# Control and stabilization properties for a singular heat equation with an inverse-square potential

Sylvain Ervedoza*

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**Abstract**

The goal of this article is to analyze control properties of parabolic equations with a singular potential $-\mu/|x|^2$, where $\mu$ is a real number. When $\mu \leq (N-2)^2/4$, it was proved in [19] that the equation can be controlled to zero with a distributed control which surrounds the singularity. In the present work, using Carleman estimates, we will prove that this assumption is not necessary, and that we can control the equation from any open subset as for the heat equation. Then we will study the case $\mu > (N-2)^2/4$, and prove that the situation changes completely: Indeed, we will consider a sequence of regularized potentials $\mu/(|x|^2+\varepsilon^2)$, and prove that we cannot stabilize the corresponding systems uniformly with respect to $\varepsilon > 0$, due to the presence of explosive modes which concentrate around the singularity.

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*Laboratoire de Mathématiques de Versailles, Université de Versailles Saint-Quentin-en-Yvelines, 45, avenue des États Unis, 78035 Versailles Cedex, France

# 1 Introduction

Let $N \geq 3$ and consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^N$ such that $0 \in \Omega$, and let $\omega \subset \Omega$ be a non-empty open set.
We are interested in the control and stabilization properties of the following equation

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in L^2(\Omega)$. Here, $f \in L^2((0,T); H^{-1}(\Omega))$ is the control that we assume to be null in $\Omega \backslash \bar{\omega}$, that is

$$\forall \theta \in \mathcal{D}(\Omega \backslash \bar{\omega}), \quad \theta f = 0 \quad \text{in } L^2((0,T); H^{-1}(\Omega)). \tag{1.2}$$

First of all, let us briefly mention that the Cauchy problem with such singular potential is not straightforward. Indeed, it has been proved that there is a critical value $\mu^*(N) = (N-2)^2/4$ of $\mu$ which determines the well-posedness of (1.1). Actually, this problem is strongly related to the Hardy inequality:

$$\forall u \in H_0^1(\Omega), \qquad \mu^*(N) \int_\Omega \frac{u^2}{|x|^2} \,\mathrm{dx} \leq \int_\Omega |\nabla u|^2 \,\mathrm{dx}, \tag{1.3}$$

where $\mu^*(N)$ is the optimal constant. Note that equality in (1.3) is not attained.

The first work [1] on the Cauchy problem was considering positive initial data. In [1], it was proved that if $\mu \leq \mu^*(N)$ and if the initial data $u_0$ is positive, then equation (1.1) has a global weak solution whereas if $\mu > \mu^*(N)$, then equation (1.1) has no solution if $u_0 > 0$ and $f \geq 0$, even locally in time (see also [4]).

Actually, the Cauchy problem properties for equation (1.1) can be deduced from generalizations of the Hardy inequality (1.3). Studying more precisely (1.3), it is proved in [20] that the Cauchy problem is well-posed in $L^2(\Omega)$ for any $\mu \leq \mu^*(N)$. A precise functional setting is given even in the special case $\mu = \mu^*(N)$ (see [20]).

The objective of the present paper is twofold. First, when $\mu \leq \mu^*(N)$, we will prove the null-controllability of (1.1) with a control $f \in L^2((0,T); L^2(\omega))$. Second, we will show that when $\mu > \mu^*(N)$, there is no way to stabilize system (1.1) with a control supported in $\omega$ in a reasonable sense when $0 \notin \bar{\omega}$.

The null-controllability problem reads as follows: Given any $u_0 \in L^2(\Omega)$, find a function $f \in L^2(\omega \times (0,T))$ such that the solution of (1.1) satisfies

$$u(x,T) = 0, \qquad x \in \Omega. \tag{1.4}$$

The controllability issue was already discussed under the assumption $\mu \leq \mu^*(N)$ in the recent work [19], in the special case where $\omega$ contains an annulus centered in the singularity. The authors of [19] need this assumption since their proof strongly uses a decomposition in spherical harmonics which allows to reduce the problem to the study of 1-d singular equations. J. Le Rousseau mentioned an argument in [19] to relax this strong geometric assumption into these two conditions: $\omega$ circles the singularity, and the exterior part of $\omega$ contains an annular set centered in the singularity. Even with this improvement, a non-trivial geometric assumption on $\omega$ is needed. Our purpose is to prove that we can actually remove this assumption and consider any non-empty open subset $\omega$ of $\Omega$.

**Theorem 1.1.** *Let $\mu$ be a real number such that $\mu \leq \mu^*(N)$.*

*Given any non-empty open set $\omega \subset \Omega$, for any $T > 0$ and $u_0 \in L^2(\Omega)$, there exists a control $f \in L^2((0,T)\times\omega)$ such that the solution of* (1.1) *satisfies* (1.4). *Besides, there exists a constant $C_T$ such that*

$$\|f\|_{L^2((0,T)\times\omega)} \leq C_T \|u_0\|_{L^2(\Omega)}. \tag{1.5}$$

Following the by now classical HUM method ([16]), the controllability property is equivalent to an observability inequality for the adjoint system

$$\begin{cases} \partial_t w + \Delta w + \dfrac{\mu}{|x|^2} w = 0, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,T) = w_T(x), & x \in \Omega. \end{cases} \tag{1.6}$$

More precisely, when $\mu \leq \mu^*(N)$, we need to prove that there exists a constant $C$ such that for all $w_T \in L^2(\Omega)$, the solution of (1.6) satisfies

$$\int\limits_{\Omega} |w(x,0)|^2 \text{ dx} \leq C \iint\limits_{\omega\times(0,T)} |w(x,t)|^2 \text{ dx dt}. \tag{1.7}$$

In order to prove (1.7), we will use a particular Carleman estimate, which is by now a classical technique in control theory, see for instance [2, 9, 10, 11, 12, 13, 14]... Indeed, the Carleman estimate we will derive later implies that for any solution $w$ of (1.6)

$$\iint\limits_{\Omega\times(\frac{T}{4},\frac{3T}{4})} |w(x,t)|^2 \text{ dx dt} \leq C \iint\limits_{\omega\times(0,T)} |w(x,t)|^2 \text{ dx dt}, \tag{1.8}$$

which directly implies inequality (1.7) since $t \mapsto \|w(t,.)\|^2_{L^2(\Omega)}$ is increasing by the Hardy inequality (1.3).

The Carleman estimate derived here is inspired by the works [5, 17] on 1-d degenerate heat equations, the recent paper [19] which is inspired from the methods and results in [5, 17] to obtain radial estimates, and the article [13] on the controllability of the heat equation in any dimension. As in [5, 17, 19, 13], the major difficulty is to choose a special weight function appearing in the Carleman estimate. In [19], this has been done in the 1d case only, using spherical harmonics to recover results in the multi-d case, but with an extra geometric condition on the support of the control region. We thus adapt the results in [19] to derive directly Carleman estimates without using a spherical harmonics decomposition, in order to avoid the use of the geometric condition needed in [19].

Let us briefly present the existing results concerning the observability properties of a parabolic equation with a potential $V$:

$$\begin{cases} \partial_t z + \Delta z + V\, z = 0, & (x,t) \in \Omega \times (0,T), \\ z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ z(T) = z_T \in L^2(\Omega). \end{cases} \tag{1.9}$$

It has been proved in [13] using Carleman estimates that, for potentials $V \in L^\infty(\Omega \times (0,T))$, such systems are observable in the sense of (1.7) for any open set $\omega \subset \Omega$. Later, in [14], this result has been extended to the case $V \in L^\infty((0,T); L^{2N/3}(\Omega))$. To our knowledge, the case $V \in L^\infty((0,T); L^{N/2+\epsilon}(\Omega))$ with $\epsilon > 0$ is still open. Note that our work presents a case in which the potential $V = \mu/|x|^2$ is not in $L^{N/2}(\Omega)$, and therefore none of these results applies. In this context, it is worth mentioning the work [15] which proves the strong unique continuation property for system (1.9) for a general potential $V \in L^{(N+1)/2}(\Omega \times (0,T))$.

The second part of this work is devoted to the case $\mu > \mu^*(N)$. In this case, the Cauchy problem is severely ill-posed as proved in [1] and [4]. Indeed, if $u_0$ is positive and $f = 0$ in (1.1), there is complete instantaneous blow-up, which makes impossible to define a reasonable solution. However, it does not answer to the following stabilization problem:

Given $u_0 \in L^2(\Omega)$, can we find a control $f \in L^2((0,T); H^{-1}(\Omega))$ localized in $\omega$ such that there exists a solution $u \in L^2((0,T); H_0^1(\Omega))$ of (1.1) ?

In other words, we ask whether it is possible or not to prevent from blow-up phenomena by acting only on a subset. Before going further, note that if $u \in L^2((0,T); H_0^1(\Omega))$ satisfies (1.1) with $f \in L^2((0,T); H^{-1}(\Omega))$, then $\partial_t u \in L^2((0,T); H^{-1}(\Omega))$, and therefore $u \in C([0,T]; L^2(\Omega))$, and the equality $u(0) = u_0$ in (1.1) makes sense.

Following the ideas of optimal control, for any $u_0 \in L^2(\Omega)$, we consider the functional

$$J_{u_0}(u,f) = \frac{1}{2} \iint\limits_{\Omega \times (0,T)} |u(t,x)|^2 \ \mathrm{dx\ dt} + \frac{1}{2} \int_0^T \|f(t)\|^2_{H^{-1}(\Omega)} \ \mathrm{dt}, \tag{1.10}$$

defined on the set

$$\mathcal{C}(u_0) = \Big\{(u,f) \in L^2((0,T); H_0^1(\Omega)) \times L^2((0,T); H^{-1}(\Omega)) \text{ such that } u \\ \text{satisfies (1.1) with } f \text{ as in } (1.2)\Big\}. \quad (1.11)$$

We say that we can stabilize system (1.1) if we can find a constant $C$ such that

$$\forall u_0 \in L^2(\Omega), \qquad \inf_{(u,f)\in\mathcal{C}(u_0)} J_{u_0}(u,f) \leq C \left\|u_0\right\|^2_{L^2(\Omega)}. \quad (1.12)$$

Of course, this property strongly depends on the set $\omega$ where the stabilization is effective. Especially, when $0 \in \omega$, (1.12) holds (see Section 4 B1).

When $0 \notin \bar{\omega}$, the situation is more intricate. Therefore we focus our study on this particular case, and give a severe obstruction, in this case, to the stabilization property (1.12).

More precisely, for $\varepsilon > 0$, we approximate (1.1) by the systems

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2+\varepsilon^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \quad (1.13)$$

For these approximate problems, the Cauchy problem is well-posed. Therefore we can consider the functionals

$$J^\varepsilon_{u_0}(f) = \frac{1}{2} \iint\limits_{\Omega\times(0,T)} |u(x,t)|^2 \,\mathrm{dx}\,\mathrm{dt} + \frac{1}{2}\int_0^T \left\|f(t)\right\|^2_{H^{-1}(\Omega)} \,\mathrm{dt}, \quad (1.14)$$

where $f \in L^2((0,T); H^{-1}(\Omega))$ is localized in $\omega$ in the sense of (1.2) and $u$ is the corresponding solution of (1.13). We prove the following:

**Theorem 1.2.** *Assume that $\mu > \mu^*(N)$, and that $0 \notin \bar{\omega}$.*

*There is no constant $C$ such that for all $\varepsilon > 0$, and for all $u_0 \in L^2(\Omega)$,*

$$\inf_{\substack{f \,\in\, L^2((0,T);\, H^{-1}(\Omega)) \\ f \text{ as in } (1.2)}} J^\varepsilon_{u_0}(f) \leq C \left\|u_0\right\|^2_{L^2(\Omega)}. \quad (1.15)$$

In particular, this result implies that the stabilization of (1.1) is impossible to attain through regularization processes when $\mu > \mu^*(N)$ and $0 \notin \bar{\omega}$, and that we cannot prevent the system from blowing up.

Let us briefly mention the related work [12], which presents a study of the control properties of weakly blowing-up semi-linear heat equations, which deals with a similar question as the one asked here. In particular, in [12], examples of systems are given for which blow up may occur in finite time, but this blow-up can be controlled in any time for any initial data.

The structure of the paper is the following. In Section 2, we give the proof of Theorem 1.1 for $\mu \leq \mu^*(N)$, or, to be more precise, of inequality (1.7) for the solutions of the adjoint equation (1.6). In Section 3, we prove that when $\mu > \mu^*(N)$ we cannot uniformly stabilize system (1.1), in the sense of Theorem 1.2. In Section 4, we add some comments.

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# 2 Null controllability in the case $\mu \leq \mu^*(N)$

First of all, to simplify the presentation, we assume that $0 \notin \bar{\omega}$, that can always be done, taking if necessary a smaller set. We also assume that the unit ball $\bar{B}(0,1)$ is included in $\Omega$ and $\bar{B}(0,1) \cap \bar{\omega}$ is empty. This can always be done by a scaling argument.

## 2.1 Carleman estimate

As said in the introduction, the main tool we use to address the observability inequality (1.8) is a Carleman estimate. However, since it is based on tedious computations, we postpone the proofs of several technical lemmas in Subsection 2.3.

The major problem when designing a Carleman estimate is the choice of a smooth weight function $\sigma$, which is in general assumed to be positive, and to blow up as $t$ goes to zero and as $t$ goes to $T$. Hence we are looking for a weight function $\sigma$ that satisfies:

$$\left\{ \begin{array}{l} \sigma(t,x) > 0, \quad (x,t) \in \Omega \times (0,T), \\ \lim\limits_{t \to 0^+} \sigma(t,x) = \lim\limits_{t \to T^-} \sigma(t,x) = +\infty, \quad x \in \Omega. \end{array} \right. \tag{2.1}$$

More precisely, we propose the weight

$$\sigma(t,x) = s\theta(t)\Big(e^{2\lambda \sup \psi} - \frac{1}{2}|x|^2 - e^{\lambda\psi(x)}\Big) \tag{2.2}$$

where $s$ and $\lambda$ are positive parameters aimed at being large,

$$\theta(t) = \Big(\frac{1}{t(T-t)}\Big)^3, \tag{2.3}$$

and $\psi$ is a function satisfying

$$\left\{ \begin{array}{ll} \psi(x) = \ln(|x|), & x \in B(0,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \backslash \bar{B}(0,1), \end{array} \right. \tag{2.4}$$

and there exists an open set $\omega_0$ such that $\bar{\omega}_0 \subset \omega$ and $\delta > 0$ such that

$$|\nabla \psi(x)| \geq \delta, \qquad x \in \bar{\Omega} \backslash \omega_0. \tag{2.5}$$

The existence of such function $\psi$ is not straightforward but can be easily deduced from the construction given in [13].

Indeed, there exists a smooth function which extends $\ln(|x|)$ outside the ball, which vanishes on the boundary, and with finitely many critical points, since this property is generically true. Then it is sufficient to consider such a function, and to move its critical points into $\omega_0$ without modifying the function in $B(0,1)$. This can be done following the construction given in [13].

Note that the weight function $\sigma$ defined by (2.2) indeed satisfies (2.1) and is smooth (at least in $C^4((0,T) \times \bar{\Omega})$) when $\lambda$ is large enough.

To explain this choice for the weight function $\sigma$, we point out that in the ball $B(0,1)$, since $\psi$ is negative, the weight function $\sigma$ behaves like

$$s\theta(t)(C - \frac{1}{2}|x|^2)$$

when $\lambda$ is large. This corresponds precisely to the weight given in [17] for dealing with singular 1-d heat-type equation and in [19] when dealing with the observability around the singularity. On the contrary, outside the unit ball, since $\psi$ is positive, when $\lambda$ is large enough, the weight is very close to the one used for the observability of the heat equation in [13].

To simplify notations, let us denote by $\phi$ the function

$$\phi(x) = e^{\lambda \psi(x)}, \tag{2.6}$$

by $\mathcal{O}$ the open set $\Omega \backslash (\bar{B}(0,1) \cup \bar{\omega}_0)$ and by $\tilde{\mathcal{O}}$ the open set $\Omega \backslash \bar{B}(0,1)$.

We are now in position to state the Carleman estimate.

**Theorem 2.1.** *There exist positive constants $K$ and $\lambda_0$ such that for $\lambda \geq \lambda_0$, there exists $s_0(\lambda)$ such that for all $s \geq s_0$, any $w$ solution of* (1.6) *satisfies*

$$\begin{aligned} s\lambda^2 \iint_{\tilde{\mathcal{O}} \times (0,T)} \theta \phi e^{-2\sigma} |\nabla w|^2 \text{ dx dt} + s \iint_{\Omega \times (0,T)} \theta e^{-2\sigma} \frac{|w|^2}{|x|} \text{ dx dt} \\ + s^3 \iint_{\Omega \times (0,T)} \theta^3 e^{-2\sigma} |x|^2 |w|^2 \text{ dx dt} + s^3 \lambda^4 \iint_{\tilde{\mathcal{O}} \times (0,T)} \theta^3 \phi^3 e^{-2\sigma} |w|^2 \text{ dx dt} \\ \leq K \Bigg( s\lambda^2 \iint_{\omega_0 \times (0,T)} \theta \phi e^{-2\sigma} |\nabla w|^2 \text{ dx dt} \\ + s^3 \lambda^4 \iint_{\omega_0 \times (0,T)} \theta^3 \phi^3 e^{-2\sigma} |w|^2 \text{ dx dt} \Bigg). \end{aligned} \tag{2.7}$$

*Remark* 2.2. Following the proof carefully, one can check that there exists a constant $s_1(\psi) > 0$ such that the choice

$$s_0(\lambda) = s_1 e^{3\lambda \sup \psi}$$

is convenient in Theorem 2.1.

*Remark* 2.3. We stated the Carleman estimate (2.7) in the restrictive setting that we need, but we can handle a source term. To be more precise, for any $w \in \mathcal{D}([0,T] \times \Omega)$, taking $s$ and $\lambda$ large enough, the following holds:

$$\begin{aligned}
s\lambda^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi e^{-2\sigma} |\nabla w|^2 \text{ dx dt} + s \iint\limits_{\Omega\times(0,T)} \theta e^{-2\sigma} \frac{|w|^2}{|x|} \text{ dx dt} \\
+ s^3 \iint\limits_{\Omega\times(0,T)} \theta^3 e^{-2\sigma} |x|^2 |w|^2 \text{ dx dt} + s^3\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^3 e^{-2\sigma} |w|^2 \text{ dx dt} \\
+ s(\mu^*(N) - \mu) \iint\limits_{\Omega\times(0,T)} \theta e^{-2\sigma} \frac{|w|^2}{|x|^2} \text{ dx dt} \\
\leq K\Bigg( \iint\limits_{\Omega\times(0,T)} e^{-2\sigma} \Big|\partial_t w + \Delta w + \frac{\mu}{|x|^2} w\Big|^2 \text{ dx dt} \\
+ s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\phi e^{-2\sigma} |\nabla w|^2 \text{ dx dt} + s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3\phi^3 e^{-2\sigma} |w|^2 \text{ dx dt}\Bigg).
\end{aligned}$$

*Proof.* We present the main ideas and steps of the proof of Theorem 2.1, using several technical Lemmas, that are proved later in Subsection 2.3.

Let us first remark that using the density the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, if estimate (2.7) holds for any solution $w$ of (1.6) with initial data $w_T \in H_0^1(\Omega)$, then (2.7) also holds for any solution $w$ of (1.6) with initial data $w_T \in L^2(\Omega)$. We thus prove (2.7) only for solutions of (1.6) with initial data in $H_0^1(\Omega)$.

Now, let us assume that $w$ is a solution of (1.6) for some initial data $w_T \in H_0^1(\Omega)$, and define

$$z(t,x) = \exp(-\sigma(t,x)) w(t,x), \tag{2.8}$$

which obviously satisfies

$$z(T) = z(0) = 0 \qquad \text{in } H_0^1(\Omega) \tag{2.9}$$

due to the assumptions (2.1) on $\sigma$.

Then, plugging $w = z\exp(\sigma(t,x))$ in the equation (1.6), we obtain that $z$ satisfies

$$\begin{aligned}
\partial_t z + \Delta z + \frac{\mu}{|x|^2} z + 2\nabla z \cdot \nabla\sigma + z\Delta\sigma \\
+ z\Big(\partial_t\sigma + |\nabla\sigma|^2\Big) = 0, \quad (x,t) \in \Omega\times(0,T), \quad (2.10)
\end{aligned}$$

with the boundary condition

$$z = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{2.11}$$

Let us define a smooth positive radial function $\alpha(x) = \alpha(|x|)$ such that

$$\begin{aligned} \alpha(x) = 0, \quad |x| \le \frac{1}{2}, \qquad & \alpha(x) = \frac{1}{N}, \quad |x| \ge \frac{3}{4}, \\ 0 \le \alpha(x) \le \frac{1}{N}, \quad & \frac{1}{2} \le |x| \le \frac{3}{4}. \end{aligned} \tag{2.12}$$

Setting

$$\begin{aligned} Sz &= \Delta z + \frac{\mu}{|x|^2} z + z\Big(\partial_t \sigma + |\nabla\sigma|^2\Big), \\ Az &= \partial_t z + 2\nabla z \cdot \nabla\sigma + z\Delta\sigma\Big(1+\alpha\Big), \end{aligned} \tag{2.13}$$

one easily deduces from (2.10) that

$$Sz + Az = -\alpha z \Delta\sigma, \qquad \|Sz\|^2 + \|Az\|^2 + 2 < Sz, Az >= \|\alpha z \Delta\sigma\|^2,$$

where $\|\cdot\|$ denotes the $L^2(\Omega \times (0,T))$ norm and $< \cdot,\cdot >$ the corresponding scalar product. Especially, the quantity

$$I =< Sz, Az > -\frac{1}{2}\|\alpha z \Delta\sigma\|^2 \tag{2.14}$$

is non positive.

**Lemma 2.4.** *The following equality holds:*

$$\begin{aligned} I = &-2 \iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z, \nabla z)\ \mathrm{dx\ dt} + \iint\limits_{\partial\Omega\times(0,T)} |\partial_n z|^2\ \partial_n\sigma\ \mathrm{ds\ dt} \\ &- \iint\limits_{\Omega\times(0,T)} |\nabla z|^2 \Delta\sigma\ \alpha\ \mathrm{dx\ dt} + \frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta^2\sigma\Big(1+\alpha\Big)\ \mathrm{dx\ dt} \\ &+ \iint\limits_{\Omega\times(0,T)} |z|^2 \nabla\alpha \cdot \nabla\Delta\sigma\ \mathrm{dx\ dt} + \frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta\sigma\ \Delta\alpha\ \mathrm{dx\ dt} \\ &- \frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Big(\partial_{tt}^2\sigma + 2\partial_t\Big(|\nabla\sigma|^2\Big)\Big)\ \mathrm{dx\ dt} - 2\iint\limits_{\Omega\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma,\nabla\sigma\Big)\ \mathrm{dx\ dt} \\ &+ \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma\Big(\partial_t\sigma + |\nabla\sigma|^2\Big)\ \mathrm{dx\ dt} - \frac{1}{2}\iint\limits_{\Omega\times(0,T)} \alpha^2|z|^2|\Delta\sigma|^2\ \mathrm{dx\ dt} \\ &+ \mu \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma\ \alpha\ \mathrm{dx\ dt} + 2\mu \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^3}\partial_r\sigma, \end{aligned} \tag{2.15}$$

*where $\partial_n = \vec{n}\cdot\nabla$, $\vec{n}$ being the normal outward vector on the boundary, $\partial_r = \frac{x}{|x|}\cdot\nabla$ and* ds *denotes the trace of the Lebesgue measure on $\partial\Omega$.*

For the proof, see Subsection 2.3.

Now, we will decompose the term $I$ in (2.15) into several terms that we handle separately.

Let us define $I_l$ as the sum of the integrals linear in $\sigma$ which do not have any time derivative:

$$\begin{aligned} I_l = -2 \iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z, \nabla z) \text{ dx dt} + \iint\limits_{\partial\Omega\times(0,T)} |\partial_n z|^2 \, \partial_n \sigma \text{ ds dt} \\ - \iint\limits_{\Omega\times(0,T)} |\nabla z|^2 \Delta\sigma \; \alpha \text{ dx dt} + \frac{1}{2} \iint\limits_{\Omega\times(0,T)} |z|^2 \Delta^2\sigma \Big(1+\alpha\Big) \text{ dx dt} \\ + \iint\limits_{\Omega\times(0,T)} |z|^2 \nabla\alpha\cdot\nabla\Delta\sigma \text{ dx dt} + \frac{1}{2} \iint\limits_{\Omega\times(0,T)} |z|^2 \Delta\sigma \; \Delta\alpha \text{ dx dt} \\ + \mu \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2} \Delta\sigma \; \alpha \text{ dx dt} + 2\mu \iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^3} \partial_r \sigma \text{ dx dt}. \end{aligned} \tag{2.16}$$

Then we have the following estimate:

**Lemma 2.5.** *There exist positive constants such that for $\lambda$ large enough, we have:*

$$\begin{aligned} I_l \geq 2s \iint\limits_{\Omega\times(0,T)} \theta \frac{|z|^2}{|x|} \text{ dx dt} + sN \iint\limits_{\Omega\times(0,T)} \theta\alpha |\nabla z|^2 \text{ dx dt} \\ + C_1 s\lambda^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi |\nabla z|^2 \text{ dx dt} - C_2 s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\phi |\nabla z|^2 \text{ dx dt} \\ - C_3 s\lambda^4 \iint\limits_{\Omega\times(0,T)} \theta |z|^2 \text{ dx dt} - C_4 s\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi |z|^2 \text{ dx dt}. \end{aligned} \tag{2.17}$$

Again, the proof is given in Subsection 2.3. Note that the proof of Lemma 2.5 uses an improved form of the Hardy inequality (1.3), which can be found for instance in [18], namely:

**Lemma 2.6.** *There exists a positive constant $C_5 > 0$, such that*

$$\mu^*(N) \int_\Omega \frac{|z|^2}{|x|^2} \text{ dx} + \int_\Omega \frac{|z|^2}{|x|} \text{ dx} \leq \int_\Omega |\nabla z|^2 \text{ dx} + C_5 \int_\Omega |z|^2 \text{ dx}, \quad z \in H_0^1(\Omega). \tag{2.18}$$

Of course, this inequality also holds for $\mu < \mu^*(N)$.

We then consider the integrals involving non-linear terms in $\sigma$ and without any time derivative, that is

$$I_{nl} = -2 \iint\limits_{\Omega\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma, \nabla\sigma\Big) \text{ dx dt} + \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma|\nabla\sigma|^2 \text{ dx dt} - \frac{1}{2} \iint\limits_{\Omega\times(0,T)} \alpha^2|z|^2|\Delta\sigma|^2 \text{ dx}. \quad (2.19)$$

Then, with $\sigma$ as in (2.2), we obtain (see Subsection 2.3) that

**Lemma 2.7.** *There exist positive constants such that for $\lambda$ large enough, for $s \geq s_0(\lambda)$,*

$$I_{nl} \geq C_6 s^3 \iint\limits_{\Omega\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} + C_7 s^3\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt} - C_8 s^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt}. \quad (2.20)$$

We finally estimate the terms involving the time derivatives in $\sigma$:

$$I_t = -\frac{1}{2} \iint\limits_{\Omega\times(0,T)} |z|^2\Big(\partial_{tt}^2\sigma + 2\partial_t\Big(|\nabla\sigma|^2\Big)\Big) \text{ dx dt} + \iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma\partial_t\sigma \text{ dx dt}. \quad (2.21)$$

We also add to $I_t$ the integrals appearing in Lemma 2.5 that we want to get rid of and define

$$I_r = I_t - C_3 s\lambda^4 \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} - C_4 s\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi|z|^2 \text{ dx dt}. \quad (2.22)$$

Then we have to prove that $I_r$ is negligible with respect to the positive terms in (2.17) and (2.20).

**Lemma 2.8.** *For any $\lambda$ large enough, there exists $s_0(\lambda)$ such that for $s \geq s_0(\lambda)$,*

$$|I_r| \leq s \iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|} \text{ dx dt} + \frac{C_6}{2}s^3 \iint\limits_{\Omega\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} + \frac{C_7}{2}s^3\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt}, \quad (2.23)$$

*where $C_6$ and $C_7$ are as in* (2.20).

Using (2.14) and Lemmas 2.5, 2.7 and 2.8, whose proofs are postponed to Subsection 2.3, we obtain a Carleman estimate in the $z$ variable. Undoing the change of variable (2.8) provides the Carleman estimate (2.7). $\square$

## 2.2 From the Carleman estimate to the Observability inequality

In this Subsection, we explain why the Carleman estimate (2.7) implies the observability inequality (1.8).

We fix $\lambda > \lambda_0$ and $s > s_0(\lambda)$ such that (2.7) holds. These parameters now enter in the constant $K$ :

$$\iint_{\Omega\times(0,T)} \theta e^{-2\sigma}\frac{|w|^2}{|x|} \text{ dx dt} \leq K \iint_{\omega_0\times(0,T)} \theta\phi e^{-2\sigma}|\nabla w|^2 \text{ dx dt} \\ + K \iint_{\omega_0\times(0,T)} \theta^3\phi^3 e^{-2\sigma}|w|^2 \text{ dx dt}. \quad (2.24)$$

One easily checks the existence of a constant $C$ such that

$$\begin{array}{rclr} \theta\ e^{-2\sigma}\dfrac{1}{|x|} & \geq & C, & (x,t)\in\Omega\times\Big[\dfrac{T}{4},\dfrac{3T}{4}\Big], \\ \theta\ \phi\ e^{-2\sigma} & \leq & Ce^{-\sigma}, & (x,t)\in\omega_0\times(0,T), \\ \theta^3\phi^3 e^{-2\sigma} & \leq & C, & (x,t)\in\omega_0\times(0,T). \end{array}$$

Thus, (2.24) implies

$$\iint_{\Omega\times(T/4,3T/4)} |w|^2 \text{ dx dt} \leq K \iint_{\omega_0\times(0,T)} e^{-\sigma}|\nabla w|^2 \text{ dx dt} \\ + K \iint_{\omega_0\times(0,T)} |w|^2 \text{ dx dt}. \quad (2.25)$$

Therefore to obtain inequality (1.8), it is sufficient to prove the following lemma:

**Lemma 2.9** (Caccioppoli's inequality). *Let $\bar\sigma : (0,T)\times\bar\omega \to \mathbb{R}_+^*$ be a smooth nonnegative function such that*

$$\bar\sigma(t,x) \to +\infty \text{ as } t\to 0^+ \text{ and as } t \to T^-.$$

*There exists a constant $C$ independent of $\mu \leq \mu^*(N)$ such that any solution $w$ of* (1.6) *satisfies*

$$\iint_{\omega_0\times(0,T)} e^{-\bar\sigma}|\nabla w|^2 \text{ dx dt} \leq C \iint_{\omega\times(0,T)} |w|^2 \text{ dx dt}. \quad (2.26)$$

The proof of this lemma is given for instance in [19, Lemma III.3]. This obviously implies (1.8) by taking $\bar{\sigma} = \sigma$ in Lemma 2.9, since $\sigma$ satisfies (2.1). It follows that inequality (1.7) holds as well and, by the classical HUM duality ([16]), this proves Theorem 1.1.

## 2.3 Proofs of technical Lemmas

Here we present the proofs of the technical Lemmas stated in Subsection 2.1. This part can be skipped in a first lecture. In this subsection, all the constants are positive and independent of $\lambda$ or $s$.

*Proof of Lemma 2.4.* To make the computations easier, we define

$$\begin{array}{lll} S_1 z = \Delta z, & S_2 z = \dfrac{\mu}{|x|^2} z, & S_3 z = z\big(\partial_t \sigma + |\nabla \sigma|^2\big), \\ A_1 z = \partial_t z, & A_2 z = 2\nabla z \cdot \nabla \sigma, & A_3 z = z \Delta\sigma \big(1+\alpha\big), \end{array} \tag{2.27}$$

and denotes by $I_{ij}$ the scalar product $< S_i, A_j >$. We will compute each term using integration by parts and the boundary conditions (2.9) and (2.11).

*Computation of $I_{11}$:*

$$I_{11} = \iint\limits_{\Omega\times(0,T)} \Delta z \; \partial_t z \text{ dx dt} = -\iint\limits_{\Omega\times(0,T)} \partial_t\Big(\frac{|\nabla z|^2}{2}\Big) \text{ dx dt} = 0, \tag{2.28}$$

where the last identity is justified by (2.9).

*Computation of $I_{12}$:* Note that, since $z$ vanishes on the boundary, its gradient $\nabla z$ on the boundary is normal to the boundary, and therefore $\nabla z = \partial_n z \, \vec{n}$, where $\vec{n}$ denotes the normal outward vector on the boundary.

$$\begin{aligned} I_{12} &= 2 \iint\limits_{\Omega\times(0,T)} \Delta z \; \nabla z \cdot \nabla \sigma \text{ dx dt} \\ &= -2 \iint\limits_{\Omega\times(0,T)} \nabla z \cdot \nabla\Big(\nabla z \cdot \nabla\sigma\Big) \text{ dx dt} + 2 \iint\limits_{\partial\Omega\times(0,T)} |\partial_n z|^2 \; \partial_n \sigma \text{ ds dt}, \end{aligned}$$

Besides, one can check that

$$\nabla z \cdot \nabla\Big(\nabla z \cdot \nabla \sigma\Big) = \frac{1}{2}\nabla\Big(|\nabla z|^2\Big) \cdot \nabla\sigma + D^2\sigma(\nabla z, \nabla z).$$

It follows easily that

$$\begin{aligned} I_{12} = \iint\limits_{\Omega\times(0,T)} |\nabla z|^2 \Delta \sigma \text{ dx dt} - 2 \iint\limits_{\Omega\times(0,T)} D^2\sigma(\nabla z, \nabla z) \text{ dx dt} \\ + \iint\limits_{\partial\Omega\times(0,T)} |\partial_n z|^2 \; \partial_n \sigma \text{ ds dt}. \end{aligned} \tag{2.29}$$

*Computation of $I_{13}$:*

$$\begin{aligned} I_{13} &= \iint\limits_{\Omega\times(0,T)} \Delta z\; z\Delta\sigma\Big(1+\alpha\Big)\;\mathrm{dx}\;\mathrm{dt} \\ &= -\iint\limits_{\Omega\times(0,T)} \nabla z\cdot\nabla\Big(z\Delta\sigma\Big(1+\alpha\Big)\Big)\;\mathrm{dx}\;\mathrm{dt}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} I_{13} = &-\iint\limits_{\Omega\times(0,T)} |\nabla z|^2\Delta\sigma\Big(1+\alpha\Big)\;\mathrm{dx}\;\mathrm{dt} \\ &+\frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta^2\sigma\Big(1+\alpha\Big)\;\mathrm{dx}\;\mathrm{dt}+\iint\limits_{\Omega\times(0,T)} |z|^2\nabla\alpha\cdot\nabla\Delta\sigma\;\mathrm{dx}\;\mathrm{dt} \\ &+\frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\Delta\sigma\;\Delta\alpha\;\mathrm{dx}\;\mathrm{dt}. \end{aligned} \tag{2.30}$$

*Computation of $I_{21}$:* As in (2.28), using (2.9), one easily checks that

$$I_{21} = 0. \tag{2.31}$$

*Computation of $I_{22}$:*

$$\begin{aligned} I_{22} &= \mu\iint\limits_{\Omega\times(0,T)} \frac{1}{|x|^2}\nabla\Big(|z|^2\Big)\cdot\nabla\sigma\;\mathrm{dx}\;\mathrm{dt} \\ &= -\mu\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma\;\mathrm{dx}\;\mathrm{dt}+2\mu\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^3}\partial_r\sigma\;\mathrm{dx}\;\mathrm{dt}. \end{aligned} \tag{2.32}$$

*Computation of $I_{23}$:*

$$I_{23} = \mu\iint\limits_{\Omega\times(0,T)} \frac{|z|^2}{|x|^2}\Delta\sigma\Big(1+\alpha\Big)\;\mathrm{dx}\;\mathrm{dt}. \tag{2.33}$$

*Computation of $I_{31}$:*

$$\begin{aligned} I_{31} &= \frac{1}{2}\iint\limits_{\Omega\times(0,T)} \partial_t\Big(|z|^2\Big)\Big(\partial_t\sigma+|\nabla\sigma|^2\Big)\;\mathrm{dx}\;\mathrm{dt} \\ &= -\frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\partial_t\Big(\partial_t\sigma+|\nabla\sigma|^2\Big)\;\mathrm{dx}\;\mathrm{dt}. \end{aligned} \tag{2.34}$$

*Computation of $I_{32}$:*

$$I_{32} = \iint\limits_{\Omega\times(0,T)} \nabla\Big(|z|^2\Big)\cdot\nabla\sigma\Big(\partial_t\sigma+|\nabla\sigma|^2\Big)\;\mathrm{dx}\;\mathrm{dt}.$$

It follows that

$$I_{32} = -\iint\limits_{\Omega\times(0,T)} |z|^2 \Delta\sigma \Big(\partial_t\sigma + |\nabla\sigma|^2\Big) \text{ dx dt}$$
$$- \iint\limits_{\Omega\times(0,T)} |z|^2 \nabla\sigma\cdot\nabla\Big(\partial_t\sigma\Big) - 2\iint\limits_{\Omega\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma,\nabla\sigma\Big) \text{ dx dt.} \quad (2.35)$$

*Computation of $I_{33}$:*

$$I_{33} = \iint\limits_{\Omega\times(0,T)} |z|^2 \Delta\sigma \Big(\partial_t\sigma + |\nabla\sigma|^2\Big)\Big(1+\alpha\Big) \text{ dx dt.} \quad (2.36)$$

Lemma 2.4 follows directly from these computations. □

*Proof of Lemma 2.5.* Since the integral $I_l$ is linear in $\sigma$, we decompose $\sigma$ as

$$\sigma = s\theta(t)e^{2\lambda\sup\psi} + \sigma_{x^2}(t,x) + \sigma_\phi(t,x),$$

with

$$\sigma_{x^2}(t,x) = -s\theta(t)\frac{|x|^2}{2}, \quad \sigma_\phi(t,x) = -s\theta(t)\phi(x).$$

Note that the term $s\theta\exp(2\lambda\sup\psi)$ in $\sigma$ does not appear in the computations of $I_l$, since it is constant in the space variable, and each integral in (2.16) involves space derivatives.

We then define $I_{l,x^2}$ and $I_{l,\phi}$ as the terms in $I_l$ corresponding respectively to $\sigma_{x^2}$ and $\sigma_\phi$.

First, we compute $I_{l,x^2}$. In this case, all the computations are explicit:

$$I_{l,x^2} = 2s\iint\limits_{\Omega\times(0,T)} \theta|\nabla z|^2 \text{ dx dt} - s\iint\limits_{\partial\Omega\times(0,T)} \theta|\partial_n z|^2 \vec{x}\cdot\vec{n} \text{ ds dt}$$
$$+ sN\iint\limits_{\Omega\times(0,T)} \theta\alpha|\nabla z|^2 \text{ dx dt} - s\frac{N}{2}\iint\limits_{\Omega\times(0,T)} \theta|z|^2\Delta\alpha \text{ dx dt}$$
$$- s\mu N\iint\limits_{\Omega\times(0,T)} \theta\alpha\frac{|z|^2}{|x|^2} \text{ dx dt} - 2s\mu\iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|^2} \text{ dx dt.}$$

Thus, from the Hardy improved inequality (2.18), since $\theta$ only depends on the time variable $t$ and since $\alpha$ vanishes on $B(0,1/2)$ by (2.12), there exists a constant such that

$$I_{l,x^2} \geq 2s \iint\limits_{\Omega\times(0,T)} \theta \frac{|z|^2}{|x|} \text{ dx dt} + sN \iint\limits_{\Omega\times(0,T)} \theta\alpha|\nabla z|^2 \text{ dx dt} - s \iint\limits_{\partial\Omega\times(0,T)} \theta|\partial_n z|^2 \vec{x}\cdot\vec{n} \text{ ds dt} - Cs \iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt}. \tag{2.37}$$

Second, let us consider $I_{l,\phi}$. To simplify, we decompose this integral into the integrals $I_{l,\phi,1}$ in $B(0,1)$ and $I_{l,\phi,2}$ outside $B(0,1)$.

In the unit ball, $\phi(x) = |x|^\lambda$ and then, all the computations are explicit. Especially, $\phi$ is convex (at least for $\lambda > 1$, which can be assumed since $\lambda$ is aimed at being large), and therefore $D^2\phi(\xi,\xi)$ is a positive quadratic form in $\xi$, and $\Delta\phi > 0$. Besides, all the terms

$$\Delta^2\phi,\ \nabla\Delta\phi,\ \Delta\phi,\ \frac{\Delta\phi}{|x|^2},\ \frac{\partial_r\phi}{|x|^3}$$

are bounded by $C\lambda^4|x|^{\lambda-4}$ for $\lambda$ large enough (namely $\lambda > 4$). Then

$$I_{l,\phi,1} \geq -Cs\lambda^4 \iint\limits_{B(0,1)\times(0,T)} \theta(t)|x|^{\lambda-4}|z|^2 \text{ dx dt}. \tag{2.38}$$

Outside the unit ball, the computations are more intricate. First, let us compute the first derivative of $\phi$ :

$$\begin{aligned} &\nabla\phi = \lambda\phi\nabla\psi, \qquad \partial^2_{i,j}\phi = \lambda\phi\partial^2_{i,j}\psi + \lambda^2\phi\ \partial_i\psi\ \partial_j\psi, \\ &\Delta\phi = \lambda\phi\Delta\psi + \lambda^2\phi|\nabla\psi|^2. \end{aligned} \tag{2.39}$$

Besides, due to the particular choice of $\psi$, and especially (2.5), one can get the following estimates :

$$\begin{array}{rcll} 2D^2\phi(\xi,\xi) + \alpha\Delta\phi|\xi|^2 & \geq & C\lambda^2\phi|\xi|^2, & \xi\in\mathbb{R}^N,\ x\in\mathcal{O}, \\ \left|2D^2\phi(\xi,\xi) + \alpha\Delta\phi|\xi|^2\right| & \leq & C\lambda^2\phi|\xi|^2, & \xi\in\mathbb{R}^N,\ x\in\omega_0, \\ |\Delta^2\phi| + |\Delta\phi| + |\nabla\phi| + |\partial_r\phi| + |\nabla\Delta\phi| & \leq & C\phi\lambda^4, & x\in\tilde{\mathcal{O}}, \end{array}$$

for $\lambda$ large enough. Hence we deduce that

$$I_{l,\phi,2} \geq Cs\lambda^2 \iint\limits_{\mathcal{O}\times(0,T)} \theta\phi|\nabla z|^2 \text{ dx dt} - s\lambda \iint\limits_{\partial\Omega\times(0,T)} \theta\phi|\partial_n z|^2\nabla\psi\cdot\vec{n} \text{ ds dt} - Cs\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi|z|^2 \text{ dx dt} - s\lambda^2 \iint\limits_{\omega_0\times(0,T)} \theta\phi|\nabla z|^2 \text{ dx dt}. \tag{2.40}$$

Taking $\lambda$ large enough, due to the properties (2.4) and (2.5), the sum of boundary terms in (2.37) and in (2.40) is positive. Indeed, from (2.4) and (2.5), $\nabla\psi \cdot \vec{n} = -|\nabla\psi| \leq -\delta$, and thus the choice $\lambda \geq \operatorname{diam}(\Omega)/\delta$, where $\operatorname{diam}(\Omega)$ is the diameter of $\Omega$, is convenient.

Hence, combining (2.37), (2.38) and (2.40) gives Lemma 2.5. □

*Proof of Lemma 2.7.* Again, we handle separately the integrals $I_{nl1}$ in the unit ball and $I_{nl2}$ outside the unit ball. This is needed since the terms $|x|^2$ and $\phi$ of $\sigma$ (see (2.2)) do not have the same order inside and outside the unit ball.

Notice that, in the unit ball,

$$\begin{cases} \nabla\sigma = -s\theta x\Big(1 + \lambda|x|^{\lambda-2}\Big), \\ \Delta\sigma = -s\theta\Big(N + \lambda(N+\lambda-2)|x|^{\lambda-2}\Big). \end{cases} \tag{2.41}$$

Hence we compute explicitly the terms appearing in the integrals for a radial vector $\xi$ of $\mathbb{R}^N$, which is the case of $\nabla\sigma$ in the unit ball:

$$\begin{aligned} \alpha\Delta\sigma|\xi|^2 - 2D^2\sigma(\xi,\xi) = s\theta\Big(&(2-\alpha N)|\xi|^2 + 2\lambda|x|^{\lambda-2}|\xi|^2 \\ &+ \lambda|x|^{\lambda-4}|\xi|^2\big((2-\alpha)\lambda - 4 - \alpha(N+2)\big)\Big). \end{aligned}$$

Thus we can take $\lambda$ large enough such that

$$\begin{aligned} -2\iint\limits_{B(0,1)\times(0,T)} |z|^2 D^2\sigma\Big(\nabla\sigma, \nabla\sigma\Big)\,\mathrm{dx\,dt} + \iint\limits_{B(0,1)\times(0,T)} \alpha|z|^2\Delta\sigma|\nabla\sigma|^2\,\mathrm{dx\,dt} \\ \geq Cs \iint\limits_{B(0,1)\times(0,T)} \theta|z|^2|\nabla\sigma|^2\,\mathrm{dx\,dt} \geq s^3 \iint\limits_{B(0,1)\times(0,T)} \theta^3|x|^2|z|^2\,\mathrm{dx\,dt}. \end{aligned} \tag{2.42}$$

The last term in (2.19) can be absorbed, since from (2.41), we have

$$\big|\Delta\sigma\big|^2 \leq Cs^2\theta^2\lambda^4.$$

Indeed, combined with the assumption (2.12) on the support of $\alpha$, the last integral in (2.19) satisfies

$$\iint\limits_{B(0,1)\times(0,T)} \alpha^2|z|^2|\Delta\sigma|^2\,\mathrm{dx\,dt} \leq Cs^2\lambda^4 \iint\limits_{B(0,1)\times(0,T)} \theta^2|x|^2|z|^2\,\mathrm{dx\,dt}.$$

Then taking $s$ large, for instance $s > C\lambda^4$, we can absorb the third term in (2.19), and we obtain that

$$I_{nl1} \geq Cs^3 \iint\limits_{B(0,1)\times(0,T)} \theta^3|x|^2|z|^2\,\mathrm{dx\,dt}. \tag{2.43}$$

Outside the unit ball, due to the particular choice of $\psi$, and especially (2.5), and since $\|\alpha\|_{L^\infty(\Omega)} < 2$, as in [13] we remark that, for $s$ and $\lambda$ large enough,

$$\begin{aligned} \alpha\Delta\sigma|\nabla\sigma|^2 - 2D^2\sigma(\nabla\sigma,\nabla\sigma) &\geq Cs^3\lambda^4\theta^3\phi^3, \qquad x\in\mathcal{O}, \\ \Big|\alpha\Delta\sigma|\nabla\sigma|^2 - 2D^2\sigma(\nabla\sigma,\nabla\sigma)\Big| &\leq Cs^3\lambda^4\theta^3\phi^3, \qquad x\in\omega_0, \end{aligned}$$

and

$$|\Delta\sigma|^2 \leq Cs^2\lambda^4\theta^2\phi^2, \qquad x\in\tilde{\mathcal{O}}.$$

Then, taking $s$ large yields

$$I_{nl2} \geq Cs^3\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt} - Cs^3\lambda^4 \iint\limits_{\omega_0\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt}. \tag{2.44}$$

Hence the proof of Lemma 2.7 is completed. □

*Proof of Lemma 2.8.* First notice that

$$\Big|\theta\theta'\Big| \leq C\theta^3, \qquad \Big|\theta'\Big| \leq C\theta^3, \qquad \Big|\theta''\Big| \leq C\theta^{5/3}.$$

Then, since $\alpha$ vanishes in $B(0,1/2)$, bounding the integral in $B(0,1)$ and $\tilde{\mathcal{O}}$ using respectively (2.39) and (2.41),

$$\begin{aligned} \Big|\iint\limits_{\Omega\times(0,T)} \alpha|z|^2\Delta\sigma\partial_t\sigma \text{ dx dt}\Big| \leq Cs^2\lambda^2 e^{2\lambda\sup\psi} &\iint\limits_{B(0,1)\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} \\ &+ Cs^2\lambda^2 e^{2\lambda\sup\psi} \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi|z|^2 \text{ dx dt}. \end{aligned}$$

Similarly,

$$\begin{aligned} \Big|\iint\limits_{\Omega\times(0,T)} |z|^2\partial_t\Big(|\nabla\sigma|^2\Big) \text{ dx dt}\Big| \leq Cs^2\lambda^2 &\iint\limits_{B(0,1)\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} \\ &+ Cs^2\lambda^2 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^2|z|^2 \text{ dx dt}. \end{aligned} \tag{2.45}$$

The remaining term

$$\begin{aligned} R = -\frac{1}{2}\iint\limits_{\Omega\times(0,T)} |z|^2\partial_{tt}^2\sigma \text{ dx dt} - C_3 s\lambda^4 &\iint\limits_{\Omega\times(0,T)} \theta|z|^2 \text{ dx dt} \\ &- C_4 s\lambda^4 \iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta\phi|z|^2 \text{ dx dt} \end{aligned}$$

satisfies for $\lambda$ large enough

$$\left|R\right| \leq Cse^{2\lambda \sup \psi} \iint\limits_{\Omega\times(0,T)} \theta^{5/3}|z|^2 \text{ dx dt}. \tag{2.46}$$

Let us then estimate this last integral. Take $\beta$ a positive number that we will choose later on. Then

$$\begin{aligned}\iint\limits_{\Omega\times(0,T)} &\theta^{5/3}|z|^2 \text{ dx dt}\\ &= \iint\limits_{\Omega\times(0,T)} \left(\beta\theta|x|^{2/3}|z|^{2/3}\right)\left(\frac{1}{\beta}\theta^{2/3}|x|^{-2/3}|z|^{4/3}\right) \text{ dx dt}\\ &\leq \frac{\beta^3}{3}\iint\limits_{\Omega\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt} + \frac{2}{3\beta^{3/2}}\iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|} \text{ dx dt},\end{aligned}$$

where we used the classical convexity inequality

$$ab \leq \frac{1}{3}a^3 + \frac{2}{3}b^{3/2}.$$

Then we get three constants such that

$$\begin{aligned}|I_r| \leq c_1\Big(s^2\lambda^2 + s^2\lambda^2e^{2\lambda\sup\psi} + se^{2\lambda\sup\psi}\beta^3\Big) &\iint\limits_{\Omega\times(0,T)} \theta^3|x|^2|z|^2 \text{ dx dt}\\ + c_2\Big(s^2\lambda^2e^{2\lambda\sup\psi} + s^2\lambda^2\Big) &\iint\limits_{\tilde{\mathcal{O}}\times(0,T)} \theta^3\phi^3|z|^2 \text{ dx dt}\\ + c_3se^{2\lambda\sup\psi}\frac{1}{\beta^{3/2}} &\iint\limits_{\Omega\times(0,T)} \theta\frac{|z|^2}{|x|} \text{ dx dt}.\end{aligned} \tag{2.47}$$

Thus, for a given $\lambda > 0$, choosing $\beta$ such that

$$c_3e^{2\lambda\sup\psi} = \beta^{3/2},$$

there exists $s_0(\lambda)$ such that for any $s \geq s_0(\lambda)$, inequality (2.23) holds. $\square$

# 3 Non uniform stabilization in the case $\mu > \mu^*(N)$

The goal of this section is to prove Theorem 1.2. The proof is divided into two main steps.

First, we prove some basic estimates on the spectrum of the operator

$$L^\varepsilon = -\Delta - \frac{\mu}{|x|^2+\varepsilon^2} \tag{3.1}$$

on $\Omega$ with Dirichlet boundary conditions, especially on the first eigenvalue $\lambda_0^\varepsilon$ and the corresponding eigenfunction $\phi_0^\varepsilon$. This will be done in Subsection 3.1.

Second, we deduce Theorem 1.2 in Subsection 3.2 by giving a lower bound on the quantity $J^\varepsilon_{\phi_0^\varepsilon}$ that goes to infinity when $\varepsilon \to 0$.

## 3.1 Spectral estimates

Since for $\varepsilon > 0$, the function $1/(|x|^2 + \varepsilon^2)$ is smooth and bounded in $\Omega$, the spectrum of $L^\varepsilon$ is formed by a sequence of real eigenvalues $\lambda_0^\varepsilon \le \lambda_1^\varepsilon \le \cdots \le \lambda_n^\varepsilon \le \cdots$, with $\lambda_n^\varepsilon \to +\infty$. The corresponding eigenvectors $\phi_n^\varepsilon$ are a basis of $L^2(\Omega)$, orthonormal with respect to the $L^2$ scalar product. We choose $\phi_n^\varepsilon$ of unit $L^2$-norm.

In the sequel, we focus on the bottom of the spectrum -*the most explosive mode.*

**Proposition 3.1.** *Assume that $\mu > \mu^*(N)$. Then we have that*

$$\lim_{\varepsilon \to 0} \lambda_0^\varepsilon = -\infty. \tag{3.2}$$

*and for all $\alpha > 0$,*

$$\lim_{\varepsilon \to 0} \|\phi_0^\varepsilon\|_{H^1(\Omega \setminus \bar{B}(0,\alpha))} = 0. \tag{3.3}$$

*Proof.* We argue by contradiction, and assume that $\lambda_0^\varepsilon$ is bounded from below for a subsequence by a real number $C$. Then, from the Rayleigh formula we get

$$\forall \varepsilon > 0, \forall u \in H_0^1(\Omega), \quad \mu \int_\Omega \frac{|u|^2}{|x|^2 + \varepsilon^2} \, \mathrm{dx} \le \int_\Omega |\nabla u|^2 \, \mathrm{dx} - C \int_\Omega |u|^2 \, \mathrm{dx}.$$

Taking $u \in \mathcal{D}(\Omega)$, we pass to the limit $\varepsilon \to 0$ and get

$$\mu \int_\Omega \frac{|u|^2}{|x|^2} \, \mathrm{dx} \le \int_\Omega |\nabla u|^2 \, \mathrm{dx} - C \int_\Omega |u|^2 \, \mathrm{dx}, \tag{3.4}$$

that must therefore hold for any $u \in H_0^1(\Omega)$ by a density argument.

Now, there exists $\alpha_0 > 0$ such that $B(0, \alpha_0) \subset \Omega$. We then choose $u \in H_0^1(B(0, \alpha_0))$ that we extend by 0 on $\mathbb{R}^N$, and define for $a \ge 1$

$$u_a(r) = a^N \; u(ar).$$

These functions are in $H_0^1(B(0, \alpha_0))$, and therefore in $H_0^1(\Omega)$, and we can apply (3.4) to them:

$$a^2 \Big( \mu \int_\Omega \frac{|u|^2}{|x|^2} \, \mathrm{dx} - \int_\Omega |\nabla u|^2 \, \mathrm{dx} \Big) \le -C \int_\Omega |u|^2 \, \mathrm{dx}.$$

Passing to the limit $a \to \infty$, we obtain that

$$\forall u \in H_0^1(B(0, \alpha_0)), \qquad \mu \int_\Omega \frac{|u|^2}{|x|^2} \, \mathrm{dx} \le \int_\Omega |\nabla u|^2 \, \mathrm{dx}.$$

Therefore we should have that $\mu \le \mu^*(N)$, since this is the Hardy inequality (1.3) in the set $B(0,\alpha_0)$, and then we have a contradiction.

Now, consider the first eigenvector $\phi_0^\varepsilon \in H_0^1(\Omega)$ of $L^\varepsilon$:

$$-\Delta\phi_0^\varepsilon - \frac{\mu}{|x|^2+\varepsilon^2}\phi_0^\varepsilon = \lambda_0^\varepsilon\phi_0^\varepsilon, \quad \text{in } \Omega. \tag{3.5}$$

Remark that since the potential is smooth in $\Omega$, the function $\phi_0^\varepsilon$ is smooth by classical elliptic estimates.

Set $\alpha > 0$. Let $\eta_\alpha$ be a nonnegative smooth function that vanishes in $B(0,\alpha/2)$ and equals 1 in $\mathbb{R}^N \backslash B(0,\alpha)$ with $\|\eta_\alpha\|_\infty \le 1$. Multiplying (3.5) by $\eta_\alpha\phi_0^\varepsilon$, we get:

$$\int_\Omega \eta_\alpha|\nabla\phi_0^\varepsilon|^2 \text{ dx} + |\lambda_0^\varepsilon|\int_\Omega \eta_\alpha|\phi_0^\varepsilon|^2 = \mu\int_\Omega \eta_\alpha\frac{|\phi_0^\varepsilon|^2}{|x|^2+\varepsilon^2} \text{ dx} + \frac{1}{2}\int_\Omega \Delta\eta_\alpha|\phi_0^\varepsilon|^2 \text{ dx}. \tag{3.6}$$

Therefore, since $\phi_0^\varepsilon$ is of unit $L^2$-norm, due to the particular choice of $\eta_\alpha$, we get

$$|\lambda_0^\varepsilon|\int_{\Omega\backslash B(0,\alpha)} |\phi_0^\varepsilon|^2 \text{ dx} \le \frac{4\mu}{\alpha^2} + \frac{1}{2}\|\Delta\eta_\alpha\|_{L^\infty(\Omega)}.$$

Since $|\lambda_0^\varepsilon| \to \infty$ when $\varepsilon \to 0$, we get that for any $\alpha > 0$,

$$\lim_{\varepsilon\to 0}\int_{\Omega\backslash B(0,\alpha)} |\phi_0^\varepsilon|^2 \text{ dx} = 0. \tag{3.7}$$

Besides, still using (3.6) and the particular form of $\eta_\alpha$

$$\int_{\Omega\backslash B(0,\alpha)} |\nabla\phi_0^\varepsilon|^2 \text{ dx} \le \left(\frac{4\mu}{\alpha^2} + \frac{1}{2}\|\Delta\eta_\alpha\|_{L^\infty(\Omega)}\right)\int_{\Omega\backslash B(0,\alpha/2)} |\phi_0^\varepsilon|^2 \text{ dx}.$$

Therefore the proof of (3.3) is completed by using (3.7) for $\alpha/2$ instead of $\alpha$. $\square$

## 3.2 Proof of Theorem 1.2

Fix $\varepsilon > 0$, and choose $u_0^\varepsilon = \phi_0^\varepsilon$, which is of unit $L^2$-norm. Our goal is to prove that

$$\inf_{\substack{f \in L^2((0,T);H^{-1}(\Omega)) \\ f \text{ as in } (1.2)}} J_{u_0^\varepsilon}^\varepsilon(f) \underset{\varepsilon\to 0}{\longrightarrow} \infty. \tag{3.8}$$

Let $f \in L^2((0,T);H^{-1}(\Omega))$ as in (1.2), and consider $u$ the corresponding solution of (1.13) with initial data $u_0^\epsilon = \phi_0^\epsilon$.

Set

$$a(t) = \int_\Omega u(t,x)\phi_0^\varepsilon(x) \text{ dx}, \qquad b(t) = < f(t), \phi_0^\varepsilon >_{H^{-1}(\Omega)\times H_0^1(\Omega)}.$$

Then $a(t)$ satisfies the equation

$$a'(t) + \lambda_0^\varepsilon a(t) = b(t), \qquad a(0) = 1.$$

Duhamel's formula gives

$$a(t) = \exp(-\lambda_0^\varepsilon t) + \int_0^t \exp(-\lambda_0^\varepsilon (t-s))\ b(s)\ \mathrm{ds}.$$

Therefore

$$\begin{aligned}
\iint\limits_{\Omega\times(0,T)} |u(t,x)|^2\ \mathrm{dx}\ \mathrm{dt} &\geq \int_0^T a(t)^2\ \mathrm{dt} \\
&\geq \frac{1}{2}\int_0^T \exp(-2\lambda_0^\varepsilon t)\ \mathrm{dt} - \int_0^T \Big(\int_0^t \exp(-\lambda_0^\varepsilon (t-s)) b(s)\ \mathrm{ds}\Big)^2\ \mathrm{dt}. \quad (3.9)
\end{aligned}$$

Of course,

$$\frac{1}{2}\int_0^T \exp(-2\lambda_0^\varepsilon t)\ \mathrm{dt} = \frac{1}{4|\lambda_0^\varepsilon|}\Big(\exp(2|\lambda_0^\varepsilon|T) - 1\Big).$$

The other term satisfies

$$\begin{aligned}
&\int_0^T \Big(\int_0^t \exp(-\lambda_0^\varepsilon (t-s)) b(s)\ \mathrm{ds}\Big)^2\ \mathrm{dt} \\
&\qquad \leq \int_0^T \Big(\int_0^t \exp(-2\lambda_0^\varepsilon (t-s))\ \mathrm{ds}\Big)\Big(\int_0^t |b(s)|^2\ \mathrm{ds}\Big)\ \mathrm{dt} \\
&\qquad \leq \int_0^T \frac{1}{2|\lambda_0^\varepsilon|} \exp(2|\lambda_0^\varepsilon| t)\Big(\int_0^t |b(s)|^2\ \mathrm{ds}\Big)\ \mathrm{dt} \\
&\qquad \leq \frac{1}{4|\lambda_0^\varepsilon|^2} \exp(2|\lambda_0^\varepsilon| T) \int_0^T |b(s)|^2\ \mathrm{ds}.
\end{aligned}$$

Besides, from the definition of $b$ and the assumption (1.2), we get that

$$|b(t)|^2 \leq \|f(t)\|^2_{H^{-1}(\Omega)} \|\phi_0^\varepsilon\|^2_{H^1(\omega)}.$$

Hence we deduce from (3.9) that

$$\frac{1}{4|\lambda_0^\varepsilon|}\Big(e^{2|\lambda_0^\varepsilon|T} - 1\Big) \leq \iint\limits_{\Omega\times(0,T)} |u(t,x)|^2\ \mathrm{dx}\,\mathrm{dt} + \frac{\|\phi_0^\varepsilon\|^2_{H^1(\omega)}}{4|\lambda_0^\varepsilon|^2} e^{2|\lambda_0^\varepsilon|T} \int_0^T \|f(t)\|^2_{H^{-1}(\Omega)}\ \mathrm{dt}.$$

Therefore, either

$$\frac{1}{8|\lambda_0^\varepsilon|}\Big(e^{2|\lambda_0^\varepsilon|T} - 1\Big) \leq \iint\limits_{\Omega\times(0,T)} |u(t,x)|^2\ \mathrm{dx}\ \mathrm{dt}$$

or

$$\frac{1}{8|\lambda_0^\varepsilon|}\Big(e^{2|\lambda_0^\varepsilon|T} - 1\Big) \leq \frac{\|\phi_0^\varepsilon\|^2_{H^1(\omega)}}{4|\lambda_0^\varepsilon|^2} e^{2|\lambda_0^\varepsilon|T} \int_0^T \|f(t)\|^2_{H^{-1}(\Omega)}\ \mathrm{dt},$$

and in any case, for any $f$ as in (1.2), we get

$$J^\varepsilon_{u^\varepsilon_0}(f) \geq \inf\left\{ \frac{e^{2|\lambda^\varepsilon_0|T}-1}{16|\lambda^\varepsilon_0|},\ \frac{|\lambda^\varepsilon_0|}{4\left\|\phi^\varepsilon_0\right\|^2_{H^1(\omega)}}\Big(1-e^{-2|\lambda^\varepsilon_0|T}\Big)\right\}.$$

This bound blows up when $\varepsilon \to 0$ from the estimates (3.2). Indeed, since $0 \notin \bar{\omega}$, we can choose $\alpha > 0$ small enough such that $\omega \subset \Omega \backslash B(0,\alpha)$ and therefore

$$\|\phi^\varepsilon_0\|_{H^1(\omega)} \leq \|\phi^\varepsilon_0\|_{H^1(\Omega\backslash B(0,\alpha))} \underset{\varepsilon\to 0}{\longrightarrow} 0. \qquad \square$$

# 4 Comments

In this article we proposed a study of a parabolic equation with an inverse-square potential $-\mu/|x|^2$ from a control point of view, in the two cases $\mu \leq \mu^*(N)$, which corresponds to a subcritical case, and $\mu > \mu^*(N)$, the surcritical case.

**A.** When $\mu \leq \mu^*(N)$, we have addressed the null-controllability problem for a distributed control in an arbitrary open subset of $\Omega$. To this end, we have derived a new Carleman inequality (2.7) inspired by the articles [19] and [13].

1. Our arguments can be adapted in much more general settings than presented here. For instance, one can handle several inverse-square singularities:

$$\begin{cases} \partial_t u - \Delta u - \displaystyle\sum_i \frac{\mu_i}{|x-x_i|^2} u = f, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{4.1}$$

where $\mu_i \leq \mu^*(N)$ for each $i$ and $f$ is localized in some open subset $\omega \subset \Omega$ in the sense of (1.2). In this case, the difficulty will again come from the choice of the weight. Let us assume that the points $x_i$ satisfy the following properties

$$|x_i - x_j| \geq 3, \quad i \neq j, \qquad d(x_i, \partial\Omega) \geq 3.$$

Note that by a scaling argument, this can be assumed as soon as the set $\{x_i\}_i$ does not have any accumulation point in $\bar{\Omega}$, which is equivalent to say that they are in finite numbers since $\Omega$ is bounded. In this case, we propose a weight of the form

$$\sigma(t,x) = s\theta\Big(e^{2\lambda \sup\psi} - \frac{1}{2}\sum_i |x-x_i|^2\gamma(x-x_i) - e^{\lambda\psi(x)}\Big),$$

where $\lambda$ and $s$ are positive parameters, $\theta$ is as in (2.3), $\psi$ satisfies

$$\begin{cases} \psi(x) = \ln(|x-x_i|), & x \in B(x_i,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega\backslash\Big(\cup_i \bar{B}(x_i,1)\Big), \end{cases}$$

and (2.5), and $\gamma = \gamma(|x|)$ is a smooth cut-off function such that

$$\gamma(x) = 1, \quad |x| \leq 1, \qquad \gamma(x) = 0, \quad |x| \geq 3/2.$$

Using this weight and following the proof of Theorem 2.1, one can prove a Carleman estimate for the adjoint system of (4.1), which still directly implies (1.8). However it may occur that the system (4.1) is not dissipative (see [8] where a necessary and sufficient condition is given for a multipolar potential to be positive on $\mathbb{R}^n$), and therefore we need to explain why inequality (1.7) is still implied by (1.8). Following for instance [6, Lemma 2.1], one can prove that

$$F(t) = \int_\Omega |w(t,x)|^2 \, \mathrm{dx}$$

satisfies

$$F'(t) \geq -CF(t).$$

Thus a Gronwall inequality allows us to conclude (1.7) from (1.8).

2. Note also the dispersive properties (that is Strichartz estimates) of the operators $i\partial_t + P$ and $\partial_{tt}^2 + P$, with

$$P = -\Delta - \frac{\mu}{|x|^2},$$

were studied in the whole space $\mathbb{R}^N$, $N \geq 3$, in [3]. In [3], it is proved that Strichartz estimates hold for the Schrödinger and the wave equations provided $\mu < \mu^*(N)$. This result was generalized to the critical case $\mu = \mu^*(N)$ and to the multipolar case in [6]. To complete this picture, we mention [7], in which a positive potential $V$ of order

$$\frac{\log(|x|)^2}{|x|^2}$$

was constructed in such a way that there exist quasi-modes for $P = -\Delta + V$ localized around the singularity. Note that in this case, the operator $P$ is strongly elliptic since $V$ is positive. To our knowledge, the controllability properties for the wave or Schrödinger equations with an inverse-square potential are widely open. Especially, it would be interesting to understand precisely the behavior of the rays of Geometric Optics around the singularities.

**B.** When $\mu > \mu^*(N)$, we have shown that we cannot uniformly stabilize regularized approximations of (1.1) with a control supported in $\omega$ when $0 \notin \bar{\omega}$.

1. To complete this result, we comment the case $0 \in \omega$, for which the stabilization property (1.10) holds. Given $u_0 \in L^2(\Omega)$, we claim that we can find $u \in L^2((0,T); H_0^1(\Omega))$ and $f \in L^2((0,T); H^{-1}(\Omega))$ as in (1.2) such that $u$ is the solution of (1.1) and that $J_{u_0}(u,f) \leq C \|u_0\|^2_{L^2(\Omega)}$ (see (1.10)).

Indeed, denote by $\chi$ a smooth function that equals 1 in a neighborhood of 0 and vanishing outside $\omega$. Then consider the solution $u$ of

$$\begin{cases} \partial_t u - \Delta u - (1-\chi)\dfrac{\mu}{|x|^2} u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

which satisfies $u \in L^2((0,T); H_0^1(\Omega))$, and $\|u\|_{L^2(0,T;H_0^1(\Omega))} \leq C \|u_0\|_{L^2}$ for some constant $C$. Then taking $f = \mu \chi u/|x|^2 \in L^2((0,T); H^{-1}(\Omega))$ provides an admissible stabilizer with the required property (1.2).

The same argument can also be applied to derive the null-controllability property for (1.1) when $0 \in \omega$. Indeed, the results in [13] proves that there exists a control $v \in L^2((0,T) \times \omega)$ such that the solution of

$$\begin{cases} \partial_t u - \Delta u - (1-\chi)\dfrac{\mu}{|x|^2} u = v, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

satisfies $u(T) = 0$. Besides, the norms of $v$ in $L^2((0,T)\times\omega)$ and $u$ in $L^2((0,T); H_0^1(\Omega))$ are bounded by the norm of $u_0$ in $L^2(\Omega)$. Then, taking $f = v + \mu \chi u/|x|^2$ provides a control in $L^2((0,T); H^{-1}(\Omega))$ for (1.1) that drives the solution to 0 in time $T$.

2. Since we proved that we cannot uniformly stabilize (1.13) when $0 \notin \bar{\omega}$, there is no uniform observability properties such as (1.7) for the corresponding adjoint regularized systems.

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