On the observability of abstract time-discrete linear parabolic equations

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Abstract

This article aims at analyzing the observability properties of time-discrete approximation schemes of abstract parabolic equations $\dot{z} + Az = 0$, where A is a self-adjoint positive definite operator with dense domain and compact resolvent. We analyze the observability properties of these diffusive systems for an observation operator $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$ with $\nu < 1/2$. Assuming that the continuous system is observable, we prove uniform observability results for suitable time-discretization schemes within the class of conveniently filtered data. We also propose a HUM type algorithm to compute discrete approximations of the exact controls. Our approach also applies to sequences of operators which are uniformly observable. In particular, our results can be combined with the existing ones on the observability of space semi-discrete systems, yielding observability properties for fully discrete approximation schemes.

Key words: Time discretization, Observability, Controllability, Parabolic equations, Filtering techniques.

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1 Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a positive definite self-adjoint operator with dense domain and compact resolvent.

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We consider the following abstract system:

$$\dot{z}(t) + Az(t) = 0, \quad t \in [0, T], \quad z(0) = z_0. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, and $z = z(t)$ is the *state* of the system.

Such equations are often used as models for diffusive systems and especially heat equations.

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$, with $\nu \leq 1/2$, be an observation operator, and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

To give a sense to (1.2), we will assume that B is an admissible observation operator, i.e. for every $T > 0$ there exists a constant $K_T > 0$ such that any solution of system (1.1) with initial data $z_0 \in \mathcal{D}(A)$ satisfies

$$\int_0^T \|Bz(t)\|_Y^2 \, dt \leq K_T \, \|z_0\|_X^2 \, . \tag{1.3}$$

Under this assumption, the output function y in (1.2) is well-defined as a function in $L^2((0, T); Y)$ for any solution of (1.1) with initial data $z_0 \in X$.

Actually, this property is automatically satisfied when $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$ with $\nu \leq 1/2$ (see, e.g., [25] and Theorem 2.2 below), which we will always assume in the following.

The exact observability property for system (1.1)-(1.2) can be formulated as follows:

Definition 1.1. *System* (1.1)-(1.2) *is exactly observable in time T^* if there exists $k_* > 0$ such that any solution of system* (1.1) *with initial data $z_0 \in X$ satisfies*

$$k_* \, \|z(T^*)\|_X^2 \leq \int_0^{T^*} \|Bz(t)\|_Y^2 \, dt. \tag{1.4}$$

Moreover, system (1.1)-(1.2) *is said to be exactly observable if it is exactly observable in some time $T^* > 0$.*

Inequalities (1.3) and (1.4) are relevant in controllability theory due to the duality argument given by the Hilbert Uniqueness Method (HUM in short), see [18] and Section 4.

In the following, we assume that (1.4) holds for the continuous system. Such results have been proved, often by means of Carleman estimates, for various models including the heat equation [10, 12, 16], the Stokes equations [9], and some other singular models such as [3, 7, 20, 24].

This article aims at studying the admissibility and observability properties for time-discrete approximation schemes of (1.1)-(1.2), and corresponding controllability properties.

To be more precise, we consider a linear time-discretization of (1.1)-(1.2), which takes the general form

$$\left\{\begin{array}{rcl} z^{k+1} & = & \mathbb{T}_{\triangle t} z^k, \quad k \in \mathbb{N}, \\ z^0 & = & z_0, \end{array}\right. \qquad y^k = B z^k, \tag{1.5}$$

where $\triangle t > 0$ is the time-discretization parameter, z^k is an approximation of z at time $k\triangle t$, and $\mathbb{T}_{\triangle t}$ is an approximation of $\exp(-(\triangle t)A)$, in a sense we will make precise below.

Since we assume A to be a positive definite self-adjoint operator with compact resolvent, its spectrum is explicitly given by an increasing sequence $(\mu_j)_{j\in\mathbb{N}}$ of positive real numbers with $\lim_j \mu_j = +\infty$ and corresponding eigenvectors $(\Phi_j)_j$ such that

$$A\Phi_j = \mu_j \Phi_j.$$

Besides, one can further impose the sequence $(\Phi_j)_{j\in\mathbb{N}}$ to be an orthonormal basis of X.

We can now present more precisely the assumptions on $\mathbb{T}_{\triangle t}$. We shall assume that the discrete operators $\mathbb{T}_{\triangle t}$ preserve the eigenvectors, and that there exists a function $f : [0, R) \to \mathbb{R}_+$ such that, for any $\triangle t > 0$,

$$\forall j \in \mathbb{N}, \ s.t. \quad \mu_j < \frac{R}{\triangle t}, \quad \left\{\begin{array}{l} \mathbb{T}_{\triangle t}\Phi_j = \exp(-(\triangle t)\lambda_{j,\triangle t})\Phi_j, \\ \qquad \text{where } \lambda_{j,\triangle t} = \dfrac{1}{\triangle t} f((\triangle t)\mu_j), \end{array}\right. \tag{1.6}$$

or, in a more concise form, $\mathbb{T}_{\triangle t} = \exp(-f((\triangle t)A))$. Such assumptions are of a very general nature and are satisfied for many time-discrete approximation schemes, for instance the Euler implicit and explicit methods, the Crank-Nicolson scheme, the θ-methods, the Runge-Kutta discretizations... see, e.g., the textbook [4] and the examples presented in Subsection 2.1. In other words, the function f describes the action of a given numerical method, and R corresponds to the limit of stability for the numerical method.

We also assume that f is smooth (actually C^2 is sufficient), and that

$$\lim_{\eta\to 0} \frac{f(\eta)}{\eta} = 1 \quad \text{and} \quad \forall \eta \in (0, R), \quad f(\eta) > 0. \tag{1.7}$$

The first assumption is equivalent to the consistency of the scheme. The second one ensures that the numerical scheme damps out every frequency, which is needed for numerical stability.

In the following, we assume that the discrete operators $\mathbb{T}_{\triangle t}$ satisfy these conditions.

We investigate the admissibility and observability properties which are needed for controllability purposes (see [18] and [27, 28]).

• *Uniform admissibility*: To find positive constants T and K_T such that for all $\triangle t > 0$, any solution z of (1.5) with initial data z_0 in an appropriate class $X_{\triangle t}$ satisfies

$$\triangle t \sum_{k=0}^{\lceil T/\triangle t \rceil} \left\| B z^k \right\|_Y^2 \leq K_T \left\| z_0 \right\|_X^2 . \tag{1.8}$$

Note that, using that $k \mapsto \left\| z^k \right\|_X^2$ decays, one easily checks that, if (1.8) holds for some $(\tilde{T}, K_{\tilde{T}})$, it holds as well for any time T, taking $K_T = K_{\tilde{T}}(1 + T/\tilde{T})$.
• *Uniform observability*: To find positive constants T^* and k_* such that for all $\triangle t > 0$, any solution z of (1.5) with initial data z_0 in an appropriate class $X_{\triangle t}$ satisfies

$$k_* \left\| z^{\lceil T^*/\triangle t \rceil} \right\|_X^2 \leq \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B z^k \right\|_Y^2 . \tag{1.9}$$

Of course, our interest is to make the class $X_{\triangle t}$ as big as possible. However, $X_{\triangle t} = X$ is out of reach in general, even when $R = \infty$ (see [26]). But at least we want $X_{\triangle t} \to_{\triangle t \to 0} X$ in some sense, in order to recover the admissibility and observability properties (1.3) and (1.4) of the continuous system when $\triangle t \to 0$.

We emphasize that inequalities (1.8)-(1.9) shall hold uniformly in $\triangle t$. Indeed, this is needed for controllability issues to ensure the convergence of the discrete controls (see [28] and the examples therein). Precise statements will be given in Section 4.

It is by now well-known that discretization processes may create high frequency spurious solutions which might lead to non-uniform observability properties.

For *time* semi-discrete approximations of parabolic systems, the only work we are aware of is [26], which is based on the spectral estimates proved in [16, 17]. Using a standard duality argument, one can easily check that the results in [26] read as an observability inequality similar to (1.9) in a class of filtered data in the special case where the operator $A = -\Delta_D$ is the Laplace operator with Dirichlet boundary conditions in some smooth bounded domain Ω and B is the restriction operator to ω. In [26], we emphasize that it is also proved that filtering the initial data is needed to obtain uniform observability results for time semi-discrete heat equations.

Let us also mention that the *space* semi-discrete heat equation in 1-d has been studied in [19] using, as in [8], Müntz Szász type theorem. In this case, observability properties hold uniformly with respect to the *space* mesh-size [19]. A more general result has been derived in [14], which provides a weak observability inequality in a very general setting inspired by [15]. The weak observability property in [14] suffices to derive an explicit numerical method for computing approximations of exact controls for the continuous system. We will actually follow the methodology in [14] and derive weak forms of (1.9) for time semi-discrete systems, which still are relevant for the exact controllability problem.

Note however that the results presented below and in [14] are more precise than the ones in [15] (see also [1]). These references are indeed dealing with the linear quadratic regulator (LQR) problem, which rather corresponds to an optimal control approach (in particular, the final point is not fix).

Note that the counterexample of Kavian in [28] also emphasizes the need of filtering the initial data to obtain uniform observability results for space semi-discrete approximations of the heat equation in the 2-d square.

Let us also mention that several works have been devoted to analyze observability and controllability properties for *space* semi-discrete *wave* equations [13, 28]. In the context of *time* semi-discrete and *fully* discrete *conservative* equations, we refer to [6], which deals with very general time-approximation schemes for *conservative* linear systems, and closely related to the question addressed here.

Let us now introduce, for $s \in \mathbb{R}_+$, the following filtered space:

$$\mathcal{C}(s) = \operatorname{span}\Big\{\Phi_j \ : \ \text{the corresponding } \mu_j \text{ satisfies } \mu_j \leq s\Big\}. \tag{1.10}$$

We are now in position to state the main result of our paper:

Theorem 1.2. *Let A be a self-adjoint positive definite operator with dense domain and compact resolvent, and $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$, with $\nu < 1/2$. Let $\mathbb{T}_{\triangle t}$ be a numerical scheme satisfying* (1.6) *and* (1.7). *Also assume that system* (1.1)-(1.2) *is exactly observable in some time T^*. Set*

$$\beta = \min\{2, 1 - 2\nu\}. \tag{1.11}$$

Then, given any $\delta \in (0, R)$, there exist positive constants K_δ, k_δ and C_δ such that, for all $\triangle t > 0$, any solution z^k of (1.5) with initial data $z_0 \in \mathcal{C}(\delta/\triangle t)$ satisfies

$$k_\delta \left\| z^{\lceil T^*/\triangle t \rceil} \right\|_X^2 \leq \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B z^k \right\|_Y^2 + C_\delta (\triangle t)^\beta \left\| z_0 \right\|_X^2 \leq K_\delta \left\| z_0 \right\|_X^2. \tag{1.12}$$

The admissibility inequality in (1.12) can be derived as in the continuous case, see Theorem 2.2. We will also explain in Remark 2.3 why filtering the data are needed when looking at the admissibility properties of (1.5).

Note that the observability inequality in (1.12) is a weak form of (1.9), due to the presence of the term $(\triangle t)^\beta \left\| z_0 \right\|_X^2$. We will explain in Subsection 3.3 why such a term is needed in our general setting, even after filtering the initial data.

However the weak observability inequality (1.12) is sufficient to derive an efficient computational technique for controllability problems. This method is inspired by previous works on space semi-discrete heat equations [14] and on Tychonov regularization techniques for wave equations [5, 11, 28].

One of the interesting features of our approach is that it can be applied to families of operator which are uniformly observable. In particular, we can derive uniform (in both space and time-discretization parameters) observability properties for fully discrete approximation schemes under the condition that the space semi-discrete approximation schemes are uniformly observable with respect to the space discretization parameter.

We will present such applications derived for the finite difference 1-d heat equation, for which observability properties hold uniformly with respect to the space mesh size (see [19]). We will also exhibit an example of application of our results to the observability properties of fully discrete schemes derived from the finite element method, for which, to our knowledge, only weak observability properties have been proved so far in a general setting, see [14].

This article is organized as follows. In Section 2, we present some examples of time-discrete schemes, which fit the abstract framework presented above, and we state a discrete admissibility result. In Section 3, we prove Theorem 1.2. In Section 4, we show that Theorem 1.2 can be applied to derive controllability properties. In Section 5, we present applications of Theorem 1.2 to the study of the observability properties of fully discrete approximation schemes of (1.1)-(1.2). Finally, we end up with some further comments.

2 Preliminaries

We will first present several well-known schemes which fit the abstract framework proposed here. Second, we will derive some basic estimates on $\lambda_{j,\triangle t}$, which will be useful all along the paper. Third, we will focus on the admissibility inequality in the discrete setting and prove a precise admissibility result.

2.1 Examples of numerical schemes

This subsection presents several time-discrete approximation schemes (see for instance [4] for the analysis of their convergence properties) which enter in the abstract framework presented above in (1.6)-(1.7). Routines computations are left to the reader.

• *The θ-methods:* Set $\theta \in [0,1]$. The θ-method is given by

$$\frac{z^{k+1}-z^k}{\triangle t} = -A(\theta z^{k+1} + (1-\theta)z^k), \quad k \in \mathbb{N}, \quad z^0 = z_0.$$

This is a generalization of the Crank-Nicolson method ($\theta = 1/2$) and the Euler methods ($\theta = 0$ for the explicit Euler method and $\theta = 1$ for the implicit Euler method).

The operator $\mathbb{T}_{\triangle t}$ is then given by

$$\mathbb{T}_{\triangle t} = (I + \theta(\triangle t)A)^{-1}\left(I - (1-\theta)(\triangle t)A\right).$$

Thus, the function f_θ is defined by

$$f_\theta : \left[0, \frac{1}{1-\theta}\right) \to \mathbb{R}_+, \quad \eta \mapsto \ln\left(\frac{1+\theta\eta}{1-(1-\theta)\eta}\right), \qquad \left(R_\theta = \frac{1}{1-\theta}\right).$$

In particular, for implicit Euler method, $\theta = 1$ and $R_\theta = \infty$.

• *The Runge-Kutta methods:* Assume $q \in \mathbb{N}^*$. The Runge-Kutta time-discrete schemes take the form of

$$z^{k+1} = z^k + \triangle t \sum_{1 \le i \le q} b_i \kappa_i \text{ with } \kappa_i = -A \left(z^k + \triangle t \sum_{j=1}^{q} a_{ij} \kappa_j \right), \quad \forall i \in \{1, ..., q\},$$

with appropriate real numbers $(b_i)_{1 \le i \le q}$ and $(a_{ij})_{1 \le i,j \le q}$, verifying $\sum_{i=1}^{q} b_i = 1$. For instance, we can consider the (explicit) fourth-order Runge-Kutta method given by $q = 4$, $b = (1/6, 2/6, 2/6, 1/6)$ and

$$(a_{ij})_{i,j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case, the operator $\mathbb{T}_{\triangle t}$ is defined by

$$\mathbb{T}_{\triangle t} = I - (\triangle t)A + \frac{(\triangle t)^2}{2} A^2 - \frac{(\triangle t)^3}{6} A^3 + \frac{(\triangle t)^4}{24} A^4,$$

and the function f simply is

$$f : [0, \alpha] \to \mathbb{R}_+, \quad \eta \mapsto -\ln \left(1 - \eta + \frac{\eta^2}{2} - \frac{\eta^3}{6} + \frac{\eta^4}{24} \right), \qquad (R = \alpha),$$

where α is solution of $1 - \frac{\alpha}{2} + \frac{\alpha^2}{6} - \frac{\alpha^3}{24} = 0$.

We refer to [4] for more examples of explicit and implicit Runge-Kutta methods.

2.2 Rough estimates on $\lambda_{j,\triangle t}$

We shall state some basic estimates:

Proposition 2.1. *Given any $\delta \in (0, R)$, there exist positive constants m_δ, M_δ and S_δ such that for all $0 < \mu_j \le \delta / \triangle t$, the estimates*

$$m_\delta \le \frac{\lambda_{j,\triangle t}}{\mu_j} \le M_\delta, \qquad \left| \lambda_{j,\triangle t} - \mu_j \right| \le (\triangle t) \; S_\delta \mu_j^2, \tag{2.1}$$

hold uniformly in $\triangle t$.

Proof. The first inequality is a consequence of assumption (1.7). Indeed, (1.7) guarantees the continuity of the function $\eta \mapsto f(\eta)/\eta$ on $(0, \delta]$ extended by 1 at $\eta = 0$, and this function does not vanish on $[0, \delta]$.

The second inequality in (2.1) is a consequence of Taylor's formula. Indeed, assumption (1.7) implies $f(0) = 0$ and $f'(0) = 1$ and therefore

$$|\lambda_{j,\triangle t} - \mu_j| = \frac{1}{\triangle t} \left| f(\mu_j \triangle t) - \mu_j \triangle t \right| \le \sup_{\zeta \in [0,\delta]} \{ |f''(\zeta)| \} \frac{\mu_j^2 (\triangle t)}{2}.$$

□

2.3 Admissibility

For convenience, we introduce the Hilbert spaces $X_s = \mathcal{D}(A^s)$ for $s \geq 0$ and $X_s = X^*_{-s}$ for $s < 0$, endowed, respectively, with the norms $\|\cdot\|_s$ defined, for $z = \sum_j a_j \Phi_j$, by

$$\|z\|_s^2 = \sum_j |a_j|^2 \mu_j^{2s}.$$

Note that, for $s = 0$, $X_0 = X$ and $\|.\|_0 = \|.\|_X$.

We now prove the following theorem, which in particular implies the admissibility property stated in Theorem 1.2.

Theorem 2.2. *Assume that $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$, with $\nu \leq 1/2$. Then there exists a positive constant K_0 such that any solution of (1.1) with initial data $z_0 \in X_{\nu-1/2}$ satisfies*

$$\int_0^\infty \|Bz(t)\|_Y^2 \leq K_0 \|z_0\|_{\nu-1/2}^2 . \tag{2.2}$$

Besides, given any $\delta \in (0, R)$, there exists a positive constant K_δ, which only depends on $\|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}$, f and δ, such that for any $\triangle t > 0$, any solution z of (1.5) with initial data $z_0 \in \mathcal{C}(\delta/\triangle t)$ satisfies

$$\triangle t \sum_{k=0}^\infty \left\|Bz^k\right\|_Y^2 \leq K_\delta \|z_0\|_{\nu-1/2}^2 . \tag{2.3}$$

Proof. The continuous case is classical and is left to the reader. It may also be deduced from our result in the discrete setting.

Let z be a solution of (1.5) with initial data $z_0 = \sum_j a_j \Phi_j \in \mathcal{C}(\delta/\triangle t)$. Then

$$z^k = \sum_{\mu_j \leq \delta/\triangle t} a_j \exp(-(\triangle t)k\lambda_{j,\triangle t})\Phi_j.$$

In particular, for all $k \in \mathbb{N}$, z^k belongs to $\mathcal{D}(A^\nu)$ since it is a finite combination of eigenvectors of A. Thus

$$\left\|Bz^k\right\|_Y^2 \leq \|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}^2 \sum_{\mu_j \leq \delta/\triangle t} |a_j|^2 \mu_j^{2\nu} \exp(-2(\triangle t)k\lambda_{j,\triangle t}).$$

Therefore, inequality (2.3) holds uniformly in $\triangle t$ within the class $\mathcal{C}(\delta/\triangle t)$ if, for j such that $\mu_j \triangle t \leq \delta$, the quantity

$$\triangle t \sum_{k=0}^\infty \mu_j \exp(-2(\triangle t)k\lambda_{j,\triangle t}) = \frac{(\triangle t)\mu_j}{1 - \exp(-2(\triangle t)\lambda_{j,\triangle t})}$$

is bounded uniformly in $\triangle t$ (recall that, from (1.7), for $\mu_j < \delta/\triangle t$, $\lambda_{j,\triangle t} > 0$). This is indeed the case since the function $\eta \mapsto \eta/(1-\exp(-2f(\eta)))$ is continuous on $(0, \delta]$ and can be extended continuously by $1/2$ in $\eta = 0$ due to (1.7). $\square$

Remark 2.3. *When $R < \infty$, it is natural to consider only initial data in the class $\mathcal{C}(\delta/\triangle t)$ for $\delta < R$. But, when $R = \infty$, for instance for the Euler implicit method, this condition might seem too restrictive. This is actually not the case. For instance, if $B = A^{1/2}$, then the continuous system* (1.1)-(1.2) *is admissible. However, for the solution $z^k = \exp(-\lambda_{j,\triangle t} k \triangle t)\Phi_j$ of* (1.5)*, we have*

$$\|z_0\|_X^2 = 1 \text{ and } \triangle t \sum_{k=0}^{\lceil T/\triangle t \rceil} \left\| A^{1/2} z^k \right\|_X^2 = (\triangle t)\mu_j \frac{1 - e^{-2T\lambda_{j,\triangle t}}}{1 - e^{-2(\triangle t)\lambda_{j,\triangle t}}}.$$

But, with the Euler implicit method, $(\triangle t)\lambda_{j,\triangle t} = \ln(1 + (\triangle t)\mu_j)$, and then, if $\mu_j \triangle t \simeq \delta$, we have $(\triangle t)\lambda_{j,\triangle t} \simeq \ln(1+\delta)$ and

$$\triangle t \sum_{k=0}^{\lceil T/\triangle t \rceil} \left\| A^{1/2} z^k \right\|_X^2 \underset{\triangle t \to 0}{\simeq} \frac{\delta}{1 - 1/(1+\delta)^2} \underset{\delta \to \infty}{\longrightarrow} +\infty.$$

This indicates that filtering processes are needed in general to ensure uniform admissibility properties, even in the case $R = \infty$.

3 Observability

In this section, we prove the observability estimate in Theorem 1.2, the admissibility one being a consequence of Theorem 2.2. Below, we present the proof of Theorem 1.2 using two lemmas whose proofs are postponed to Subsections 3.1 and 3.2. We will also comment our observability result in Subsection 3.3.

In the following, we shall make explicit all the dependences on the time discretization parameter $\triangle t$, and then the constants will always be independent of $\triangle t$.

Proof of Theorem 1.2. First, we introduce the linear operator $A_{\triangle t}$ defined by

$$A_{\triangle t}\Phi_j = \lambda_{j,\triangle t}\Phi_j, \quad \forall j \in \mathbb{N} \ s.t. \quad \mu_j \triangle t < R,$$

which satisfies the following property: for any $z_0 \in \mathcal{C}(R/\triangle t)$,

$$\exp(-(\triangle t)kA_{\triangle t})z_0 = \mathbb{T}_{\triangle t}^k z_0. \tag{3.1}$$

The first step of our proof establishes a link between the (time continuous) semi-groups generated by $-A_{\triangle t}$ and $-A$.

Lemma 3.1. *Assume that $B \in \mathcal{L}(\mathcal{D}(A^\nu), Y)$ with $\nu < 1/2$.*

Then, given any $\delta \in (0, R)$ and any $T > 0$, there exists a positive constant $C_{\delta,T}$, which only depends on δ, T, f and $\|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}$, such that for any $\triangle t > 0$, for any $z_0 \in \mathcal{C}(\delta/\triangle t)$, the following estimates hold:

$$\begin{aligned} &\left\| e^{-TA_{\triangle t}} z_0 \right\|_X^2 \le 2 \left\| e^{-TA} z_0 \right\|_X^2 + C_{\delta,T}(\triangle t)^2 \|z_0\|_X^2, \\ &\int_0^T \left\| Be^{-tA} z_0 \right\|_Y^2 dt \le 2 \int_0^T \left\| Be^{-tA_{\triangle t}} z_0 \right\|_Y^2 dt + C_{\delta,T}(\triangle t)^{1-2\nu} \|z_0\|_X^2. \end{aligned} \tag{3.2}$$

In a second step, we need to evaluate precisely the difference between the continuous integral in (3.2) and the discrete integral in (1.12).

Lemma 3.2. *Assume that $B \in \mathcal{L}(\mathcal{D}(A^\nu), Y)$ with $\nu < 1/2$.*

Then, given any $\delta \in (0, R)$ and $T > 0$, there exists a positive constant $C_{\delta,T}$, which only depends on δ, T, f and $\|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}$, such that for any $\triangle t > 0$, for any $z_0 \in \mathcal{C}(\delta/\triangle t)$, the following estimate holds:

$$\int_0^T \left\|Be^{-tA_{\triangle t}} z_0\right\|_Y^2 \, dt \leq 2\triangle t \sum_{k=0}^{\lceil T/\triangle t \rceil} \left\|B\mathbb{T}_{\triangle t}^k z_0\right\|_Y^2 + C_{\delta,T} (\triangle t)^{1-2\nu} \|z_0\|_X^2 . \quad (3.3)$$

The proof of Theorem 1.2 then follows directly from the combination of Lemmas 3.1 and 3.2 and the observability property (1.4) for system (1.1)-(1.2) in time $T = T^*$. Indeed, if (1.4) holds, i.e. if there exists a positive constant k_* such that

$$k_* \left\|e^{-T^* A} z_0\right\|_X^2 \leq \int_0^{T^*} \left\|Be^{-tA} z_0\right\|_Y^2 dt,$$

then, applying Lemmas 3.1 and 3.2 with $T = T^*$, we obtain the existence of positive constants k_δ and C_δ such that (1.12) holds. □

3.1 Proof of Lemma 3.1

Proof of Lemma 3.1. In this subsection, we work under the assumptions of Lemma 3.1, and $\delta \in (0, R)$ is fixed.

Expand $z_0 \in \mathcal{C}(\delta/\triangle t)$ as $z_0 = \sum_{\mu_j \triangle t \leq \delta} a_j \Phi_j$. Then for any $t > 0$,

$$e^{-tA} z_0 = \sum_{\mu_j \triangle t \leq \delta} a_j e^{-\mu_j t} \Phi_j, \qquad e^{-tA_{\triangle t}} z_0 = \sum_{\mu_j \triangle t \leq \delta} a_j e^{-\lambda_{j,\triangle t} t} \Phi_j.$$

Therefore, using the inequality

$$\forall (a, b) \in \mathbb{R}_+^2, \quad \left|e^{-a} - e^{-b}\right| \leq |a - b| e^{-\inf\{a,b\}},$$

and the estimates (2.1), we obtain, for any $s \in \mathbb{R}$,

$$\begin{aligned} \left\|\left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0\right\|_s^2 &\leq \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \mu_j^{2s} \left(e^{-t\mu_j} - e^{-t\lambda_{j,\triangle t}}\right)^2 \\ &\leq \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \mu_j^{2s} t^2 |\mu_j - \lambda_{j,\triangle t}|^2 e^{-2t \inf\{\lambda_{j,\triangle t}, \mu_j\}} \\ &\leq S_\delta^2 (\triangle t)^2 \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \mu_j^{2s+4} t^2 e^{-2t\mu_j \inf\{m_\delta, 1\}}. \end{aligned} \quad (3.4)$$

In particular, this implies

$$\left\|e^{-TA} z_0 - e^{-TA_{\triangle t}} z_0\right\|_X^2 \leq S_\delta^2 (\triangle t)^2 T^2 \|z_0\|_X^2 \sup_{\mu_j \triangle t \leq \delta} \left\{\mu_j^4 e^{-2T\mu_j \inf\{m_\delta,1\}}\right\}.$$

Since $\eta \mapsto \eta^4 e^{-2\eta\alpha}$, with $\alpha > 0$, is bounded on $\mathbb{R}_+$, it follows that there exists a constant C such that

$$\left\| e^{-TA} z_0 - e^{-TA_{\triangle t}} z_0 \right\|_X^2 \le C(\triangle t)^2 \left\| z_0 \right\|_X^2 , \tag{3.5}$$

from which we deduce the first estimate in (3.2).

To study the second estimate in (3.2), we use $B \in \mathfrak{L}(\mathcal{D}(A^\nu), Y)$:

$$\left\| B\left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0 \right\|_Y^2 \le \|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}^2 \left\| \left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0 \right\|_\nu^2 , \quad t > 0.$$

Hence, from (3.4) we deduce that

$$\int_0^T \left\| B\left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0 \right\|_Y^2 \, dt$$
$$\le \|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}^2 S_\delta^2 (\triangle t)^2 \sum_{\mu_j \triangle t \le \delta} |a_j|^2 \mu_j^{2\nu+4} \int_0^T \left(t^2 e^{-2t\mu_j \inf\{m_\delta, 1\}} \right) dt.$$

But, for $a > 0$,

$$0 \le \int_0^T t^2 e^{-ta} \, dt \le \frac{2}{a^3}.$$

Therefore we obtain

$$\int_0^T \left\| B\left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0 \right\|_Y^2 \, dt$$
$$\le \|B\|_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}^2 \frac{S_\delta^2}{4 \inf\{m_\delta, 1\}^3} \sum_{\mu_j \triangle t \le \delta} |a_j|^2 (\triangle t)^2 \mu_j^{2\nu+1}. \tag{3.6}$$

But, when $(\triangle t)\mu_j \le \delta$, $\mu_j^{2\nu+1} (\triangle t)^2 \le \delta^{2\nu+1} (\triangle t)^{1-2\nu}$, and then (3.6) implies that there exists a constant C such that

$$\int_0^T \left\| B\left(e^{-tA} - e^{-tA_{\triangle t}}\right) z_0 \right\|_Y^2 \, dt \le C(\triangle t)^{1-2\nu} \left\| z_0 \right\|_X^2 . \tag{3.7}$$

The second estimate in (3.2) follows. □

3.2 Proof of Lemma 3.2

Proof of Lemma 3.2. First, let us recall the following classical estimates on Riemann sums for a function $g \in W^{1,1}(0, T)$:

$$\left| \int_0^T g(t) dt - \triangle t \sum_{k=0}^{\lceil T/\triangle t \rceil} g(k\triangle t) \right| \le \sum_{k=0}^{\lceil T/\triangle t \rceil} \iint_{[k\triangle t, (k+1)\triangle t]^2} \chi_{\{s<t\}} |\dot{g}(s)| \, ds \, dt$$
$$\le \triangle t \int_0^T |\dot{g}| \, dt. \tag{3.8}$$

Therefore, for any $z_0 \in \mathcal{C}(\delta/\triangle t)$, taking the smooth function

$$g(t) = \|B\exp(-tA_{\triangle t})z_0\|_Y^2$$

in (3.8), we obtain

$$\begin{aligned}
&\Big| \int_0^T \left\|Be^{-tA_{\triangle t}} z_0\right\|_Y^2 dt - \triangle t \sum_{k=0}^{\lceil T/\triangle t\rceil} \left\|Be^{-k(\triangle t)A_{\triangle t}} z_0\right\|_Y^2 \Big| \\
&\qquad \leq \ 2\triangle t \int_0^T \left| \left\langle Be^{-tA_{\triangle t}} z_0, Be^{-tA_{\triangle t}} A_{\triangle t}\ z_0 \right\rangle_Y \right| dt \\
&\qquad \leq \ \frac{1}{2}\int_0^T \left\|Be^{-tA_{\triangle t}} z_0\right\|_Y^2 dt + 2(\triangle t)^2 \int_0^T \left\|Be^{-tA_{\triangle t}} A_{\triangle t} z_0\right\|_Y^2 dt. \quad (3.9)
\end{aligned}$$

But, according to Proposition 2.1, for any $z = \sum_j a_j \Phi_j \in \mathcal{C}(\delta/\triangle t)$,

$$\|Bz\|_Y^2 \leq \|B\|^2_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)} \sum_j |a_j|^2 \mu_j^{2\nu} \leq \frac{\|B\|^2_{\mathfrak{L}(\mathcal{D}(A^\nu),Y)}}{m_\delta^{2\nu}} \sum_j |a_j|^2 \lambda_{j,\triangle t}^{2\nu}.$$

It follows that the operator B is continuous from the space $\mathcal{D}(A^\nu_{\triangle t}) \cap \mathcal{C}(\delta/\triangle t)$ endowed with the norm $\left\|A^\nu_{\triangle t}\ \cdot\ \right\|_X$ to Y and its corresponding operator norm is uniformly bounded with respect to $\triangle t$.

Therefore, from Theorem 2.2, there exists a constant K_0 such that

$$\begin{aligned}
(\triangle t)^2 \int_0^T \left\|e^{-tA_{\triangle t}} A_{\triangle t} z_0\right\|_Y^2 dt &\leq K_0 (\triangle t)^2 \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \lambda_{j,\triangle t}^{2(\nu-1/2)} \lambda_{j,\triangle t}^2 \\
&\leq K_0 M_\delta^{2\nu+1} (\triangle t)^2 \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \mu_j^{2\nu+1}, \quad (3.10)
\end{aligned}$$

where we again used (2.1). Since $\nu < 1/2$, $\mu_j^{2\nu+1}(\triangle t)^2 \leq \delta^{2\nu+1}(\triangle t)^{1-2\nu}$, estimate (3.9) implies the existence of a constant C_δ such that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \left\|Be^{-tA_{\triangle t}} z_0\right\|_Y^2 dt \\
&\qquad \leq \triangle t \sum_{k=0}^{\lceil T/\triangle t\rceil} \left\|Be^{-k(\triangle t)A_{\triangle t}} z_0\right\|_Y^2 + C_\delta (\triangle t)^{1-2\nu} \|z_0\|_X^2, \quad (3.11)
\end{aligned}$$

and the proof of Lemma 3.2 is complete, due to the remark in (3.1). □

3.3 Comments

On the necessity of the remaining term. An interesting case to be considered corresponds to the case $B = \exp(-A)$. This operator B obviously is a smoothing operator: for all $\nu < 0$, $B \in \mathfrak{L}(\mathcal{D}(A^\nu), X)$.

In this case, the continuous model (1.1) is exactly observable in any time $T^* > 1$, since

$$\left\| e^{-TA} \Big(\sum_j a_j \Phi_j \Big) \right\|_X^2 = \sum_j |a_j|^2 e^{-2\mu_j T},$$

$$\int_0^T \left\| e^{-A} e^{-tA} \Big(\sum_j a_j \Phi_j \Big) \right\|_X^2 \, dt = \sum_j |a_j|^2 \frac{e^{-2\mu_j}}{2\mu_j} \left(1 - e^{-2\mu_j T}\right).$$

But in the discrete case, this is more intricate:

$$\left\| e^{-TA_{\triangle t}} \Big(\sum_j a_j \Phi_j \Big) \right\|_X^2 = \sum_j |a_j|^2 e^{-2\lambda_{j,\triangle t} T},$$

$$\triangle t \sum_{k=0}^{T/\triangle t} \left\| e^{-A} e^{-k(\triangle t) A_{\triangle t}} \Big(\sum_j a_j \Phi_j \Big) \right\|_X^2 = \sum_j |a_j|^2 e^{-2\mu_j} (\triangle t) \left(\frac{1 - e^{-2\lambda_{j,\triangle t} T}}{1 - e^{-2\lambda_{j,\triangle t}(\triangle t)}} \right).$$

Hence, when considering for instance the Euler implicit method, and $\mu_j \triangle t \simeq \delta$, which corresponds to $\lambda_{j,\triangle t} \triangle t \simeq \ln(1+\delta)$, we have

$$\left\| e^{-TA_{\triangle t}} \Phi_j \right\|_X^2 \simeq \exp\left(-\frac{2}{\triangle t} T \ln(1+\delta) \right),$$

$$\triangle t \sum_{k=0}^{T/\triangle t} \left\| e^{-A} e^{-k(\triangle t) A} \Phi_j \right\|_X^2 \simeq \exp\left(-\frac{2\delta}{\triangle t} \right) \frac{(\triangle t)}{1 - 1/(1+\delta)^2}.$$

Therefore, in that case, (1.9) cannot be satisfied uniformly with respect to $\triangle t$ in the class $\mathcal{C}(\delta/\triangle t)$ for any δ, T such that $\delta > T \ln(1+\delta)$. These explicit computations actually show that, in this case, there exist positive constants C_δ, $C_{2,\delta}$ and k_δ such that, for $z_0 \in \mathcal{C}(\delta/\triangle t)$,

$$k_\delta \left\| z^{\lceil T^*/\triangle t \rceil} \right\|_X^2 \le \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B z^k \right\|_Y^2 + C_\delta e^{-C_{2,\delta}/\triangle t} \left\| z_0 \right\|_X^2,$$

whereas our results yield a remaining term of the form $(\triangle t)^2 \left\| z_0 \right\|_X^2$.

To sum up, this example shows that, in general, a remaining term in (1.12) cannot be avoided.

The case $R = \infty$ *and* $B \in \mathfrak{L}(X, Y)$. In that case, if we further assume that $\lim_{\delta \to \infty} f(\delta) = +\infty$, we can prove that (1.12) holds uniformly with respect to the time discretization parameter $\triangle t > 0$ for any solution of (1.5) without any filtering condition. Under this assumption, the admissibility property in (1.12) is obvious since B is continuous on X.

Let us now deal with the observability inequality in (1.12). Choose first $\delta = 1$ and apply Theorem 1.2. Then, consider a solution z^k of (1.5) with initial data $z_0 \in X$. Define $\pi_{1/\triangle t}$ as the orthogonal projection in X on $\mathcal{C}(1/\triangle t)$. Then (1.12) applies to $\pi_{1/\triangle t} z^k$. But

$$\triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B \pi_{1/\triangle t} z^k \right\|_Y^2 \leq 2 \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B z^k \right\|_Y^2 + 2 \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B (I - \pi_{1/\triangle t}) z^k \right\|_Y^2$$

and then we only have to check that there exists a constant C such that

$$\max \left\{ \left\| (I - \pi_{1/\triangle t}) z^{\lceil T^*/\triangle t \rceil} \right\|_X^2 , \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B (I - \pi_{1/\triangle t}) z^k \right\|_Y^2 \right\} \leq C(\triangle t) \left\| z_0 \right\|_X^2 . \tag{3.12}$$

Writing $z_0 = \sum_j a_j \Phi_j$,

$$\left\| (I - \pi_{1/\triangle t}) z^k \right\|_X^2 = \sum_{\mu_j \triangle t \geq 1} |a_j|^2 e^{-2k(\triangle t) \lambda_{j,\triangle t}}.$$

But, since $\lim\limits_{+\infty} f = +\infty$ and f is strictly positive on $\mathbb{R}_+^*$, $\inf_{[1,\infty)} f = c > 0$ and then

$$\left\| (I - \pi_{1/\triangle t}) z^k \right\|_X^2 \leq \sum_{\mu_j \triangle t \geq 1} |a_j|^2 e^{-2kc} \leq e^{-2kc} \left\| z_0 \right\|_X^2 ,$$

which easily implies (3.12).

4 Controllability

In this section, we apply Theorem 1.2 to derive controllability properties for time semi-discrete schemes. We first recall briefly how to obtain controllability results from (1.3)-(1.4) in the continuous setting. Then we modify the methodology in the continuous case to deal with the time semi-discrete one.

4.1 The continuous setting

Let us consider the following controllability problem: Given $u_0 \in X$, to find a control function $v \in L^2((0,T);Y)$ such that the solution of

$$\dot{u} + Au = B^* v, \quad t \in (0,\, T), \qquad u(0) = u_0 \in X, \tag{4.1}$$

satisfies

$$u(T) = 0. \tag{4.2}$$

This problem might be solved for several $v \in L^2((0,T);Y)$, and it is then natural to try to find, among all possible controls, the one of minimal $L^2((0,T);Y)$-norm. This control is the so-called HUM-control v_{HUM} (see [18]), and can be computed as follows. Consider the functional, defined for $\psi_T \in X$ by

$$J_T(\psi_T) = \frac{1}{2} \int_0^T \left\| B\psi(t) \right\|_Y^2 \, dt + \langle \psi(0), u_0 \rangle_X , \tag{4.3}$$

where $\psi(t)$ is the solution of the adjoint (backward) equation

$$\dot{\psi} - A\psi = 0, \quad t \in (0,T), \qquad \psi(T) = \psi_T. \tag{4.4}$$

When estimates (1.3)-(1.4) hold, the functional J_T is well-defined, continuous, strictly convex and coercive with respect to the norm

$$\|\psi_T\|_{obs}^2 = \int_0^T \|B\psi(t)\|_Y^2 \; dt.$$

We thus define $\bar{X}$ as the completion of X for $\|\cdot\|_{obs}$. On $\bar{X}$, using the admissibility inequality (1.3) and the density of X in $\bar{X}$, we can define a map $\Theta : \bar{X} \to L^2((0,T);Y)$ which coincides on X with the map $\psi_T \mapsto B\psi(t)$. Combined with the observability inequality (1.4) which guarantees that the map which associates $\psi_T \in \bar{X}$ to $\psi(0)$ is continuous, we can thus consider the functional J_T in (4.3) on $\bar{X}$.

Now, the functional J_T is coercive on $\bar{X}$ and thus the existence of a minimizer $\varphi_T \in \bar{X}$ for J_T is guaranteed. The HUM-control is then explicitly given by

$$v_{HUM}(t) = \Theta\varphi_T. \tag{4.5}$$

Note that, when $\varphi_T \in X$, the HUM control is then simply given by $v_{HUM}(t) = B\varphi(t)$, where φ is the corresponding solution of (4.4).

4.2 The time semi-discrete setting: Results

In this subsection, we propose a numerical method which computes a discrete approximation of an exact control for the continuous system (4.1). For this purpose, it is natural to consider the controllability properties for the semi-discrete problem:

$$u^{k+1} = \mathbb{T}_{\triangle t} u^k + (\triangle t)\pi_{\delta/\triangle t} B^* v_{\triangle t}^{k+1}, \quad 0 \le k\triangle t \le T, \qquad u^0 = u_{0,\triangle t}, \tag{4.6}$$

where $\delta \in (0,R)$, the operator $\pi_{\delta/\triangle t}$ is the orthogonal projection in X on $\mathcal{C}(\delta/\triangle t)$ and $u_{0,\triangle t} \in \mathcal{C}(\delta/\triangle t)$ is an approximation of $u_0 \in X$. In the following, $\delta \in (0,R)$ is fixed.

Assume that the system (1.1)-(1.2) is exactly observable in some time T^*. In the following, we fix $T = T^*$.

Following the methodology of the continuous setting, we introduce, in the same spirit as in [14], the functional $J_{T,\triangle t}$, defined for $\psi_T \in \mathcal{C}(\delta/\triangle t)$ by

$$J_{T,\triangle t}(\psi_T) = \frac{\triangle t}{2} \sum_{k=1}^{\lceil T/\triangle t \rceil} \left\| B\psi^k \right\|_Y^2 + \frac{1}{2}(\triangle t)^\beta \|\psi_T\|_X^2 + \left\langle \psi^0, u_{0,\triangle t} \right\rangle_X, \tag{4.7}$$

where β is as in (1.11) and ψ^k denotes the solution of the backward problem

$$\psi^k = \mathbb{T}_{\triangle t}\psi^{k+1}, \quad 0 \le k \le \lfloor T/\triangle t \rfloor, \qquad \psi^{\lceil T/\triangle t \rceil} = \psi_T. \tag{4.8}$$

Then the following proposition is an easy consequence of Theorem 1.2:

Proposition 4.1. *For each $\triangle t > 0$, the functional $J_{T,\triangle t}$ defined in* (4.7) *has a unique minimizer $\varphi_{T,\triangle t} \in \mathcal{C}(\delta/\triangle t)$.*

Moreover, setting $\varphi^k_{\triangle t}$ the corresponding solution of (4.8) and $v^k_{\triangle t} = B\varphi^k_{\triangle t}$, the solution of (4.6) satisfies

$$u^{\lceil T/\triangle t\rceil} = \pi_{\delta/\triangle t} u^{\lceil T/\triangle t\rceil} = -(\triangle t)^{\beta} \varphi_{T,\triangle t}. \tag{4.9}$$

Besides, there exists a positive constant C such that for all $\triangle t > 0$,

$$\frac{\triangle t}{2} \sum_{k=1}^{\lceil T/\triangle t\rceil} \left\| B\varphi^k_{\triangle t} \right\|_Y^2 + \frac{1}{2}(\triangle t)^{\beta} \left\| \varphi_{T,\triangle t} \right\|_X^2 \le C \left\| u_{0,\triangle t} \right\|_X^2. \tag{4.10}$$

This proposition gives an approximate controllability result for the discrete schemes. But the size of the error done on the final state (the target state here is 0) is of order $(\triangle t)^{\beta/2}$, and goes to zero when the time discretization parameter $\triangle t > 0$ goes to zero.

The proof of Proposition 4.1 is based on the HUM duality process, and will be described in the next subsection.

It is natural to think that, if the continuous system (1.1)-(1.2) is exactly observable and if $u_{0,\triangle t}$ converges to u_0 in X, then the sequence of discrete controls $v_{\triangle t}$ given by Proposition 4.1 converges to a control for (4.1). We will prove that this is indeed the case, up to extractions.

To state our results properly, we need to introduce the classical extension operators $E_{\triangle t}$ which extend discrete functions $f_{\triangle t} = (f^k)_{0 \le k\triangle t \le T}$ as piecewise affine continuous functions on $[0, T]$:

$$E_{\triangle t}(f_{\triangle t})(t) = \left(\frac{f^{k+1} - f^k}{\triangle t} \right)(t - k\triangle t) + f^k, \quad \forall t \in [k\triangle t, (k+1)\triangle t].$$

Theorem 4.2. *Assume that $B \in \mathcal{L}(\mathcal{D}(A^{\nu}), Y)$ with $\nu < 1/2$ and that system (1.1)-(1.2) is exactly observable in some time T. Consider $u_0 \in X$ and $(u_{0,\triangle t})$ a sequence of elements of X such that $u_{0,\triangle t} \in \mathcal{C}(\delta/\triangle t)$ for all $\triangle t > 0$, and $(u_{0,\triangle t}) \to u_0$ in X as $\triangle t \to 0$. For $\triangle t > 0$, let $v_{\triangle t}$ be the discrete control computed in Proposition 4.1.*

Then the sequence $(E_{\triangle t} v_{\triangle t})$ is bounded in $L^2((0,T);Y)$ and any weak accumulation point v in $L^2((0,T);Y)$ of the sequence $(E_{\triangle t} v_{\triangle t})$ is a control for (4.1). Besides, the corresponding solutions $(u_{\triangle t})$ of (4.6) converge in the following sense:

$$\begin{cases} E_{\triangle t} u_{\triangle t} \underset{\triangle t \to 0}{\longrightarrow} u, & \text{in } C([0,T]; X_{-1/2}), \\ E_{\triangle t} u_{\triangle t} \underset{\triangle t \to 0}{\rightharpoonup} u, & \text{in } L^{\infty}((0,T);X) - w*, \end{cases}$$

where u satisfies (4.1) *and* (4.2).

4.3 Proof of Proposition 4.1

Proof of Proposition 4.1. For any $\triangle t > 0$, the functional $J_{T,\triangle t}$ defined in (4.7) is obviously strictly convex from (1.12). Moreover, from Theorem 1.2, $J_{T,\triangle t}$ is coercive, and therefore has a unique minimizer $\varphi_{T,\triangle t}$ in the closed (finite dimensional) vector space $\mathcal{C}(\delta/\triangle t)$.

To get the uniform bound (4.10), we use that

$$J_{T,\triangle t}(\varphi_{T,\triangle t}) \leq J_{T,\triangle t}(0) = 0,$$

which obviously implies (4.10) due to Theorem 1.2 and the uniform observability inequality (1.12).

Since $\varphi_{T,\triangle t}$ is the minimizer of $J_{T,\triangle t}$ in $\mathcal{C}(\delta/\triangle t)$, the Fréchet derivative of $J_{T,\triangle t}$ vanishes at $\varphi_{T,\triangle t}$. This implies that any solution of (4.8) with initial data $\psi_T \in \mathcal{C}(\delta/\triangle t)$ satisfies:

$$0 = \triangle t \sum_{k=1}^{\lceil T/\triangle t\rceil} \left\langle B\varphi_{\triangle t}^k, B\psi^k \right\rangle_Y + (\triangle t)^\beta \left\langle \psi_T, \varphi_{T,\triangle t}\right\rangle_X + \left\langle \psi^0, u_{0,\triangle t}\right\rangle_X. \quad (4.11)$$

Now, let us consider a solution u of (4.6). Then for all ψ, we have

$$\begin{aligned}
\left\langle \psi^{\lceil T/\triangle t\rceil}, u^{\lceil T/\triangle t\rceil}\right\rangle - \left\langle \psi^0, u^0\right\rangle &= \sum_{k=0}^{\lfloor T/\triangle t\rfloor} \left\langle \psi^{k+1}, u^{k+1}\right\rangle - \left\langle \psi^k, u^k\right\rangle \\
&= \sum_{k=0}^{\lfloor T/\triangle t\rfloor} \left\langle \psi^{k+1}, \mathbb{T}_{\triangle t} u^k + (\triangle t)\pi_{\delta/\triangle t} B^* v_{\triangle t}^{k+1}\right\rangle - \left\langle \psi^k, u^k\right\rangle \\
&= \sum_{k=0}^{\lfloor T/\triangle t\rfloor} \left\langle \mathbb{T}_{\triangle t}\psi^{k+1} - \psi^k, u^k\right\rangle + \triangle t \sum_{k=1}^{\lceil T/\triangle t\rceil} \left\langle v_{\triangle t}^k, B\pi_{\delta/\triangle t}\psi^k\right\rangle_Y.
\end{aligned}$$

In particular, when ψ is a solution of (4.8) with $\psi_T \in \mathcal{C}(\delta/\triangle t)$, one has

$$\left\langle \psi_T, u^{\lceil T/\triangle t\rceil}\right\rangle = \left\langle \psi^0, u^0\right\rangle + \triangle t \sum_{k=1}^{\lceil T/\triangle t\rceil} \left\langle v_{\triangle t}^k, B\psi^k\right\rangle_Y. \quad (4.12)$$

Choosing $v_{\triangle t}^k = B\varphi_{\triangle t}^k$, identities (4.11) and (4.12) give (4.9). $\square$

Also note that, due to the choice of $v_{\triangle t}$, (4.10) implies that

$$\triangle t \sum_{k=1}^{\lceil T/\triangle t\rceil} \left\| v_{\triangle t}^k \right\|_Y^2 \leq C \left\| u_{0,\triangle t}\right\|_X^2. \quad (4.13)$$

This estimate will be crucial next.

4.4 Convergence results

Proof of Theorem 4.2. Inequality (4.13) implies that the sequence $(E_{\triangle t} v_{\triangle t})$ is uniformly bounded in $L^2((0,T);Y)$, and therefore there exists $v \in L^2((0,T);Y)$ such that, up to a subsequence, $E_{\triangle t} v_{\triangle t} \rightharpoonup v$ in $L^2((0,T);Y)$ as $\triangle t \to 0$. It follows that $E_{\triangle t} B^* v_{\triangle t} \rightharpoonup B^* v$ in $L^2((0,T);X_{-\nu})$ as $\triangle t \to 0$. Using the density of finite linear combination of eigenfunctions in X, we easily see that

$$E_{\triangle t} \pi_{\delta/\triangle t} B^* v_{\triangle t} \underset{\triangle t \to 0}{\rightharpoonup} B^* v, \quad \text{in } L^2((0,T);X_{-\nu}). \tag{4.14}$$

Besides, inequality (4.10) implies that $(\triangle t)^{\beta/2} \varphi_{T,\triangle t}$ is bounded in X, and thus, by (4.9) and (4.10),

$$\left\| \pi_{\delta/\triangle t} u^{\lceil T/\triangle t \rceil} \right\|_X \leq C (\triangle t)^{\beta/2} \left\| u_{0,\triangle t} \right\|_X \to 0. \tag{4.15}$$

Therefore Theorem 4.2 mainly deals with convergence properties of the discrete solutions of (4.6) toward the solution of the continuous system (4.1).

We thus need the following lemma:

Lemma 4.3 (Convergence)**.** *Assume that $u_{0,\triangle t} \in \mathcal{C}(\delta/\triangle t)$ strongly converges to $u_0 \in X$ as $\triangle t \to 0$, and that $E_{\triangle t} g_{\triangle t} \in L^2((0,T);X_{-1/2})$ weakly converges to g in $L^2((0,T);X_{-1/2})$ as $\triangle t \to 0$.*

Then, if for all $\triangle t > 0$, $u_{\triangle t}$ denotes the solution of

$$u^{k+1} = \mathbb{T}_{\triangle t} u^k + (\triangle t) \pi_{\delta/\triangle t} g_{\triangle t}^{k+1}, \quad 0 \leq k \triangle t \leq T, \qquad u^0 = u_{0,\triangle t}, \tag{4.16}$$

the following convergence results hold:

$$\begin{cases} E_{\triangle t} u_{\triangle t} \underset{\triangle t \to 0}{\longrightarrow} u & \text{in } C([0,T];X_{-1/2}), \\ E_{\triangle t} u_{\triangle t} \underset{\triangle t \to 0}{\rightharpoonup} u & \text{in } L^\infty((0,T);X) - w*, \end{cases} \tag{4.17}$$

where u satisfies

$$\dot{u} + Au = g, \ \text{in } (0,T), \qquad u(0) = u_0. \tag{4.18}$$

Indeed, assuming Lemma 4.3 which will be proved hereafter, Theorem 4.2 easily follows since the convergences in (4.17) and (4.15) imply $u(T) = 0$. $\square$

Proof. The proof of Lemma 4.3 is classical. The main idea is to derive some *a priori* estimates on the discrete solution, and then pass to the limit.

We first show that $(E_{\triangle t} u_{\triangle t})$ is uniformly bounded in $L^\infty((0,T);X)$. Taking the inner product in X of (4.16), we obtain

$$\left\| u_{\triangle t}^{k+1} \right\|_X^2 = \left\| \mathbb{T}_{\triangle t} u_{\triangle t}^k \right\|_X^2 + 2\triangle t \left\langle \mathbb{T}_{\triangle t} u_{\triangle t}^k, \, \pi_{\delta/\triangle t} g_{\triangle t}^{k+1} \right\rangle_X + (\triangle t)^2 \left\| \pi_{\delta/\triangle t} g_{\triangle t}^{k+1} \right\|_X^2. \tag{4.19}$$

Note then that

$$\begin{aligned}\left\|\mathbb{T}_{\triangle t}u^k_{\triangle t}\right\|^2_X &= \left\|u^k_{\triangle t} - \triangle t\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)u^k_{\triangle t}\right\|^2_X \\ &= \left\|u^k_{\triangle t}\right\|^2_X - 2\triangle t\left\langle u^k_{\triangle t},\, \left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)u^k_{\triangle t}\right\rangle_X \\ &\qquad +(\triangle t)^2\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)u^k_{\triangle t}\right\|^2_X .\end{aligned} \tag{4.20}$$

But, due to the explicit expression of $(I-\mathbb{T}_{\triangle t})/\triangle t$ on the basis Φ_j, one can check that $(I-\mathbb{T}_{\triangle t})/\triangle t$ defines a self-adjoint positive definite operator which commutes with A.

Below, we show

$$\left\|\mathbb{T}_{\triangle t}u^k_{\triangle t}\right\|^2_X \leq \left\|u^k_{\triangle t}\right\|^2_X - \triangle t\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2}u^k_{\triangle t}\right\|^2_X . \tag{4.21}$$

To verify this, since $u^k_{\triangle t}$ belongs to $\mathcal{C}(\delta/\triangle t)$, and due to the orthogonality properties of the eigenvectors, it is sufficient to prove that estimate (4.21) holds for any eigenvector Φ_j with $\mu_j\triangle t \leq \delta$,

$$\left\|\mathbb{T}_{\triangle t}\Phi_j\right\|^2_X \leq \left\|\Phi_j\right\|^2_X - \triangle t\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2}\Phi_j\right\|^2_X .$$

Moreover,

$$\begin{aligned}-2\triangle t\left\langle \Phi_j,\, \left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)\Phi_j\right\rangle_X + (\triangle t)^2\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)\Phi_j\right\|^2_X \\ = -2(1-e^{-\lambda_{j,\triangle t}(\triangle t)}) + (1-e^{-\lambda_{j,\triangle t}(\triangle t)})^2.\end{aligned}$$

Since $\lambda_{j,\triangle t}(\triangle t) \geq 0$,

$$\begin{aligned}-2\triangle t\left\langle \Phi_j,\, \left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)\Phi_j\right\rangle_X + (\triangle t)^2\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)\Phi_j\right\|^2_X \\ \leq -\triangle t\left\|\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2}\Phi_j\right\|^2_X ,\end{aligned}$$

which leads to (4.21), in view of (4.20).

For the second term in (4.19), we use Cauchy's inequality: for any $\alpha > 0$,

$$\left|2\triangle t\left\langle \mathbb{T}_{\triangle t}u^k_{\triangle t},\, \pi_{\delta/\triangle t}g^{k+1}_{\triangle t}\right\rangle_X\right| \leq \alpha^2\triangle t\left\|A^{1/2}\mathbb{T}_{\triangle t}u^k_{\triangle t}\right\|^2_X + \frac{1}{\alpha^2}\triangle t\left\|g^{k+1}_{\triangle t}\right\|^2_{-1/2} .$$

Moreover, as $u^k_{\triangle t}$ belongs to $\mathcal{C}(\delta/\triangle t)$, there exists a positive constant C_δ such that for all $k \in \mathbb{N}$,

$$\left\| A^{1/2} \mathbb{T}_{\triangle t} u^k_{\triangle t} \right\|^2_X \leq C_\delta \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} u^k_{\triangle t} \right\|^2_X .$$

Indeed, for Φ_j such that $\mu_j \triangle t \leq \delta$:

$$\left\| A^{1/2} \mathbb{T}_{\triangle t} \Phi_j \right\|^2_X = \mu_j e^{-2(\triangle t)\lambda_{j,\triangle t}}, \qquad \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} \Phi_j \right\|^2_X = \frac{1 - e^{-(\triangle t)\lambda_{j,\triangle t}}}{\triangle t},$$

and the quantity

$$\left(\frac{(\triangle t)\mu_j}{1 - e^{-(\triangle t)\lambda_{j,\triangle t}}} \right) e^{-2(\triangle t)\lambda_{j,\triangle t}}$$

is bounded by a constant which depends only on the filtering parameter $\delta > 0$.

Thus we obtain

$$\left| 2\triangle t \left\langle \mathbb{T}_{\triangle t} u^k_{\triangle t}, \, \pi_{\delta/\triangle t} g^{k+1}_{\triangle t} \right\rangle \right| \leq \alpha^2 C_\delta \triangle t \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} u^k_{\triangle t} \right\|^2_X + \frac{1}{\alpha^2} \triangle t \left\| g^{k+1}_{\triangle t} \right\|^2_{-1/2}, \quad (4.22)$$

for any $\alpha > 0$.

For the last term in (4.19), we notice that for all $\varphi \in \mathcal{C}(\delta/\triangle t)$,

$$(\triangle t)^2 \left\| \varphi \right\|^2_X \leq (\triangle t)\delta \left\| \varphi \right\|^2_{-1/2} . \quad (4.23)$$

Again, this can be deduced directly from the properties of the eigenvectors Φ_j satisfying $\mu_j \triangle t \leq \delta$, since $\|\Phi_j\|^2_{-1/2} = 1/\mu_j$ and $\|\Phi_j\|^2_X = 1$.

Therefore, combining (4.21), (4.22), (4.23) and (4.19), we obtain, for any $\alpha > 0$,

$$\left\| u^{k+1}_{\triangle t} \right\|^2_X - \left\| u^k_{\triangle t} \right\|^2_X + \triangle t \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} u^k_{\triangle t} \right\|^2_X \leq \alpha^2 C_\delta \triangle t \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} u^k_{\triangle t} \right\|^2_X + \left(\frac{1}{\alpha^2} + \delta \right) \triangle t \left\| g^{k+1}_{\triangle t} \right\|^2_{-1/2} .$$

By choosing α such that $\alpha^2 C_\delta = 1/2$, we thus obtain

$$\frac{\left\| u^{k+1}_{\triangle t} \right\|^2_X - \left\| u^k_{\triangle t} \right\|^2_X}{\triangle t} + \frac{1}{2} \left\| \left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t} \right)^{1/2} u^k_{\triangle t} \right\|^2_X \leq \left(\frac{1}{\alpha^2} + \delta \right) \left\| g^{k+1}_{\triangle t} \right\|^2_{-1/2} .$$

It then follows that there exists a constant C independent of $\triangle t > 0$ such that

$$\begin{aligned} &\left\|u_{\triangle t}^{k}\right\|_X^2 \leq \left\|u_{\triangle t}^{0}\right\|_X^2 + C\triangle t \sum_{k=0}^{\lceil T/\triangle t\rceil} \left\|g_{\triangle t}^{k+1}\right\|_{-1/2}^2, \quad 0 \leq k\triangle t \leq T, \\ &\triangle t \sum_{k=0}^{\lceil T/\triangle t\rceil} \left\|\left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2} u_{\triangle t}^{k}\right\|_X^2 \leq \left\|u_{\triangle t}^{0}\right\|_X^2 + C\triangle t \sum_{k=0}^{\lceil T/\triangle t\rceil} \left\|g_{\triangle t}^{k+1}\right\|_{-1/2}^2. \end{aligned} \tag{4.24}$$

But $(u_{\triangle t}^{0})$ is bounded in X and $(E_{\triangle t} g_{\triangle t})$ is bounded in $L^2((0,T);X_{-1/2})$. Thus we deduce from (4.24) that $(E_{\triangle t} u_{\triangle t})$ is bounded in $L^\infty((0,T);X)$ and that $E_{\triangle t}\left(\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2} u_{\triangle t}^{k}\right)$ is bounded in $L^2((0,T);X)$.

We now prove that $(\frac{d}{dt}(E_{\triangle t} u_{\triangle t}))$ is uniformly bounded in $L^2((0,T);X_{-1/2})$. To prove this, recall that $u_{\triangle t}$ is solution of (4.16) and that $(\pi_{\delta/\triangle t} g_{\triangle t}^{k+1})$ is bounded in $L^2((0,T);X_{-1/2})$. We now prove that $\frac{d}{dt}E_{\triangle t} u_{\triangle t}$, which equals $\left(\frac{\mathbb{T}_{\triangle t}-I}{\triangle t}\right) u_{\triangle t}^{k} + \pi_{\delta/\triangle t} g_{\triangle t}^{k+1}$ on $(k\triangle t, (k+1)\triangle t)$, is bounded in $L^2((0,T);X_{-1/2})$. If $z = \sum_{\mu_j \triangle t \leq \delta} a_j \Phi_j \in \mathcal{C}(\delta/\triangle t)$,

$$\begin{aligned} \left\|\left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t}\right) z\right\|_{-1/2}^2 &= \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \left|\frac{1 - e^{-\lambda_{j,\triangle t}(\triangle t)}}{\triangle t}\right|^2 \frac{1}{\mu_j} \\ &= \sum_{\mu_j \triangle t \leq \delta} |a_j|^2 \left(\frac{1 - e^{-\lambda_{j,\triangle t}(\triangle t)}}{\triangle t}\right)\left(\frac{1 - e^{-f(\mu_j(\triangle t))}}{\mu_j \triangle t}\right) \\ &\leq \sup_{\eta \in [0,\delta]} \left\{\left(\frac{1 - e^{-f(\eta)}}{\eta}\right)\right\} \left\|\left(\frac{I - \mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2} z\right\|_X^2. \end{aligned}$$

Thus, since $E_{\triangle t}\left(\left(\frac{I-\mathbb{T}_{\triangle t}}{\triangle t}\right)^{1/2} u_{\triangle t}\right)$ is bounded in $L^2((0,T);X)$ from (4.24), $E_{\triangle t}\left(\left(\frac{\mathbb{T}_{\triangle t}-I}{\triangle t}\right) u_{\triangle t}\right)$ is bounded in $L^2((0,T);X_{-1/2})$.

Since $\frac{d}{dt}(E_{\triangle t} u_{\triangle t})$ and $E_{\triangle t} u_{\triangle t}$ are bounded in the spaces $L^2((0,T);X_{-1/2})$ and $L^\infty((0,T);X)$ respectively, and since the embedding $X \subset X_{-1/2}$ is compact, we obtain (see [23]) that $E_{\triangle t} u_{\triangle t}$ converges to u in $C([0,T], X_{-1/2})$ as $\triangle t \to 0$.

We can then compute $u(0)$. On one hand, $E_{\triangle t} u_{\triangle t}(0) \to u(0)$ in $X_{-1/2}$ as $\triangle t \to 0$. But we also have $E_{\triangle t} u_{\triangle t}(0) = u_{0,\triangle t} \to u_0$ in X as $\triangle t \to 0$, and then $u(0) = u_0$.

Since $\frac{d}{dt}(E_{\triangle t} u_{\triangle t})$ is bounded in $L^2((0,T);X_{-1/2})$, we can also conclude that $\frac{d}{dt}(E_{\triangle t} u_{\triangle t}) \rightharpoonup \dot{u}$ in $L^2((0,T);X_{-1/2})$ as $\triangle t \to 0$.

Besides, since, for $\varphi \in X_2$, $\left(\frac{\mathbb{T}_{\triangle t}-I}{\triangle t}\right) \pi_{\delta/\triangle t} \varphi \to -A\varphi$ strongly in X as $\triangle t \to 0$, we obtain by duality that $\left(\frac{\mathbb{T}_{\triangle t}-I}{\triangle t}\right) E_{\triangle t} u_{\triangle t}(t) \rightharpoonup -Au(t) \quad$ in $L^2((0,T);X_{-2})$ as $\triangle t \to 0$. Then, passing to the limit in (4.16), the limit function u satisfies (4.18). $\square$

Remark 4.4. *When $R = \infty$, that is when the time-discrete scheme under consideration is unconditionnally stable, as in [26], one can consider, instead of (4.6), the semi-discrete problem*

$$u^{k+1} = \mathbb{T}_{\triangle t} u^k + (\triangle t) B^* v_{\triangle t}^{k+1}.$$

The HUM duality process is then the same as before, and Theorem 4.2 can easily be adapted to this case.

5 Fully discrete schemes

5.1 General setting

In this section, we consider time-discrete approximation schemes for families of operators (A, B). In particular, the operators A and B can depend on an extra parameter, which may correspond to a space discretization parameter.

It will then be convenient to denote by $\mathcal{C}(\delta/\triangle t)[A]$ the filtered class $\mathcal{C}(\delta/\triangle t)$ corresponding to the operator A.

To state our results, we introduce the following class of operators:

Definition 5.1. *For any $(\nu, K_B, T^*, k_*) \in (-\infty, 1/2) \times \left(\mathbb{R}_+^*\right)^3$, we define the class $\mathcal{F}(\nu, K_B, T^*, k_*)$ of operators (A, B) satisfying:*

• *The operator A is self-adjoint, positive definite with dense domain and compact resolvent.*

• *The operator B belongs to $\mathcal{L}(\mathcal{D}(A^\nu); Y)$ with $\|B\|_{\mathcal{L}(\mathcal{D}(A^\nu);Y)} \leq K_B$.*

• *The pair of operators (A, B) satisfies the observability inequality (1.4) in time T^* with positive constant $k_* > 0$.*

In this class, Theorem 1.2 applies and provides uniform admissibility and observability results for any of the time semi-discrete approximation schemes described by (1.5). Indeed, all the constants in Theorem 1.2 are explicit and only depend on ν, K_B, T^*, k_*, f and δ. We can then state:

Theorem 5.2. *Set $(\nu, K_B, T^*, k_*) \in (-\infty, 1/2) \times \left(\mathbb{R}_+^*\right)^3$. Let $(A_i, B_i)_{i \in I}$ be a family of operators in $\mathcal{F}(\nu, K_B, T^*, k_*)$. Set β as in (1.11). Then, for any $\delta \in (0, R)$, there exist positive constants K_δ, k_δ and C_δ such that, for any $\triangle t > 0$ and $i \in I$, any solution z^k of $(1.5)_i$ with initial data $z_0 \in \mathcal{C}(\delta/\triangle t)[A_i]$ satisfies $(1.12)_i$, where $(1.5)_i$ corresponds to system (1.5) with $\mathbb{T}_{\triangle t} = \mathbb{T}_{\triangle t, i} = \exp(-f((\triangle t) A_i))$ and $(1.12)_i$ corresponds to (1.12) with $B = B_i$.*

We now explain how Theorem 5.2 can be used when dealing with fully-discrete approximation schemes of (1.1)-(1.2). First, we introduce the space semi-discrete approximation scheme of (1.1)-(1.2). For $h > 0$, the approximation space is a finite dimensional subspace X_h, endowed with the norm $\|\cdot\|_h$, on which the continuous model (1.1)-(1.2) is approximated by

$$\dot{z_h} + A_h z_h = 0, \quad 0 \leq t \leq T, \quad z_h(0) = z_{h,0} \in X_h, \quad y_h(t) = B_h z_h(t), \tag{5.1}$$

where A_h and B_h are approximations of A and B in the discrete setting. In the following, we denote by $\mathcal{C}_h(\delta/\triangle t)$ the filtered class $\mathcal{C}(\delta/\triangle t)[A_h]$.

Thus, if one wants to study fully discrete approximation schemes of (1.1)-(1.2) deduced from (5.1) and their admissibility and observability properties, Theorem 5.2 suggests the following two-steps strategy:

1. Study the time continuous system (5.1) for every mesh-size h and prove the existence of $(\nu, K_B, T^*, k_*) \in (-\infty, 1/2) \times \left(\mathbb{R}_+^*\right)^3$ such that for all $h > 0$, $(A_h, B_h) \in \mathcal{F}(\nu, K_B, T^*, k_*)$ uniformly. In particular, one shall have, for all $h > 0$ and any solution of (5.1),

$$k_* \left\| z_h(T^*) \right\|_h^2 \leq \int_0^{T^*} \left\| B_h z_h \right\|_{Y_h}^2 dt. \tag{5.2}$$

2. Apply then Theorem 5.2 to obtain admissibility and observability results for the following fully discrete schemes

$$z_h^{k+1} = \mathbb{T}_{\triangle t,h} z_h^k, \quad 0 \leq k \leq \lfloor T/\triangle t \rfloor, \qquad z_h^0 = z_{h,0} \in X_h, \tag{5.3}$$

where $\mathbb{T}_{\triangle t,h} = \exp(-f((\triangle t)A_h))$: setting β as in (1.11), for any $\delta \in (0, R)$, there exist positive constants K_δ, k_δ and C_δ such that, for any $\triangle t > 0$ and $h > 0$, any solution z_h^k of (5.3) with initial data $z_{h,0} \in X_h \cap \mathcal{C}_h(\delta/\triangle t)$ satisfies

$$k_\delta \left\| z_h^{\lceil T^*/\triangle t \rceil} \right\|_h^2 \leq \triangle t \sum_{k=0}^{\lceil T^*/\triangle t \rceil} \left\| B_h z_h^k \right\|_{Y_h}^2 + C_\delta (\triangle t)^\beta \left\| z_{h,0} \right\|_h^2 \leq K_\delta \left\| z_{h,0} \right\|_h^2 .$$

Note that, if $\|A_h\|_{\mathcal{L}(X_h)} \leq \delta/\triangle t$, then $X_h \cap \mathcal{C}_h(\delta/\triangle t) = X_h$ and no filtering condition is required. This corresponds to a CFL type condition since $\|A_h\|_{\mathcal{L}(X_h)}$ usually is of the form C/h^α for some positive α.

In the following, we give some precise examples of applications.

5.2 The 1-d heat equation

Consider the following system

$$\begin{cases} \partial_t z(x,t) - \partial_{xx}^2 z(x,t) = 0, & 0 < x < 1,\ t > 0, \\ z(0,t) = z(1,t) = 0, & t > 0, \\ z(x,0) = z_0(x), & 0 < x < 1. \end{cases} \tag{5.4}$$

Equation (5.4) obviously has the form (1.1) by taking $A = -\partial_{xx}^2$ with Dirichlet boundary conditions, of domain $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$ on $X = L^2(0,1)$. For (a,b) a subset of $(0,1)$, we define the output function by

$$y(t) = z_{|(a,b)}(t), \quad \forall t > 0,$$

where $z_{|(a,b)}$ means the restriction of z to the interval (a,b). This obviously defines a continuous observation operator B from $X = L^2(0,1)$ to $Y = L^2(a,b)$.

It is classical that this system is observable in any time $T^* > 0$, see for instance [22].

Now, we consider the space semi-discrete approximation scheme of (5.4) derived by the finite-difference method. More precisely, for $N \in \mathbb{N}^*$ given and $h = 1/(N+1)$, we consider the following scheme:

$$\begin{cases} \dot{z}_j - \dfrac{z_{j+1} - 2z_j + z_{j-1}}{h^2} = 0, & 0 < t < T,\ j = 1, ..., N, \\ z_0 = z_{N+1} = 0, & 0 < t < T, \\ z_j(0) = z_{j,0}, & j = 1, ..., N. \end{cases} \tag{5.5}$$

Here, $z_j(t)$ denotes the approximation of the solution z of (5.4) at the point $x_j = jh$.

System (5.5) is a system of N linear differential equations. Moreover, if we denote the unknown $z_h(t) = (z_j(t))_{1\leq j\leq N}^T$, the system (5.5) can be rewritten in vector form as (5.1) with $A_h \in \mathcal{M}_N(\mathbb{R})$. This matrix A_h can be easily deduced from (5.5), and is self-adjoint and positive definite. The approximation space is then $X_h = \mathbb{R}^N$, with corresponding norm

$$\|z_h\|_h^2 = h\sum_{j=1}^{N} |z_j|^2.$$

As a discretization of the output, we choose

$$B_h z_h = (z_j)_{j\in\{\lfloor a/h\rfloor, ..., \lceil b/h\rceil\}}.$$

The rank of the operator B_h is the space $Y_h = \mathbb{R}^{\lceil b/h\rceil - \lfloor a/h\rfloor}$ with the norm

$$\|z_h\|_{Y_h}^2 = h \sum_{j=\lfloor a/h\rfloor}^{\lceil b/h\rceil} |z_j|^2.$$

Following [19] (which was dealing with a boundary observability rather than a distributed one), one can indeed prove the following discrete observability inequality in any time T^*: there exists $k_* > 0$ independent of $h > 0$ such that, for any $h > 0$, any solution z_h of (5.5) satisfies

$$k_* \|z_h(T^*)\|_h^2 \leq \int_0^{T^*} \|B_h z_h\|_{Y_h}^2 \, dt. \tag{5.6}$$

Consequently, for any $T^* > 0$, there exists $k_* > 0$ such that the pairs (A_h, B_h) belong to $\mathcal{F}(0, 1, T^*, k_*)$, and thus applying Theorem 5.2, we obtain: for any $\delta \in (0, R)$, there exist positive constants k_δ and C_δ such that for all $\triangle t > 0$ and $h > 0$, any $z_{h,0} \in X_h \cap \mathcal{C}_h(\delta/\triangle t)$ satisfies

$$k_\delta \left\| \mathbb{T}_{\triangle t,h}^{\lceil T^*/\triangle t\rceil} z_{h,0} \right\|_h^2 \leq \triangle t \sum_{k=0}^{\lceil T^*/\triangle t\rceil} \left\| B_h \mathbb{T}_{\triangle t,h}^k z_{h,0} \right\|_{Y_h}^2 + C_\delta(\triangle t) \|z_{h,0}\|_h^2,$$

where $\mathbb{T}_{\triangle t,h} = \exp(-f((\triangle t)A_h))$.

5.3 The finite element method

In the literature, there are very few results concerning exact observability properties for general space semi-discrete dissipative systems. However, as in our case, there are results of "weak" observability properties which have been proved to hold in many situations [14]. We now explain how these weak observability results can also be combined with our results to derive weak observability properties for fully discrete schemes.

Let us introduce the finite element method for (1.1) (see [21] for more details). Let $(X_h)_{h>0}$ be a family of finite dimensional spaces, which are embedded into X by a map $\rho_h : X_h \to X$ such that $\rho_h(X_h) \subset \mathcal{D}(A^{1/2})$. For $h > 0$, the space X_h is endowed with the inner product $\langle \cdot, \cdot \rangle_h = \langle \rho_h \cdot, \rho_h \cdot \rangle_X$, induced by ρ_h. For $h > 0$, we define $A_h : X_h \to X_h$ by:

$$\langle A_h \varphi_h, \psi_h \rangle_h = \left\langle A^{1/2} \rho_h \varphi_h, A^{1/2} \rho_h \psi_h \right\rangle_X, \quad \forall (\varphi_h, \psi_h) \in X_h^2. \tag{5.7}$$

This operator A_h corresponds to the space discrete approximation of A given by the finite element method and is obviously self-adjoint and positive definite.

Assume now that $B : X \to Y$ is continuous, and consider the observation operators B_h defined by $B_h = B\rho_h$. Remark then that $\|B_h\|_{\mathfrak{L}(X_h,Y)} \leq \|B\|_{\mathfrak{L}(X,Y)}$ uniformly in h.

We now make precise the assumptions we have on ρ_h. The embedding ρ_h describes the finite element approximation we have chosen. In particular we shall assume that the family of spaces $(X_h)_h$ approximates $\mathcal{D}(A^{1/2})$ in the following sense: there exist $C > 0$ and $\theta > 0$ such that

$$\left\{ \begin{array}{ll} \|(Id_X - \rho_h \rho_h^*)\varphi\|_X \leq C h^{2\theta} \|A\varphi\|_X, & \forall \varphi \in \mathcal{D}(A), \\ \left\|A^{1/2} \left(Id_X - \rho_h \rho_h^*\right) \varphi\right\|_X \leq C h^{\theta} \|A\varphi\|_X, & \forall \varphi \in \mathcal{D}(A). \end{array} \right. \tag{5.8}$$

Note that estimates (5.8) are, in particular, satisfied for $\theta = 1$, when using regular mesh (in the sense of finite elements) for the Laplace operator with Dirichlet boundary conditions (see [21] for instance).

For the space semi-discrete approximation schemes

$$\dot{z}_h + A_h z_h = 0, \quad 0 \leq t \leq T, \quad z_h(0) = z_{h,0} \in X_h, \quad y_h(t) = B_h z_h(t), \tag{5.9}$$

the results in [14] yield:

Theorem 5.3. *Assume that* (1.1)-(1.2) *is exactly observable in some time* T^* *and* $B \in \mathfrak{L}(X,Y)$*. Under the condition (5.8), for all* $h > 0$ *small enough, there exist* $k_* > 0$*,* $C > 0$ *and* $\gamma > 0$ *independent of* h *such that any solution* z_h *of (5.9) with initial data* $z_{h,0}$ *satisfies*

$$k_* \|z_h(T^*)\|_h^2 \leq \int_0^{T^*} \|B_h z_h(t)\|_Y^2 \, dt + C h^{\gamma} \|z_{h,0}\|_h^2. \tag{5.10}$$

Note that [14] gives more general results under more general assumptions on A and B. We refer to [14] for more details.

Remark that the difference between (5.10) and (5.2) is the term $h^\gamma \left\|\psi_{h,T}\right\|_h^2$. As in Theorem 4.2, this is sufficient for controllability purposes [14].

As the constants in (5.10) are independent of h, we can follow the proof of Theorem 1.2 (see Lemmas 3.1-3.2 and Theorem 2.2) to obtain weak observability estimates for fully discrete approximations of (1.1)-(1.2).

Theorem 5.4. *For any $\delta \in (0, R)$, there exist positive constants K_δ, k_δ and C_δ such that for all $\triangle t > 0$ and $h > 0$, for any $z_{h,0} \in X_h \cap \mathcal{C}_h(\delta/\triangle t)$,*

$$k_\delta \left\| \mathbb{T}_{\triangle t,h}^{\lceil T^*/\triangle t\rceil} z_{h,0} \right\|_h^2 \leq \triangle t \sum_{k=0}^{\lceil T^*/\triangle t\rceil} \left\| B_h \mathbb{T}_{\triangle t,h}^k z_{h,0} \right\|_Y^2 + C_\delta \left[(\triangle t) + h^\gamma\right] \left\| z_{h,0} \right\|_h^2 . \tag{5.11}$$

where $\mathbb{T}_{\triangle t,h} = \exp(-f((\triangle t)A_h))$.

The estimate (5.11) is a weak observability inequality due to the presence of the term $C_\delta[(\triangle t) + h^\gamma] \left\| z_{h,0} \right\|_h^2$.

As a direct application, one can for instance tackle the following problem.

Let $d \geq 1$ be an integer, Ω a smooth bounded convex domain of $\mathbb{R}^d$, $\Gamma = \partial\Omega$, $c \in L^\infty(\Omega)$ a nonnegative function. Consider the following system

$$\begin{cases} \partial_t z - \Delta z + c(x)z = 0, & \text{in } (0,T)\times\Omega, \\ z(x,t) = 0, & \text{on } [0,T]\times\Gamma, \\ z(0,x) = z_0(x) & \text{in } \Omega, \end{cases} \tag{5.12}$$

where $z_0 \in L^2(\Omega)$.

Equation (5.12) obviously has the form (1.1) where the self-adjoint operator A is defined by $Az = -\Delta z + c(x)z$ on $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$ and $X = L^2(\Omega)$. For ω a subset of Ω, we define the output function by $y(t) = z_{|\omega}(t), \quad \forall t > 0$, where $z_{|\omega}$ denotes the restriction of z to ω. This defines a continuous observation operator B from $X = L^2(\Omega)$ to $Y = L^2(\omega)$.

It is well-known that system (5.12) observed by $y(t) = z_{|\omega}(t)$ is observable in any time $T^* > 0$, see [16, 10].

We then consider triangulations $\mathcal{T}_h$ of the domain Ω which we assume to be regular in the sense of [21]. Roughly speaking, this assumption imposes that the triangles of $(\mathcal{T}_h)$ are not too flat. In this case, estimates (5.8) hold with $\theta = 1$ (see [21]). Estimate (5.11) is then verified for the solutions of the corresponding fully discrete schemes, uniformly with respect to both discretization parameters $\triangle t$ and h.

6 Comments

1. In this article, we assumed A to be self-adjoint, positive definite with dense domain and compact resolvent. One can actually weaken the hypothesis of positivity of A and replace it by the following one: there exists $\alpha \in \mathbb{R}_+$ such

that $A + \alpha I$ is positive definite. Indeed, the admissibility and observability properties for systems

$$\dot{z} + Az = 0, \quad z(0) = z_0, \quad y(t) = Bz(t), \tag{6.1}$$

and

$$\dot{\tilde{z}} + (A + \alpha I)\tilde{z} = 0, \quad \tilde{z}(0) = z_0, \quad \tilde{y}(t) = B\tilde{z}(t), \tag{6.2}$$

are linked by the change of variable $\tilde{z}(t) = e^{-\alpha t} z(t)$. Since system (6.2) fits the abstract setting of this article, one can derive immediately admissibility and observability properties for system (6.1).

2. Note that in [26], the study of the controllability of the heat equation discretized in time is done for several time-discretization schemes, and yields better results than ours, obtaining the discrete observability inequality (1.9) for the Euler implicit method with a bounded operator when taking initial data in $\mathcal{C}(1/(\triangle t)^{2-\epsilon})$ $(\epsilon > 0)$. Though, the study in [26] is based on a good knowledge of the spectrum of the Laplace operator, and in particular the spectral inequality obtained in [17], which are not proved so far in the space discrete setting. However, recently, in the 1-d case, this issue has been successfully addressed in [2] by means of discrete Carleman estimates.

Besides, as shown by the example in Subsection 3.3, the extra term in (1.12) is needed when no further assumptions is available. In this sense, our approach is more robust: it can be applied directly to any observable parabolic systems (even Stokes equations), and does not require the explicit knowledge of the eigenvalues and eigenvectors. This is indeed an interesting feature since it allows to derive instantaneously uniform observability properties for fully discrete dissipative systems from the ones of the space semi-discrete (and time continuous) schemes.

3. In this article, we need the assumption $B \in \mathcal{L}(\mathcal{D}(A^{\nu}), Y)$ with $\nu < 1/2$. There are several cases of interests in which this condition is not satisfied, for instance when considering the classical problem of the observability of the heat equation by the normal derivative on the boundary. It would then be interesting to address the case $B \in \mathcal{L}(\mathcal{D}(A), Y)$ with more details.

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