e^{s\varphi} \xi^{-3/2} \partial_x \rho_f$ from the ones on u in (3.23) and the fact that $\hat{u} \in Y_{s, \lambda, R_u}$. Here, we emphasize that the worst term in (4.15) is the one coming from the boundary $m_f(t)$, and more precisely to $\partial_x u(t, 0)$, see (4.10). But there is a gain of λ in the spatial derivative at $x = 0$ in estimates (3.24), which makes the fixed point strategy work.

Also note that, for $u \in Y_{s, \lambda, R_u}$,

$$e^{C_0 s \lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \leq e^{C_0 s \lambda e^{4\lambda} R_u} e^{-\inf\{\varphi\} s/2} e^{C_0 s \lambda e^{4\lambda} R_{in}}.$$

Thus, the first term in the right-hand side is bounded uniformly for $R_u \in (0, 1)$ and $s, \lambda \geq 1$. Hence there exists a constant C_1 such that for $s, \lambda \geq 1$ and $R_u \in (0, 1)$, for all $u \in Y_{s, \lambda, R_u}$,

$$e^{C_0 s \lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \leq C_1 (1 + \mathcal{O}_{s, \lambda}(R_{in})), \quad (4.16)$$

where $\mathcal{O}_{s, \lambda}(R_{in})$ is a function which for any s, λ , can be made arbitrarily small by taking R_{in} small enough. Of course, this will impose an order in the choice of the parameters at the end, but this does not prevent the fixed point approach from working.

Remark 4.1. *The rigorous proof of Theorem 1.1, though relying exactly on the above ideas, requires a slightly improved form of the estimate (4.15).*

Indeed, there are quadratic terms in h coming from the initial data, for instance $\Lambda\rho_{in}\partial_t u$ (see (4.8)), which are too singular in time and cannot be handled directly as in (4.15). These terms, which we called h_1 in [6] and given by

$$\bar{\rho}h_1(t, x) = -\frac{1}{\bar{\rho}}\Lambda\rho_{in}\partial_{xx}\hat{u} - \Lambda\rho_{in}\partial_x\hat{u},$$

require a special treatment based on the following remarks:

- There exists a constant C such that

$$\|e^{s\varphi}\xi^{-3/2}\left(\int_0^t h_1(\tilde{\tau}, X(\tilde{\tau}, t, x))e^{-\int_0^{\tilde{\tau}} k(\tau, X(\tau, t, x))d\tau}d\tilde{\tau}\right)\|_{L^2} \leq Ce^{C_0s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}e^{2\lambda}\|e^{s\varphi}\xi^{-2}h_1\|_{L^2}.$$

- If u belongs to Y_{s,λ,R_u} , using (4.16) and the explicit form of h_1 ,

$$Ce^{C_0s\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}e^{2\lambda}\|e^{s\varphi}\xi^{-2}h_1\|_{L^2} \leq C_1(1 + \mathcal{O}_{s,\lambda}(R_{in}))e^{2\lambda}R_uR_{in} \leq C_2R_u\mathcal{O}_{s,\lambda}(R_{in}).$$

- R_{in} will be chosen after the parameters s and λ .

Let us finally remark that this discussion is not needed when getting estimate on μ_b on $(3T_0, T)$ since h_1 there vanishes.

Getting further in the estimates on ρ can then be done using these estimates on $\partial_x\rho$ and the formula (4.12). We do not give the details of these tedious computations that can be found in [6]. We only indicate the set in which ρ will be looking for to run the fixed point argument:

$$X_{s,\lambda,R_\rho} = \left\{ \rho \text{ such that } \begin{array}{l} \xi^{-1}e^{s\varphi}\rho \in L^2((0, T) \times (0, L)) \\ \xi^{-3/2}e^{s\varphi}\partial_x\rho \in L^2((0, T) \times (0, L)) \\ \partial_t\rho \in L^2((0, T) \times (0, L)) \\ e^{s\tilde{\varphi}/2}\rho \in L^\infty((0, T) \times (0, L)) \\ e^{s\tilde{\varphi}/2}\partial_x\rho \in L^\infty((0, T); L^2(0, L)) \\ (\xi^{-3/2}e^{s\varphi}\rho)(\cdot, 0) \in L^2(0, T) \\ (\xi^{-3/2}e^{s\varphi}\rho)(\cdot, L) \in L^2(0, T) \end{array} \right. \left. \begin{array}{l} \text{with } \|\xi^{-1}e^{s\varphi}\rho\|_{L^2((0, T) \times (0, L))} \leq R_\rho, \\ \text{with } \|\xi^{-3/2}e^{s\varphi}\partial_x\rho\|_{L^2((0, T) \times (0, L))} \leq R_\rho, \\ \text{with } \|\partial_t\rho\|_{L^2((0, T) \times (0, L))} \leq R_\rho, \\ \text{with } \|e^{s\tilde{\varphi}/2}\rho\|_{L^\infty((0, T) \times (0, L))} \leq R_\rho, \\ \text{with } \|e^{s\tilde{\varphi}/2}\partial_x\rho\|_{L^\infty((0, T); L^2(0, L))} \leq R_\rho, \\ \text{with } \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot, 0)\|_{L^2(0, T)} \leq R_\rho, \\ \text{with } \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot, L)\|_{L^2(0, T)} \leq R_\rho, \end{array} \right\} \quad (4.17)$$

where

$$\tilde{\varphi}(t) := \min_{x \in [0, L]} \varphi(t, x) = \varphi(t, 0), \quad t \in (0, T) \quad (4.18)$$

and φ and ξ are the weight functions given by (3.19)–(3.20).

5 The fixed point argument

Of course, everything has been done to guarantee that the fixed point argument runs smoothly. But still, one has to check that for $(\hat{\rho}, \hat{u}) \in X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$, $(\rho, u) = F(\hat{\rho}, \hat{u})$, which is defined as the solutions of the control problems (2.6)–(2.7) for u and (2.8)–(2.9) on ρ , belongs to $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$.

Tedious computations (for instance, one should get a bound on the L^2 -norm of $e^{s\varphi}\xi^{-3/2}g(\hat{\rho}, \hat{u})$) show that this is indeed the case provided $R_\rho = \alpha R_u$, where $\alpha > 0$ is small enough, R_u is small enough, the parameters s and λ are large enough and the parameter R_{in} is small enough, see [6]. (As indicated above in Section 4 (see also Remark 4.1), this parameter R_{in} , which comes from the initial data, is chosen last.)

We then use Schauder's fixed point theorem. In order to do that, we endow the set $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ with the topology of $L^2((0, T) \times (0, L))^2$ and show that the map F is continuous on $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ endowed with that topology. Besides, one easily checks that this set is convex, closed, and compact with respect to that topology.

Schauder's fixed point theorem then applies and yields the result. Details can be found in [6].

6 Conclusion

Several questions remain open:

- The initial data in Theorem 1.1 should be in an $H^3 \times H^3$ -neighborhood of the target equilibrium. This regularity is surprising and seems to be a consequence of the fact that the Carleman weight blows up as $t \rightarrow 0$. We are currently working on getting Carleman weights for the heat equation with a weight which does not blow up at time $t = 0$.
- Getting a local exact controllability result around smooth target trajectory should be possible under a suitable geometric condition. A possible way to do that is to include the target flow in the Carleman weight itself. This issue is currently under investigation.
- Our approach is based on the linearized compressible Navier-Stokes equation and consider the non-linear effects as a perturbation. Of course, another way to proceed would be to think all the way around by using the non-linear effects to control the fluid. This is the idea beyond the return method of J.-M. Coron [3, 4], which has already been used several times in the control theory of fluid flows. Whether or not these ideas can be applied in the context of compressible Navier Stokes equation is an interesting open problem.
- Our result applies in the 1d case only. Indeed, our method fails to provide estimates on the density in higher dimension. In particular, we did not find so far a quantity similar to μ in (4.6), which we strongly use to get suitable estimates on $\partial_x \rho$.
- Other controllability issues can be considered. For instance, as far as we know, approximate controllability for such system is completely open.

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