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A SYSTEMATIC METHOD FOR BUILDING SMOOTH CONTROLS FOR SMOOTH DATA

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Abstract. We prove a regularity result for an abstract control problem $z' = Az + Bv$ with initial datum $z(0) = z_0$ in which the goal is to determine a control v such that $z(T) = 0$. Under standard admissibility and observability assumptions on the adjoint system, when A generates a C^0 group, we develop a method to compute algorithmically a control function v that inherits the regularity of the initial datum to be controlled. In particular, the controlled equation is satisfied in a strong sense when the initial datum is smooth. In this way, the controlled trajectory is smooth as well. Our method applies mainly to time-reversible infinite-dimensional systems and, in particular, to the wave equation, but fails to be valid in the parabolic frame.

1. **Introduction.** Since the pioneering work of D. L. Russell and coworkers summarized in his celebrated 1978 paper in SIAM Review [19], the control and stabilization of wave processes have undergone a significant progress. It would be impossible to summarize here the variety and the depth of the results developed after his influential work. More recently, the subject has been developed even further using the so-called Hilbert Uniqueness Method (HUM) developed by J. L. Lions and presented, in particular, in his SIAM Review article of 1988 [16]. Using HUM one can transform observability inequalities on the adjoint system (that can be derived using multipliers, non-harmonic Fourier series techniques, microlocal analysis, Carleman inequalities, etc) into controllability ones by means of a flexible variational technique. This allows for instance proving the existence of controls of minimal norm in different functional frameworks. However, it is natural to ask whether the controls

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that have been built to be optimal (of minimal norm) in a given functional setting, L^2 for instance, happen to be smooth when the data to be controlled are smooth. In other words, the control map has been built to assign a control to each initial datum in a given functional setting, but is it able to preserve other properties of the data to be controlled, and in particular smoothness ? This question does not seem to have had a systematic answer. The main goal of this paper is to fill this gap. We show that, by applying HUM with suitable time depending weights, controls do indeed inherit the regularity of the data to be controlled.

This question is relevant because of its many applications. In particular, often, to deal with nonlinear control problems, the controls need to be smooth (see for instance the recent work [4]). The same happens when trying to derive convergence rates for numerical controls. As it often occurs in numerical analysis of PDE, the derivation of convergence rates requires more regular solutions and, in our context, this means that the controls need to be smooth when the data are smooth. This turns out to require control maps that are stable in two different functional frameworks simultaneously.

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$ be a strongly continuous group on X, with generator $A : \mathcal{D}(A) \subset X \to X$.

For convenience, we further assume that A is invertible with continuous inverse in X (see Remark 3.3). Define then the Hilbert space $X_1 = \mathcal{D}(A)$ of elements of X such that $\|Ax\|_X < \infty$, endowed with the norm $\|\cdot\|_1 = \|A\cdot\|_X$. Also define X_{-1} as the completion of X with respect to the norm $\|\cdot\|_{-1} = \left\|A^{-1}\cdot\right\|_X$.

Let us then consider the control system

$$z' = Az + Bv, \quad t \geq 0, \qquad z(0) = z_0 \in X, \tag{1.1}$$

where $B \in \mathfrak{L}(U, X_{-1})$, U is an Hilbert space which describes the possible actions of the control, and $v \in L^2_{loc}([0,\infty); U)$ is a control function.

We assume that the operator B is admissible in the sense of [21, Def. 4.2.1]:

Definition 1.1. The operator $B \in \mathfrak{L}(U, X_{-1})$ is said to be an admissible control operator for $\mathbb{T}$ if for some $\tau > 0$, the operator Φ_τ defined by

$$\Phi_\tau v = \int_0^\tau \mathbb{T}_{\tau - s} B v(s)\, ds$$

satisfies $\operatorname{Ran}\Phi_\tau \subset X$, where $\operatorname{Ran}\Phi_\tau$ denotes the range of the map Φ_τ.

When B is an admissible control operator for $\mathbb{T}$, system (1.1) is called admissible.

Note that, obviously, if B is a bounded operator, that is if $B \in \mathfrak{L}(U, X)$, then B is admissible for $\mathbb{T}$.

But there are non-trivial examples as, for instance, the boundary control of the wave equation with Dirichlet boundary conditions, in which B is unbounded but admissible, see [15].

In the following, we will always assume that B is an admissible control operator for $\mathbb{T}$.

In this case, see [21, Prop. 4.2.5], for every $z_0 \in X$ and $v \in L^2_{loc}([0,\infty); U)$, equation (1.1) has a unique mild solution z which belongs to $C([0,\infty); X)$.

Our purpose is to study the exact controllability of system (1.1):

Definition 1.2. System (1.1) is said to be exactly controllable in time T^* if for any $z_0 \in X$, there exists a control function $v \in L^2(0, T^*; U)$ such that the solution of (1.1) satisfies

$$z(T^*) = 0. \tag{1.2}$$

System (1.1) is said to be exactly controllable if it is exactly controllable in some time $T^* > 0$.

To be more precise, we assume that B is an admissible operator for $\mathbb{T}$ and that system (1.1) is exactly controllable in some time T^*. Then, using the Hilbert Uniqueness Method (HUM in short), introduced by Lions [15], (see also [5]), one can compute the control of minimal $L^2(0,T^*;U)$-norm by using the adjoint system and minimizing a functional which turns out to be strictly convex and coercive under these assumptions.

Our purpose is to explain how HUM behaves with respect to the regularity property of the initial data to be controlled. To be more precise, assuming that $z_0 \in X_1$, can we prove that the control obtained by HUM is more regular ?

The answer to that question is delicate. Curiously, the original HUM is intricate, but it can be slightly modified so that it behaves nicely with respect to the regularity property of the initial data.

We thus propose an alternate method, based on HUM, which yields a control V of minimal norm in some weighted L^2 space, and for which we prove that if $z_0 \in X_1$, then the control function V belongs to $H^1_0(0,T;U)$, with no further assumption, thus including for instance the case of boundary controls. In particular, this implies that the controlled solution z of (1.1) belongs to $C^1([0,T],X)$ and also, in various situations (see Section 5), to a strict subspace of X for all time $t \in [0,T]$.

Fix $T > T^*$, and choose $\delta > 0$ such that $T - 2\delta \geq T^*$. Let $\eta = \eta(t) \in L^\infty(\mathbb{R})$ be such that

$$\eta : \mathbb{R} \to [0,1], \qquad \eta(t) = \begin{cases} 0 & \text{if } t \notin (0,T), \\ 1 & \text{if } t \in [\delta, T-\delta]. \end{cases} \tag{1.3}$$

Then define the functional J by

$$J(y_T) = \frac{1}{2}\int_0^T \eta(t) \left\| B^* y(t) \right\|_U^2 \, dt + \langle z_0, y(0) \rangle_X , \tag{1.4}$$

where y denotes the solution of the adjoint system

$$-y' = A^* y, \quad t \leq T, \qquad y(T) = y_T \in X, \tag{1.5}$$

and A^* is the adjoint operator of A.

Note that, according to [21, Th. 4.4.3], the functional J is well-defined for any $y_T \in X$, since the admissibility assumption in the sense of Definition 1.1 implies the existence of a constant K_T such that any solution y of (1.5) satisfies

$$\int_0^T \left\| B^* y(t) \right\|_U^2 \, dt \leq K_T \left\| y_T \right\|_X^2 . \tag{1.6}$$

Moreover, it is by now well-known (see [5, 15]) that system (1.1) is exactly controllable in time T^* if and only if there exists a positive constant $k > 0$ such that any solution of (1.5) satisfies

$$k \left\| y(0) \right\|_X^2 \leq \int_0^{T^*} \left\| B^* y(t) \right\|_U^2 \, dt. \tag{1.7}$$

Inequality (1.7) is called an observability estimate.

In particular, this implies that there exists a positive constant k_* such that any solution of (1.5) satisfies

$$k_* \left\| y(0) \right\|_X^2 \leq \int_0^T \eta(t) \left\| B^* y(t) \right\|_U^2 \, dt. \tag{1.8}$$

Inequality (1.8) then implies the strict convexity of the functional J and its coercivity, but with respect to the norm

$$\|y_T\|_{obs}^2 = \int_0^T \eta(t) \|B^* y(t)\|_U^2 \, dt. \tag{1.9}$$

Let us now remark that, since we assumed that $\mathbb{T}$ is a strongly continuous group, $\mathbb{T}^*$ has the same properties and then assumptions (1.6) and (1.8) imply that the three norms $\|y_T\|_X$, $\|y(0)\|_X$ and $\|y_T\|_{obs}$ are equivalent.

We are now in position to state our first result:

Proposition 1.3. *Let $z_0 \in X$. Assume that system (1.1) is admissible and exactly observable in some time T^*. Let $T > T^*$ and $\eta \in L^\infty(\mathbb{R})$ as in (1.3).*

Then the functional J has a unique minimizer $Y_T \in X$ on X. Besides, the function V given by

$$V(t) = \eta(t) B^* Y(t), \tag{1.10}$$

where $Y(t)$ is the solution of (1.5) with initial datum Y_T, is a control function for system (1.1). This control can also be characterized as the one of minimal $L^2(0, T; dt/\eta; U)$-norm among all possible controls for which the solution of (1.1) satisfies the control requirement (1.2). Besides,

$$\int_0^T \|V(t)\|_U^2 \frac{dt}{\eta(t)} = \|Y_T\|_{obs}^2 \leq \frac{1}{k_*} \|z_0\|_X^2, \tag{1.11}$$

where k_* is the constant in the observability inequality (1.8).

Moreover, this process defines linear maps

$$\mathbb{V} : \left\{ \begin{array}{l} X \longrightarrow X^* = X \\ z_0 \mapsto Y_T \end{array} \right. \quad \text{and} \quad \mathcal{V} : \left\{ \begin{array}{l} X \longrightarrow L^2\left(0, T, \dfrac{dt}{\eta(t)}; U\right) \\ z_0 \mapsto V(t). \end{array} \right. \tag{1.12}$$

This result is similar to those obtained in the context of HUM, see [15], which usually takes the weight function η to be constant $\eta = 1$ on $[0, T]$.

The main novelty and advantage of using the weight function η is that, with no further assumption on the control operator B, the control inherits the regularity of the data to be controlled.

To state our results, it is convenient to introduce, for $s \in \mathbb{R}$, some notations: $\lceil s \rceil$ denotes the smallest integer satisfying $\lceil s \rceil \geq s$, $\lfloor s \rfloor$ is the largest integer satisfying $\lfloor s \rfloor \leq s$ and $\{s\} = s - \lfloor s \rfloor$. Finally, the space C^s denotes the classical Hölder space.

Theorem 1.4. *Assume that the hypotheses of Proposition 1.3 are satisfied.*

Let $s \in \mathbb{R}_+$ be a nonnegative real number and further assume that $\eta \in C^{\lceil s \rceil}(\mathbb{R})$.

If the initial datum z_0 to be controlled belongs to $\mathcal{D}(A^s)$, then the minimizer Y_T given by Proposition 1.3 and the control function V given by (1.10), respectively, belong to $\mathcal{D}((A^*)^s)$ and $H_0^s(0, T; U)$.

Besides, there exists a positive constant $C_s = C_s(\eta, k_*, K_T)$ independent of $z_0 \in \mathcal{D}(A^s)$ such that

$$\|Y_T\|_{\mathcal{D}((A^*)^s)}^2 + \|V\|_{H_0^s(0,T;U)}^2 \leq C_s \|z_0\|_{\mathcal{D}(A^s)}^2. \tag{1.13}$$

In other words, the maps $\mathbb{V}$ and $\mathcal{V}$ satisfy:

$$\mathbb{V} : \mathcal{D}(A^s) \longrightarrow \mathcal{D}(A^s), \qquad \mathcal{V} : \mathcal{D}(A^s) \longrightarrow H_0^s(0, T; U). \tag{1.14}$$

In other words, the constructive method we have proposed, strongly inspired by HUM, naturally reads the regularity of the initial data to be controlled, and provides smoother controls for smoother initial data.

Let us point out that one of the main consequences of Theorem 1.5 is the following regularity result for the controlled trajectory:

Corollary 1.5. *Under the assumptions of Theorem 1.4, if the initial datum z_0 to be controlled belongs to $\mathcal{D}(A^s)$, then the controlled solution z of (1.1) with the control function V given by Proposition 1.3 belongs to*

$$C^s([0,T];X) \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T];\mathcal{Z}_{s-k}), \tag{1.15}$$

where the spaces $(\mathcal{Z}_j)_{j\in\mathbb{N}}$ are defined by induction by

$$\mathcal{Z}_0 = X, \qquad \mathcal{Z}_j = A^{-1}(\mathcal{Z}_{j-1} + BB^*\mathcal{D}((A^*)^j)), \tag{1.16}$$

and the spaces $\mathcal{Z}_s$ for $s \in \mathbb{R}$ are defined by interpolation by

$$\mathcal{Z}_s = [\mathcal{Z}_{\lfloor s \rfloor}, \mathcal{Z}_{\lceil s \rceil}]_{\{s\}}.$$

The spaces $\mathcal{Z}_j$ are not explicit in general. However, there are several cases in which they can be shown to be included in Hilbert spaces of smooth functions, for instance $\mathcal{D}(A^j)$, see Section 5.

To our knowledge, such results are not known when the weight function is simply constant $\eta = \eta_c$:

$$\eta_c(t) = \begin{cases} 1 & \text{for } t \in [0,T], \\ 0 & \text{for } t \notin [0,T]. \end{cases} \tag{1.17}$$

However, following the strategy of the above results, we obtain similar results in this case, but adding regularity assumptions on the control operator B:

Theorem 1.6. *Assume that the hypotheses of Proposition 1.3 are satisfied and let $\eta = \eta_c$ as in (1.17).*

Let $s \in \mathbb{R}$ be a nonnegative real number and assume that

$$\forall k \in \{0, \cdots, \lceil s \rceil - 1\}, \quad BB^* \in \mathfrak{L}(\mathcal{D}((A^*)^k), \mathcal{D}(A^k)). \tag{1.18}$$

If the initial datum z_0 to be controlled belongs to $\mathcal{D}(A^s)$, then the minimizer Y_T given by Proposition 1.3 and the control function V given by (1.10), respectively, belong to $\mathcal{D}((A^*)^s)$ and $H^s(0,T;U)$.

Besides, there exists a positive constant $C_s = C_s(\eta, k_*, K_T)$ independent of $z_0 \in \mathcal{D}(A^s)$ such that

$$\|Y_T\|^2_{\mathcal{D}((A^*)^s)} + \|V\|^2_{H^s(0,T;U)}\, dt \le C_s \|z_0\|^2_{\mathcal{D}(A^s)}. \tag{1.19}$$

Note that (1.18) is a non-trivial assumption which describes the regularity properties of B with respect to A.

In particular, one easily checks that (1.18) guarantees that the spaces $\mathcal{Z}_j$ defined in (1.16) are simply given by

$$\mathcal{Z}_j = \mathcal{D}(A^j), \quad j \le \lceil s \rceil - 1. \tag{1.20}$$

However, this condition cannot be satisfied if the operator B is unbounded, and then it cannot be used to handle the case of boundary controls, whereas Theorem 1.4 and Corollary 1.5 do.

Note that Theorem 1.6 implies the following regularity result:

Corollary 1.7. *Under the assumptions of Theorem 1.6, if the initial datum z_0 to be controlled belongs to $\mathcal{D}(A^s)$, then the controlled solution z of (1.1) with the control function V given by Proposition 1.3 belongs to*

$$z \in C^s([0,T];X) \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T];\mathcal{D}(A^{s-k})). \tag{1.21}$$

As mentionned above, assumption (1.18) in Theorem 1.6 already implies (1.20). Therefore, Corollary 1.7 states that the weight function η in Corollary 1.5 is not needed when (1.18) is satisfied.

Comparing these results, we see that they represent two different phenomena:

- when using a smooth time-dependent weight function, the regularity properties are derived through the time-regularity properties of the function η.
- when using the classical function η_c, the regularity properties come from gentle compatibility properties between the operators A and B.

Let us now comment the literature related to that subject.

It is well-known in a number of different situations (for the wave equation, see [1] and in an abstract setting in [21, Section 11.3]) that under similar controllability and admissibility conditions, one can build smooth controls for smooth initial data, but this is done by suitably modifying the definition of controls, adapted to the functional setting one is looking for.

The important new contribution of Theorems 1.4 and 1.6 is that, the same controls that have been built to be optimal in an L^2 sense, are smooth when the data are smooth. As mentioned above, for doing that, we strongly use the time dependent weight function η, or suitable compatibility conditions between the operators A and B.

Such results have already been obtained for the wave equation, but in a more technical way and in a more restrictive setting in [4]: In [4], it is proved that, when considering the wave equation with a distributed control localized in a subdomain ω satisfying the Geometric Control Condition (see [1], Subsection 5.1 and Remark 5.3) by multiplication by a smooth function supported in ω approximating the characteristic function of ω, then the map $\mathbb{V}$ (computed according to our method with a smooth cut-off function $\eta(t)$) restricted to $\mathcal{D}(A^s)$ is an isomorphism from $\mathcal{D}(A^s)$ to $\mathcal{D}((A^*)^s)$. Furthermore, when considering the wave equation on a compact manifold without boundary, the map $\mathbb{V}$ is a pseudo-differential operator which preserves the regularity. Note that the results in [4] are proved using several technical tools such as microlocal analysis, pseudo-differential operators and Littlewood-Paley decomposition. In particular, [4] proves that the controlled trajectory is smoother than expected when the initial data to be controlled are smooth. Also note that these results have been illustrated numerically in [14]. We refer to Remark 5.3 for more details.

Using Theorem 1.4 and Corollary 1.5, we are able to recover this result not only in that case, but also in the context of boundary control, see Section 5.

Moreover, using Theorem 1.6, we shall see that the time-dependent weight function η is not needed to get that result, and can simply be chosen as $\eta = \eta_c$.

Let us also emphasize that our method, as the classical HUM, is independent of the way estimates (1.6), (1.7) have been obtained. Usually, the most difficult one to derive is the observability inequality (1.7), which has been proved to hold in various situations using several technical tools, for instance multiplier techniques [15],

[11], Carleman estimates [23], [7], non-harmonic Fourier series [9], [18] or pseudo-differential operators [1], [3].

The outline of the article is as follows. In Section 2, we prove Proposition 1.3. In Section 3, we prove Theorem 1.4 and Corollary 1.5. We develop the case $\eta = \eta_c$ in Section 4. In Section 5, we present our results on various instances of wave equations, and compare our method to the original HUM in the case of the 1d wave equation controlled from the boundary. We finally provide some further comments in Section 6.

2. Proof of Proposition 1.3.

To simplify the presentation, we divide the proof in three lemmas.

Lemma 2.1. *Under the assumptions of Proposition 1.3, the functional J is strictly convex and coercive on X. It therefore has a unique minimizer $Y_T \in X$.*

Proof of Lemma 2.1. First we prove the strict convexity of J on X. Let $(y_T, w_T) \in X^2$ and y and w be the corresponding solutions of (1.5) with initial data y_T, w_T respectively.

We claim that, if $w_T \neq y_T$, then

$$J\left(\frac{1}{2}(y_T + w_T)\right) < \frac{1}{2}J(y_T) + \frac{1}{2}J(w_T). \tag{2.1}$$

This is due to the parallelogram law:

$$\int_0^T \eta(t) \left\| B^* \left(\frac{y(t) + w(t)}{2} \right) \right\|_U^2 dt + \int_0^T \eta(t) \left\| B^* \left(\frac{y(t) - w(t)}{2} \right) \right\|_U^2 dt$$
$$= \frac{1}{2} \int_0^T \eta(t) \left\| B^* y(t) \right\|_U^2 dt + \frac{1}{2} \int_0^T \eta(t) \left\| B^* w(t) \right\|_U^2 dt.$$

Indeed, this implies that the identity in (2.1) holds if and only if

$$\int_0^T \eta(t) \left\| B^* \left(\frac{y(t) - w(t)}{2} \right) \right\|_U^2 dt = 0,$$

which is equivalent to $y(0) = w(0)$ thanks to (1.8), also equivalent to $y_T = w_T$ since $\mathbb{T}^*$ is a group.

To prove the coercivity of J on X, we use that (1.8) implies

$$|\langle z_0, y(0) \rangle_X| \leq \|z_0\|_X \left(\frac{1}{k_*} \int_0^T \eta(t) \left\| B^* y(t) \right\|_U^2 dt \right)^{1/2} = \frac{\|z_0\|_X}{\sqrt{k_*}} \|y_T\|_{obs}.$$

Indeed, this implies that

$$J(y_T) \geq \frac{1}{2} \|y_T\|_{obs}^2 - \frac{\|z_0\|_X}{\sqrt{k_*}} \|y_T\|_{obs},$$

and the coercivity of J in X follows from the equivalence of $\|\cdot\|_{obs}$ and $\|\cdot\|_X$. □

Lemma 2.2. *A function $v \in L^2(0,T;U)$ is a control function for system (1.1) if and only if any solution y of (1.5) with initial data $y_T \in X$ satisfies*

$$\int_0^T \langle v(t), B^* y(t) \rangle_U \, dt + \langle z_0, y(0) \rangle_X = 0. \tag{2.2}$$

Remark 2.3. Note that identity (2.2) makes sense since for any $y_T \in X$, according to (1.6), $B^* y \in L^2(0,T;U)$.

Proof of Lemma 2.2. Multiply the equation (1.1) by y solution of (1.5) with initial datum $y_T \in X$:

$$\langle z(T), y_T \rangle_X - \langle z_0, y(0) \rangle_X = \int_0^T \langle v(t), B^* y(t) \rangle_U \, dt. \tag{2.3}$$

Therefore, $z(T) = 0$ if and only if for all $y_T \in X$, identity (2.2) is satisfied. $\square$

Lemma 2.4. *Under the assumptions of Proposition 1.3, the control V given by Proposition 1.3 belongs to $L^2(0,T; dt/\eta; U)$, is the control function of minimal $L^2(0,T; dt/\eta; U)$-norm and satisfies estimate (1.11).*

Proof of Lemma 2.4. Writing that the Gâteaux differential at Y_T vanishes, we get for any $y_T \in X$ that

$$\int_0^T \eta(t) \langle B^* Y(t), B^* y(t) \rangle_U \, dt + \langle z_0, y(0) \rangle_X = 0. \tag{2.4}$$

Therefore the function $V = \eta(t) B^* Y(t)$ satisfies the condition (2.2) and then, according to Lemma 2.2, V is a control function.

Assume now that v is another control function. Then, according to Lemma 2.2, identity (2.2) is satisfied for any solution y of (1.5) with initial data $y_T \in X$. In particular, choosing $y_T = Y_T$, we obtain that

$$\int_0^T \langle v(t), V(t) \rangle_U \, \frac{dt}{\eta(t)} + \langle z_0, Y(0) \rangle_X = 0. \tag{2.5}$$

But this should also be true for $v = V$ since V is a control function. Therefore we obtain that any control function $v \in L^2(0,T;U)$ satisfies

$$\int_0^T \langle v(t), V(t) \rangle_U \, \frac{dt}{\eta(t)} = \int_0^T \|V(t)\|_U^2 \, \frac{dt}{\eta(t)}.$$

By Cauchy-Schwarz inequality, this implies that any control function v in the space $L^2(0,T;U) \cap L^2(0,T; dt/\eta; U)$ satisfies

$$\int_0^T \|V(t)\|_U^2 \, \frac{dt}{\eta(t)} \leq \int_0^T \|v(t)\|_U^2 \, \frac{dt}{\eta(t)}.$$

To obtain the estimate (1.11), we choose $v(t) = V(t) = \eta(t) B^* Y(t)$ in (2.5):

$$\int_0^T \eta(t) \left\| B^* Y(t) \right\|_U^2 \, dt = -\langle z_0, Y(0) \rangle_X \leq \|z_0\|_X \, \|Y_0\|_X \, . \tag{2.6}$$

Estimate (1.11) then follows directly from (1.8) and (2.6). This completes the proof of Lemma 2.4. $\square$

Proof of Proposition 1.3. Proposition 1.3 follows directly from Lemmas 2.1 and 2.4, except for the linearity of the maps $\mathcal{V}$ and $\mathbb{V}$.

Note that the linearity of the map $\mathcal{V}$ follows from the one of $\mathbb{V}$. We thus focus on the linearity of $\mathbb{V}$. Let z_0^1, z_0^2 in X and set $Y_T^1 = \mathbb{V}(z_0^1)$, $Y_T^2 = \mathbb{V}(z_0^2)$. Then, for any $\alpha > 0$, the linearity of (2.4) yields: $\forall y_T \in X$,

$$\int_0^T \eta(t) \langle B^*(\alpha Y^1 + Y^2)(t), B^* y(t) \rangle_U \, dt + \langle (\alpha z_0^1 + z_0^2), y(0) \rangle_X = 0. \tag{2.7}$$

This in particular implies that $\alpha Y_T^1 + Y_T^2$ is a critical point of the functional J in (1.4) corrresponding to $z_0 = \alpha z_0^1 + z_0^2$. Since J is strictly convex by Lemma 2.1,

$\alpha Y_T^1 + Y_T^2$ is the minimizer of J corrresponding to $z_0 = \alpha z_0^1 + z_0^2$. In other words, $\mathbb{V}(\alpha z_0^1 + z_0^2) = \alpha \mathbb{V}(z_0^1) + \mathbb{V}(z_0^2)$. This completes the proof. □

3. **Proof of Theorem 1.4.** For convenience, we begin with the case $s = 1$.

3.1. **The case $s = 1$.** We first present a formal proof which yields Theorem 1.4 in the case $s = 1$. We will then explain how the formal arguments developed below can be made rigorous.

Formal proof. Since $V(t) = \eta(t)B^*Y(t)$ is a control function, from Lemma 2.2, for any $y_T \in X$, identity (2.4) holds. Then, assuming that $y_T = (A^*)^2 Y_T \in X$, we get

$$\int_0^T \eta(t)\langle B^*Y(t), B^*(A^*)^2Y(t)\rangle_U \, dt + \langle z_0, (A^*)^2Y(0)\rangle_X = 0.$$

But

$$\begin{aligned}
&\int_0^T \eta(t)\langle B^*Y(t), B^*(A^*)^2Y(t)\rangle_U \, dt \\
&\quad = \int_0^T \eta(t)\langle B^*Y(t), B^*Y''(t)\rangle_U \, dt \\
&\quad = -\int_0^T \eta(t)\left\|B^*Y'(t)\right\|_U^2 \, dt - \int_0^T \eta'(t)\langle B^*Y(t), B^*Y'(t)\rangle_U \, dt
\end{aligned} \tag{3.1}$$

and

$$\langle z_0, (A^*)^2Y(0)\rangle_X = \langle Az_0, A^*Y(0)\rangle_X.$$

Therefore, assuming some regularity on Y_T, namely $Y_T \in \mathcal{D}((A^*)^2)$, one can prove

$$\int_0^T \eta(t)\left\|B^*Y'(t)\right\|_U^2 \, dt + \int_0^T \eta'(t)\langle B^*Y(t), B^*Y'(t)\rangle_U \, dt + \langle Az_0, A^*Y(0)\rangle_X = 0. \tag{3.2}$$

But, since $\eta \in C^1(\mathbb{R})$, for any $\varepsilon > 0$, (the constants C below denote various positive constants which do not depend on ε and that may change from line to line)

$$\begin{aligned}
\left|\int_0^T \eta'(t)\langle B^*Y(t), B^*Y'(t)\rangle_U \, dt\right| &\le \frac{C}{\varepsilon}\int_0^T \left\|B^*Y(t)\right\|_U^2 \, dt + \varepsilon\int_0^T \left\|B^*Y'(t)\right\|_U^2 \, dt \\
&\le \frac{C}{\varepsilon}\left\|Y_T\right\|_X^2 + C\varepsilon\left\|Y'(T)\right\|_X^2 \\
&\le \frac{C}{\varepsilon}\left\|Y_T\right\|_{obs}^2 + C\varepsilon\left\|Y'(0)\right\|_X^2 \\
&\le \frac{C}{\varepsilon}\left\|z_0\right\|_X^2 + C\varepsilon\int_0^T \eta(t)\left\|B^*Y'(t)\right\|_U^2 \, dt,
\end{aligned}$$

where we used the equivalence of the norms $\|y_T\|_X$, $\|y(0)\|_X$, $\|B^*y\|_{L^2(0,T;U)}$ and $\|y_T\|_{obs}$, the admissibility and observability inequalities (1.6) and (1.8) and estimate (1.11).

In particular, taking $\varepsilon > 0$ small enough,

$$\left|\int_0^T \eta'(t)\langle B^*Y(t), B^*Y'(t)\rangle_U \, dt\right| \le C\left\|z_0\right\|_X^2 + \frac{1}{2}\int_0^T \eta(t)\left\|B^*Y'(t)\right\|_U^2 \, dt. \tag{3.3}$$

It then follows from (3.2) that

$$\frac{1}{2}\int_0^T \eta(t)\,\|B^*Y'(t)\|_U^2\,dt \le C\,\|z_0\|_X^2 + \|z_0\|_1\,\|A^*Y(0)\|_X\,. \tag{3.4}$$

But $\|z_0\|_X \le C\,\|z_0\|_1$ and, applying the observability inequality (1.8) to $A^*Y(0)$, which reads

$$k_*\,\|A^*Y(0)\|_X^2 \le \int_0^T \eta(t)\,\|B^*Y'(t)\|_U^2\,dt,$$

we obtain

$$k_*\,\|A^*Y(0)\|_X^2 \le \int_0^T \eta(t)\,\|B^*Y'(t)\|_U^2\,dt \le C\,\|z_0\|_1^2\,. \tag{3.5}$$

Since $V' = \eta' B^*Y + \eta(t)B^*Y'$,

$$\int_0^T \|V'(t)\|_U^2\,dt \le 2\int_0^T \eta'(t)^2\,\|B^*Y(t)\|_U^2\,dt + 2\int_0^T \eta(t)^2\,\|B^*Y'(t)\|_U^2\,dt. \tag{3.6}$$

But

$$\begin{aligned}\int_0^T \eta'(t)^2\,\|B^*Y(t)\|_U^2\,dt &\le C\int_0^T \|B^*Y(t)\|_U^2\,dt\\ &\le C\,\|Y_T\|_X^2 \le C\,\|Y_T\|_{obs}^2 \le C\,\|z_0\|_X^2\,,\end{aligned} \tag{3.7}$$

where we used (1.6), the equivalence of the norms $\|Y_T\|_X$ and $\|Y(0)\|_X$, (1.8) and (1.11) in the last estimate.

Then estimate (1.13) follows from estimates (3.5), (3.6) and (3.7). □

We now give a rigorous proof of Theorem 1.4.

Proof of Theorem 1.4. The above proof requires from the beginning the regularity of Y_T. This cannot be assumed *a priori*. But the proof itself indicates that, with some technical work, the fact that Y_T is more regular can be deduced out of the estimates. Therefore, instead of putting $y_T = (A^*)^2 Y_T$ in (2.4), we put, for $\tau > 0$,

$$y_T = \frac{Y(T+\tau) - 2Y_T + Y(T-\tau)}{\tau^2}, \tag{3.8}$$

where $Y(T+\tau)$, the state of Y solution of (1.5) at time $T+\tau$, is well-defined as an element of X since $\mathbb{T}^*$ is a group. Note also that the corresponding solution of (1.5) is $(Y(t+\tau) - 2Y(t) + Y(t-\tau))/\tau^2$.

We shall then compute

$$I = \int_0^T \eta(t)\left\langle B^*Y(t), B^*\left(\frac{Y(t+\tau) - 2Y(t) + Y(t-\tau)}{\tau^2}\right)\right\rangle_U dt,$$

which, from (2.4), satisfies

$$I + \frac{1}{\tau^2}\langle z_0, Y(\tau) - 2Y(0) + Y(-\tau)\rangle_X = 0. \tag{3.9}$$

We have

$$\begin{aligned}
I &= \frac{1}{\tau}\int_0^T \eta(t)\left\langle B^*Y(t), B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\rangle_U dt \\
&\quad -\frac{1}{\tau}\int_0^T \eta(t)\left\langle B^*Y(t), B^*\left(\frac{Y(t)-Y(t-\tau)}{\tau}\right)\right\rangle_U dt \\
&= \frac{1}{\tau}\int_0^T \eta(t)\left\langle B^*Y(t), B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\rangle_U dt \\
&\quad -\frac{1}{\tau}\int_{-\tau}^{T-\tau} \eta(t+\tau)\left\langle B^*Y(t+\tau), B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\rangle_U dt \\
&= \int_{-\tau}^T \left(\frac{\eta(t)-\eta(t+\tau)}{\tau}\right)\left\langle B^*\left(\frac{Y(t)+Y(t+\tau)}{2}\right), B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\rangle_U \\
&\quad -\int_{-\tau}^T \left(\frac{\eta(t)+\eta(t+\tau)}{2}\right)\left\|B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\|_U^2 dt, \qquad (3.10)
\end{aligned}$$

where we used that $\eta(t)=0$ if $t\notin(0,T)$. This formula is similar to (3.1).

We now bound the first term in (3.10), independently of $\tau>0$. Remark that

$$\left|\frac{\eta(t+\tau)-\eta(t)}{\tau}\right| \leq \|\eta'\|_\infty ,$$

and then the first term in (3.10) can be bounded as in (3.3):

$$\begin{aligned}
&\left|\int_{-\tau}^{T+\tau}\left(\frac{\eta(t+\tau)-\eta(t)}{\tau}\right)\|B^*Y(t)\|_U^2\, dt\right| \leq C\,\|z_0\|_X^2 \\
&\qquad + \frac{1}{2}\int_{-\tau}^T \left(\frac{\eta(t)+\eta(t+\tau)}{2}\right)\left\|B^*\left(\frac{Y(t+\tau)-Y(t)}{\tau}\right)\right\|_U^2 dt. \qquad (3.11)
\end{aligned}$$

We now estimate the second term in the identity (3.9):

$$\begin{aligned}
\frac{1}{\tau^2}\langle z_0, Y(\tau)-2Y(0)+Y(-\tau)\rangle_X &= \frac{1}{\tau^2}\langle z_0, (Y(\tau)-Y(0))-(Y(0)-Y(-\tau))\rangle_X \\
&= \frac{1}{\tau^2}\langle z_0, (Y(\tau)-Y(0))\rangle_X - \frac{1}{\tau^2}\langle z_0, (Y(0)-Y(-\tau))\rangle_X \\
&= \frac{1}{\tau^2}\langle z(-\tau), (Y(0)-Y(-\tau))\rangle_X - \frac{1}{\tau^2}\langle z_0, (Y(0)-Y(-\tau))\rangle_X, \\
&= -\left\langle \frac{z_0-z(-\tau)}{\tau}, \frac{Y(0)-Y(-\tau)}{\tau}\right\rangle_X, \qquad (3.12)
\end{aligned}$$

where $z(-\tau)$ denotes the state of the solution of

$$z' = Az + BV, \quad t\in\mathbb{R}, \qquad z(0)=z_0,$$

at time $-\tau$. Note that this definition makes sense because $\mathbb{T}$ is a group, which guarantees the well-posedness of such equations. Also note that z solves $z'=Az$ on $(-\tau,0)$.

We then obtain

$$\begin{aligned}
\left|\frac{1}{\tau^2}\langle z_0, Y(\tau)-2Y(0)+Y(-\tau)\rangle_X\right| &= \left\|\frac{z_0-z(-\tau)}{\tau}\right\|_X \left\|\frac{Y(0)-Y(-\tau)}{\tau}\right\|_X \\
&\leq C\,\|Az_0\|_X \left\|\frac{Y(0)-Y(-\tau)}{\tau}\right\|_X ,
\end{aligned}$$

where we used that $z_0 \in \mathcal{D}(A)$ and that $\mathbb{T}$ is a group on X.

This estimate, combined with (3.11) and the identities (3.9), (3.10), imply that there exists a constant independent of $\tau > 0$ such that for all $\tau > 0$,

$$\int_{-\tau}^{T} \left(\frac{\eta(t) + \eta(t+\tau)}{2} \right) \left\| B^* \left(\frac{Y(t+\tau) - Y(t)}{\tau} \right) \right\|_U^2 dt \le C \left\| z_0 \right\|_1 . \tag{3.13}$$

But the left hand-side in (3.13) satisfies, for $\tau \le \delta$ (recall that δ is the parameter entering in the definition of the function η in (1.3))

$$\begin{aligned} &\int_{-\tau}^{T} \left(\frac{\eta(t) + \eta(t+\tau)}{2} \right) \left\| B^* \left(\frac{Y(t+\tau) - Y(t)}{\tau} \right) \right\|_U^2 dt \\ &\qquad \ge \int_{0}^{T} \frac{\eta(t)}{2} \left\| B^* \left(\frac{Y(t+\tau) - Y(t)}{\tau} \right) \right\|_U^2 dt \ge \frac{k_*}{2} \left\| \frac{Y(0) - Y(-\tau)}{\tau} \right\|_X^2 , \end{aligned} \tag{3.14}$$

according to the observability inequality (1.7).

We then obtain

$$\left\| \frac{Y(0) - Y(-\tau)}{\tau} \right\|_X \le C \left\| z_0 \right\|_1 . \tag{3.15}$$

But this implies $Y'(0) = -A^* Y(0) \in X$, and then that $Y \in C^1([0,T]; X)$ and that $Y'(T) = -A^* Y(T)$ satisfies

$$\left\| A^* Y(T) \right\|_X \le C \left\| z_0 \right\|_1 . \tag{3.16}$$

We then deduce (1.13) as follows.

Using (1.6), we obtain

$$\int_0^T \eta(t) \left\| B^* Y'(t) \right\|_U^2 dt \le K_T \left\| Y'(T) \right\|_X^2 \le C \left\| z_0 \right\|_1^2 ..$$

But $V' = \eta' B^* Y + \eta B^* Y'$ and therefore

$$\begin{aligned} \left\| V' \right\|_{L^2(0,T;U)} &\le \left\| \eta' \right\|_{L^\infty(0,T)} \left\| B^* Y \right\|_{L^2(0,T;U)} + \left\| \eta B^* Y' \right\|_{L^2(0,T;U)} \\ &\le C \left\| z_0 \right\|_X + C \left\| \sqrt{\eta} B^* Y' \right\|_{L^2(0,T;U)} \le C \left\| z_0 \right\|_1 , \end{aligned} \tag{3.17}$$

where we used (1.11) to estimate $\left\| B^* Y \right\|_{L^2(0,T;U)}$. □

3.2. **Theorem 1.4: The general case.**

Sketch. The general case is left to the reader. It can be shown by induction for s integer, choosing formally $y_T = (A^*)^{2s} Y_T$ in (2.4) and integrating by parts in time. In a more rigorous form, this choice corresponds to

$$y_T = \Delta_\tau^s Y_T,$$

where Δ_τ is the operator defined by

$$\Delta_\tau Y_T = \frac{1}{\tau^2} \left(Y(T+\tau) - 2 Y_T + Y(T-\tau) \right).$$

Arguing like this, one proves that $Y_T \in \mathcal{D}((A^*)^s)$ and, since

$$V^{(s)}(t) = \sum_{k=0}^{s} \binom{s}{k} \eta^{(k-s)}(t) B^* Y^{(k)}(t), \quad t \in \mathbb{R},$$

we obtain (1.13) and $V \in H_0^s(0,T;U)$.

Then the general result for $s \in \mathbb{R}_+$ follows by classical interpolation, see e.g. [20], since the maps $\mathbb{V}, \mathcal{V}$ are linear. □

Remark 3.1. Note that, for $z_0 \in \mathcal{D}(A^2)$, $Y_T \in \mathcal{D}((A^*)^2)$. In particular, $(Y(T+\tau)-Y_T)/\tau$ strongly converges to A^*Y_T, and then one can pass to the limit as $\tau \to 0$ in (3.10) and (3.9)-(3.12), and obtain the identity (3.2). Since all quantities in (3.2) depends continuously on $z_0 \in \mathcal{D}(A)$, using the density of $\mathcal{D}(A^2)$ in $\mathcal{D}(A)$, identity (3.2) holds for all $z_0 \in \mathcal{D}(A)$.

In particular, if η is C^2, one can integrate by parts once more in (3.2) and prove the following identity:

$$\int_0^T \eta(t) \left\| B^*Y'(t) \right\|_U^2 - \frac{1}{2}\int_0^T \eta''(t) \left\| B^*Y(t) \right\|_U^2 \, dt + \langle Az_0, A^*Y(0)\rangle_X = 0. \tag{3.18}$$

3.3. **Proof of Corollary 1.5.** We consider only the case $s \in \mathbb{N}$, the general result of Corollary 1.5 following immediately from interpolation theory [20].

To prove Corollary 1.5, we shall take into account the particular structure of the control function $V(t) = \eta(t)B^*Y(t)$ computed by our technique, for which we have, for $z_0 \in \mathcal{D}(A^s)$, $Y_T \in \mathcal{D}((A^*)^s)$. In particular, this implies that Y belongs to $\cap_{k=0}^s C^k([0,T];\mathcal{D}((A^*)^{s-k}))$ and V to the space

$$V_s = \bigcap_{k=0}^{s} C^k([0,T]; B^*\mathcal{D}((A^*)^{s-k})). \tag{3.19}$$

Note that $B^*Y(t)$ also belongs to $H^s(0,T;U)$ because of the admissibility condition (1.6) applied to $(A^*)^sY_T = (-1)^sY^{(s)}(T)$, which implies that $V \in H_0^s(0,T;U)$ as already stated in Theorem 1.4.

We then prove the following lemma:

Lemma 3.2. *Assume that the operator B is an admissible control operator for (1.1) in the sense of Definition 1.1. Let s be a nonnegative integer.*

If $z_0 \in \mathcal{D}(A^s)$ and $v \in H_0^s(0,T;U) \cap V_s$, the controlled solution z of (1.1) belongs to $\cap_{k=0}^s C^k([0,T],\mathcal{Z}_{s-k})$, where the spaces $\mathcal{Z}_j$ are defined by induction by (1.16).

Proof of Lemma 3.2. We argue by induction. The initialization step is the admissibility condition for the operator B.

We now take $s \geq 1$ and assume that Lemma 3.2 holds for $s-1$. We then consider $z_0 \in \mathcal{D}(A^s)$, $v \in V_s \cap H_0^s(0,T;U)$ and z the corresponding solution of (1.1). Then z' satisfies

$$(z')' = A(z') + B(v'), \quad t \geq 0, \qquad z'(0) = Az_0 \in \mathcal{D}(A^{s-1}),$$

and $v' \in V_{s-1} \cap H_0^{s-1}(0,T;U)$. Then the induction assumption applies and yields

$$z \in \bigcap_{k=1}^{s} C^k([0,T],\mathcal{Z}_{s-k}).$$

To prove that $z \in C^0([0,T];\mathcal{Z}_s)$, remark that

$$Az = z' - BV \in C^0([0,T];\mathcal{Z}_{s-1} + BB^*\mathcal{D}((A^*)^s)).$$

The result follows. □

Proof of Corollary 1.5. Corollary 1.5 then follows directly from Lemma 3.2 combined with Theorem 1.4 for $s \in \mathbb{N}$. The result for $s \in \mathbb{R}_+$ follows by interpolation. □

Remark 3.3. We claim that, when A is the generator of a group and A is not invertible but $A - \beta I$ is it for some $\beta \in \mathbb{C}$, then the same results apply.

This can be seen by using the correspondence between the solutions z of (1.1) with control v and initial datum z_0 and the solutions $\tilde{z}$ of

$$\tilde{z}' = (A-\beta)\tilde{z} + B\tilde{v}, \quad t \geq 0, \qquad \tilde{z}(0) = \tilde{z}_0, \tag{3.20}$$

with control function $\tilde{v} = v\exp(-\beta t)$ and initial datum $\tilde{z}_0 = z_0$, which coincide with $\tilde{z} = z\exp(-\beta t)$.

Our claim then is a direct consequence of Theorem 1.4 and Corollary 1.5 applied to the controlled system (3.20).

4. The case $\eta = \eta_c$.

As we explained in the introduction, adding some knowledge on the operator B (or equivalently B^*), similarly as in Theorem 1.4, we can derive Theorem 1.6 when $\eta = \eta_c$ as in (1.17).

4.1. The case $s = 1$.

We begin with the case $s = 1$:

Theorem 4.1. *Assume that the hypotheses of Proposition 1.3 are satisfied and let $\eta = \eta_c$ as in (1.17).*

Assume that B is as a bounded operator from U to X, or equivalently, that $B^* \in \mathfrak{L}(X, U)$.

If the initial datum z_0 to be controlled belongs to $\mathcal{D}(A)$, then the minimizer Y_T given by Proposition 1.3 and the control function V given by (1.10), respectively, belong to $\mathcal{D}(A^*)$ and $H^1(0,T;U)$.

Besides, there exists a positive constant $C = C(k_*, K_T)$ independent of $z_0 \in \mathcal{D}(A)$ such that

$$\|Y_T\|^2_{\mathcal{D}(A^*)} + \int_0^T \|V'(t)\|^2_U \, dt \leq C \, \|z_0\|^2_{\mathcal{D}(A)} . \tag{4.1}$$

Of course, this simply corresponds to the case $s = 1$ in Theorem 1.4, but it already requires one more assumption on the control operator B, namely its boundedness, which is not satisfied in the case of boundary controls, see Section 5.2.2.

Proof. The proof of Theorem 4.1 closely follows the one in Section 3.1. We thus only present it in a formal manner, since the rigorous proof then follows from the same ingredients as in Section 3.1.

Identity (3.1) shall now be replaced by

$$\begin{aligned}\int_0^T \langle B^*Y(t), B^*(A^*)^2 Y(t)\rangle_U \, dt &= \int_0^T \langle B^*Y(t), B^*Y''(t)\rangle_U \, dt \\ &= -\int_0^T \|B^*Y'(t)\|^2_U \, dt + \Big[\langle B^*Y(t), B^*Y'(t)\rangle_U\Big]_0^T . \end{aligned}\tag{4.2}$$

In particular, identity (3.2) should be replaced by

$$\int_0^T \|B^*Y'(t)\|^2_U \, dt - \Big[\langle B^*Y(t), B^*Y'(t)\rangle_U\Big]_0^T + \langle Az_0, A^*Y(0)\rangle_X = 0. \tag{4.3}$$

We thus only need to explain how to bound the time-boundary terms

$$\Big[\langle B^*Y(t), B^*Y'(t)\rangle_U\Big]_0^T = \langle B^*Y(T), B^*Y'(T)\rangle_U - \langle B^*Y(0), B^*Y'(0)\rangle_U, \tag{4.4}$$

the rest of the proof being as in Section 3.1.

Using the equivalence of the norms $\|y_T\|_X$, $\|y(0)\|_X$, $\|B^*y\|_{L^2(0,T;U)}$, estimate (1.11) and the boundedness of B, we obtain, for any $\varepsilon > 0$, (the constants C below denote various positive constants which do not depend on ε that may change from line to line)

$$\begin{aligned}|\langle B^*Y(T), B^*Y'(T)\rangle_U| &\leq \|B^*Y(T)\|_U \|B^*Y'(T)\|_U \leq C\|Y(T)\|_X \|Y'(T)\|_X \\ &\leq \frac{C}{\varepsilon}\|Y(T)\|_X^2 + C\varepsilon\|Y'(T)\|_X^2 \leq \frac{C}{\varepsilon}\|z_0\|_X^2 + C\varepsilon\int_0^T \|B^*Y'(t)\|_X^2\, dt.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, we obtain

$$|\langle B^*Y(T), B^*Y'(T)\rangle_U| \leq C\|z_0\|_X^2 + \frac{1}{4}\int_0^T \|B^*Y'(t)\|_U^2\, dt. \tag{4.5}$$

Doing the same for $\langle B^*Y(0), B^*Y'(0)\rangle_U$, we obtain

$$\left|\Big[\langle B^*Y(t), B^*Y'(t)\rangle_U\Big]_0^T\right| \leq C\|z_0\|_X^2 + \frac{1}{2}\int_0^T \|B^*Y'(t)\|_U^2\, dt. \tag{4.6}$$

Estimate (4.6) is similar to (3.3), and the formal proof of Theorem 4.1 then follows line to line the one of Theorem 1.4 in the case $s = 1$ and is left to the reader.

The rigorous one needs to take y_T as in (3.8) and requires the same kind of estimates as before. □

4.2. **Theorem 1.6: the general case.**

Proof. We only sketch the proof formally. As for the proof of Theorem 1.4 in Section 3.2, we first focus on the case $s \in \mathbb{N}$, using an induction argument and assuming that Theorem 1.6 holds for $s-1$, and then conclude by interpolation techniques.

The idea then is to choose $y = Y^{(2s)}$ in (2.4) and integrating by parts in time:

$$\begin{aligned} 0 &= \int_0^T \langle B^*Y(t), B^*Y^{(2s)}(t)\rangle_U\, dt + \langle z_0, (A^*)^{2s}Y(0)\rangle_X \\ &= \left[\sum_{k=0}^{s-1}(-1)^k\langle B^*Y^{(k)}(t), B^*Y^{(2s-k-1)}(t)\rangle_U\right]_0^T \\ &\quad +(-1)^s\int_0^T \left\|B^*Y^{(s)}(t)\right\|_U^2\, dt + \langle A^s z_0, (A^*)^sY(0)\rangle_X.\end{aligned}$$

Again, we need to derive suitable bounds for the terms

$$B_s(t) = \sum_{k=0}^{s-1}(-1)^k\langle B^*Y^{(k)}(t), B^*Y^{(2s-k-1)}(t)\rangle_U.$$

Namely, we shall use that, under the assumption (1.18),

$$|B_s(t)| \leq C\|Y(t)\|_{\mathcal{D}((A^*)^{s-1})}\|Y(t)\|_{\mathcal{D}((A^*)^s)}, \tag{4.7}$$

which immediately implies, by the induction argument, that

$$\left|[B_s(t)]_0^T\right| \leq C\|z_0\|_{\mathcal{D}(A^{s-1})}^2 + \frac{1}{2}\int_0^T \left\|B^*Y^{(s)}(t)\right\|_U^2\, dt,$$

and then Theorem 1.6 follows as in Section 3.2.

We then focus on the estimate (4.7). The idea is to remark that

$$\begin{aligned} B_s(t) &= (-1)^{s-1}\sum_{k=0}^{s-1}(-1)^k\langle B^*(A^*)^kY(t), B^*(A^*)^{s-k-1}Y^{(s)}(t)\rangle_U \\ &= (-1)^{s-1}\sum_{k=0}^{s-1}(-1)^k\langle A^{s-k-1}BB^*(A^*)^kY(t), Y^{(s)}(t)\rangle_U. \end{aligned}$$

But, if $Y(t) \in \mathcal{D}((A^*)^{s-1})$, $(A^*)^kY(t) \in \mathcal{D}((A^*)^{s-1-k})$ and then, according to (1.18), there exists a constant such that for all $k \in \{0, \cdots, s-1\}$,

$$\left\|A^{s-k-1}BB^*(A^*)^kY(t)\right\|_X \le C\left\|Y(t)\right\|_{\mathcal{D}((A^*)^{s-1-k})}.$$

Estimate (4.7) then follows directly. Details of the proof are left to the reader. □

4.3. **Proof of Corollary 1.7.**

Proof. Let $s \in \mathbb{N}$. Let $z_0 \in \mathcal{D}(A^s)$. By Theorem 1.6, the initial datum Y_T given by Proposition 1.3 belongs to $\mathcal{D}((A^*)^s)$ and then

$$Y \in \bigcap_{k=0}^{s} C^k([0,T];\mathcal{D}((A^*)^{s-k})).$$

Then, according to (1.18), the source term $BV = BB^*Y$ in (1.1) satisfies

$$BV \in \bigcap_{k=0}^{s} C^k([0,T];\mathcal{D}((A)^{s-k})).$$

The proof can then be done by induction as in Lemma 3.2, remarking that z' satisfies

$$(z')' = A(z') + BV', \quad t \ge 0, \qquad z'(0) = Az_0 + BV(0) \in \mathcal{D}(A^{s-1}).$$

For general $s \in \mathbb{R}_+$, the result can be deduced from interpolation theory.

Details are left to the reader. □

4.4. **Comments.**

4.4.1. *The case $T = T^*$.* When $\eta = \eta_c$ as in (1.17), Proposition 1.3, Theorem 1.6 and Corollary 1.7 apply when the time T is exactly T^*, the time in which we assumed that system (1.1) is controlable.

4.4.2. *Link with Riccatti equations.* The operator $\mathbb{V}_T : z_0 \mapsto Y_T$ is very much related to the so-called Gramian operator defined by

$$G_Ty_0 = \left(\int_0^T \exp(-tA)BB^*\exp(-tA^*)\,dt\right)y_0. \tag{4.8}$$

Indeed, $\mathbb{V}_Tz_0 = Y_T$ if and only if $z_0 = -G_T\exp(TA^*)Y_T$. The operator G_T has been extensively studied in the literature (see for instance [8]). In particular, when BB^* is bounded, using integration by parts as we did, it was proved in [8] (in the limit $T \to \infty$, but the same can be done for finite $T > 0$) that there exists a bounded operator $P \in \mathfrak{L}(X)$ such that for any $y \in \mathcal{D}(A)$,

$$AG_Ty + G_TA^*y = Py. \tag{4.9}$$

In particular, this easily shows that $G_T : \mathcal{D}(A^*) \to \mathcal{D}(A)$.

Besides, since the observability inequality ensures that G_T is invertible, one would like to multiply (4.9) by G_T^{-1} from the left and from the right to obtain

$$G_T^{-1}Az + A^*G_T^{-1}z = G_TPG_T^{-1}z. \tag{4.10}$$

However, this is true only for $z \in G_T(\mathcal{D}(A^*))$. Therefore, to conclude that G_T^{-1} maps $\mathcal{D}(A)$ to $\mathcal{D}(A^*)$ using (4.10), one should prove the density of $G_T(\mathcal{D}(A))$ in $\mathcal{D}(A)$. This follows from our result, but does not seem to be straightforward.

Note also that the idea of including a weight function in time in the Gramian has also been used in the stabilization context in [12, 22].

5. **Application: The wave equation.**

5.1. **Distributed control.**

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, and let ω be a subdomain of Ω.

We now consider the following wave equation:

$$\begin{cases} z'' - \Delta z = v\chi_\omega, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \partial\Omega \times (0,\infty), \\ (z(0), z'(0)) = (z_0, z_1) & \in H_0^1(\Omega) \times L^2(\Omega), \end{cases} \tag{5.1}$$

where $v \in L^2((0,T) \times \omega)$ is the control function, localized on ω by multiplication by the function $\chi_\omega(x)$ supported in ω, strictly positive on some $\overline{\omega_0}$, where ω_0 is an open subset of ω. For instance, χ_ω can be the characteristic function ξ_ω of ω.

Set $A_0 = -\Delta$ with domain $\mathcal{D}(A_0) = H^2 \cap H_0^1(\Omega)$.

It is well-known that system (5.1) fits the abstract setting given above, once written as a first order system in the variable $Z = (z, z') \in C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$, which satisfies

$$Z' = AZ + Bv, \text{ with } A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \begin{cases} X = H_0^1(\Omega) \times L^2(\Omega), \\ \mathcal{D}(A) = H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega), \end{cases} \tag{5.2}$$

where B is defined by

$$Bv = \begin{pmatrix} 0 \\ \chi_\omega v \end{pmatrix}.$$

The operator B is then continuous from $U = L^2(\omega)$ to $X = H_0^1(\Omega) \times L^2(\Omega)$.

It is by now well-known that the exact controllability property for (5.1) in time T^* is equivalent to the so-called *Geometric Control Condition* (GCC in short), which asserts that all the rays of Geometric Optics in Ω enter the subdomain ω_0 in a time smaller than T^*, see [1], [3].

We thus assume the GCC in time T^* in the following.

Let $T > T^*$, choose $\delta > 0$ such that $T - 2\delta \geq T^*$ and fix a function η satisfying (1.3). Note that in this case, the control operator being bounded, the weight function η can be taken to be constant, i.e. η_c.

Then the functional J introduced in (1.4) reads as

$$J(y_0, y_1) = \frac{1}{2}\int_0^T \int_\omega \eta(t)\chi_\omega^2(x)|y(x,t)|^2\,dxdt + \langle z_0, y'(\cdot,0)\rangle_{H_0^1(\Omega)\times H^{-1}(\Omega)} - \int_\Omega z_1(x)\,y(x,0)\,dx \tag{5.3}$$

where y is the solution of

$$\begin{cases} y'' - \Delta y = 0, & \text{in } \Omega \times (0,\infty), \\ y = 0, & \text{on } \partial\Omega \times (0,\infty), \\ (y(T), y'(T)) = (y_0, y_1) & \in L^2(\Omega) \times H^{-1}(\Omega). \end{cases} \tag{5.4}$$

Then our results imply the following:

Theorem 5.1. *Let η be a weight function satisfying (1.3), either smooth or simply $\eta = \eta_c$. Let χ_ω be a cut-off function as above localizing the support of the control. Then, under the controllability conditions above, given any $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique minimizer (Y_0, Y_1) of J over $L^2(\Omega) \times H^{-1}(\Omega)$. The function*

$$V(x,t) = \eta(t)\chi_\omega(x)Y(x,t) \tag{5.5}$$

is a control function for (5.1), which is characterized as the control function of minimal $L^2(0,T;dt/\eta;L^2(\omega))$-norm, defined by

$$\|v\|^2_{L^2(0,T;dt/\eta;L^2(\omega))} = \int_0^T \int_\omega |v(x,t)|^2 dx \frac{dt}{\eta(t)}.$$

Furthermore, if either the function χ_ω is smooth, all its derivatives vanish at the boundary, and $\eta = \eta_c$, or the weight function η satisfies $\eta \in C^\infty(\mathbb{R})$, then if (z_0, z_1) belongs to $\mathcal{D}(A^s)$ for some $s \in \mathbb{R}_+$, $(Y_0, Y_1) \in \mathcal{D}((A^*)^s)$.

In particular, when χ_ω is smooth and all its derivatives vanish at the boundary (η could be chosen either $C^\infty(\mathbb{R})$ or equal to η_c), the control function V given by (5.5) belongs to

$$V \in H^s(0,T;L^2(\omega)) \cap \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T]; H_0^{s-k}(\omega)), \tag{5.6}$$

the controlled solution z of (5.10) belongs to

$$(z, z') \in C^s([0,T]; X) \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T]; \mathcal{D}(A^{s-k})), \tag{5.7}$$

and, in particular,

$$(z, z') \in \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T]; H^{s+1-k}(\Omega) \times H^{s-k}(\Omega)). \tag{5.8}$$

Remark 5.2. In our theoretical results, we assumed that the operator A was defined on a Hilbert space X, which we implicitly identified with its dual. Since here, the wave operator A in (5.2) is skew-adjoint on X, Theorems 1.4 and 1.6 can be applied.

However, when working with PDE, it is more standard to identify $L^2(\Omega)$ with its dual, and Theorem 5.1 is stated with this identification in mind. But this makes necessary to distinguish between $X = H_0^1(\Omega) \times L^2(0,1)$ and its dual, see for instance [2].

Hence, in the above case, for instance, we shall define $X^* = L^2(\Omega) \times H^{-1}(\Omega)$ and the duality product defined for $(y_0, y_1) \in X^*$, $(z_0, z_1) \in X$ by

$$\left\langle \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\rangle_{X^* \times X} = \langle y_0, z_1 \rangle_{L^2} - \langle y_1, z_0 \rangle_{H^{-1} \times H_0^1},$$

where

$$\langle y, z \rangle_{H^{-1} \times H_0^1} = \langle \nabla A_0^{-1} y, \nabla z \rangle_{L^2}.$$

With this scalar product, one easily checks that

$$A^* = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \text{with} \left\{ \begin{array}{l} X^* = L^2(\Omega) \times H^{-1}(\Omega), \\ \mathcal{D}(A^*) = H_0^1(\Omega) \times L^2(\Omega), \end{array} \right.$$

and

$$B^* = (\chi_\omega \quad 0).$$

Note that, in particular, this implies that the operator BB^*, which goes from X^* to X, is given by:

$$BB^* = \begin{pmatrix} 0 & 0 \\ \chi_\omega^2 & 0 \end{pmatrix}. \tag{5.9}$$

Also note that the control map $\mathbb{V}$ now maps $X = H_0^1(\Omega) \times L^2(\Omega)$ to $X^* = L^2(\Omega) \times H^{-1}(\Omega)$, and, according to our results, $\mathcal{D}(A^s)$ to $\mathcal{D}((A^*)^s) = \mathcal{D}(A^{s-1})$.

Proof. The first part of Theorem 5.1 is a direct consequence of Proposition 1.3.

According to Theorems 1.4 and 1.6, if $(z_0, z_1) \in \mathcal{D}(A^s)$ and either χ_ω or η is smooth, $(Y_0, Y_1) \in \mathcal{D}((A^*)^s)$. Indeed, if χ_ω is smooth and all its derivative vanish at the boundary, the operator BB^*, given in (5.9), maps $\mathcal{D}((A^*)^j)$ in $\mathcal{D}(A^j)$, see Remark 5.2.

The regularity result (5.6) follows directly. It is based on (3.19) and follows directly from the fact that $B^*\mathcal{D}((A^*)^j) \subset H_0^j(\omega)$.

For the regularity property (5.7), assuming χ_ω is smooth and all its derivative vanish at the boundary, BB^* satisfies (1.18), and then $\mathcal{Z}_j \subset \mathcal{D}(A^j)$ for all $j \geq 0$.

The regularity property (5.8) then follows directly from $\mathcal{D}(A^{s-j}) \subset H^{s+1-j}(\Omega) \times H^{s-j}(\Omega)$. □

Remark 5.3. In [4] it is proved that, when the functions η and χ_ω are smooth, the map $\mathbb{V} : (z_0, z_1) \mapsto (Y_0, Y_1)$ is a bijection from $\mathcal{D}(A^s)$ to $\mathcal{D}((A^*)^s) = \mathcal{D}(A^{s-1})$ for any $s \geq 0$.

The proof in [4] is done by looking at the behavior of the Gramian operator (cf (4.8)) on each frequency block of solutions using a Littlewood-Paley decomposition.

In particular, their result shows that the Gramian operator is a pseudo-differential operator when considering the wave equation on a compact manifold without boundary. This result has been illustrated numerically in the recent work [14]. Roughly speaking, it also implies that the control maps $\mathbb{V}$ acts frequency by frequency. In particular, this result can be used (see [4]) to derive control results for semilinear wave equations.

Theorem 5.1 gives another proof of the fact that, when the initial datum (z_0, z_1) to be controlled belongs to $\mathcal{D}(A^s)$, then (Y_0, Y_1) belongs to $\mathcal{D}(A^{s-1})$ and the controlled solution with the control provided by Theorem 5.1 satisfies (5.7).

Remark that our result also applies when $\eta = \eta_c$, thus extending the previous work [4]. Also note that the method in [4] has been developed only for the wave equation. Though it may probably be developed in more general situations, it requires careful estimates in each case, whereas our method applies in a very general abstract framework.

5.2. **Boundary control.** Let Ω be a smooth bounded domain of $\mathbb{R}^N$, and let Γ_1 and Γ_2 be two open subsets of the boundary such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$.

Let χ be a function defined on $\partial\Omega$ supported in Γ_1 and non-vanishing on $\overline{\Gamma_0}$, for some open subset of the boundary $\Gamma_0 \subset \Gamma_1$.

We now consider the following wave equation:

$$\begin{cases} z'' - \Delta z = 0, & \text{in } \Omega \times (0, \infty), \\ z = \chi v, & \text{on } \Gamma_1 \times (0, \infty), \\ z = 0, & \text{on } \Gamma_2 \times (0, \infty), \\ (z(0), z'(0)) = (z_0, z_1) & \in L^2(\Omega) \times H^{-1}(\Omega), \end{cases} \tag{5.10}$$

where $v \in L^2(0, T; \Gamma_1)$ is the control function, localized on Γ_1 by multiplication by χ.

As before, set $A_0 = -\Delta$ with domain $\mathcal{D}(A_0) = H_0^1(\Omega)$.

Similarly as in Subsection 5.1, system (5.10) fits the abstract setting given above, once written as a first order system in the variable $Z = (z, z') \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$, which satisfies

$$Z' = AZ + Bv, \text{ with } A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \begin{cases} X = L^2(\Omega) \times H^{-1}(\Omega), \\ \mathcal{D}(A) = H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

but this time, B is defined by

$$Bv = \begin{pmatrix} 0 \\ A_0 \tilde{z} \end{pmatrix}, \quad \text{where} \begin{cases} -\Delta \tilde{z} = 0, & \text{in } \Omega, \\ \tilde{z} = \chi v(\cdot, t), & \text{on } \Gamma_1, \\ \tilde{z} = 0, & \text{on } \Gamma_2. \end{cases} \tag{5.11}$$

Note that the map $v \in L^2(\Gamma_1) \mapsto \tilde{z} \in H^{1/2}(\Omega)$ is continuous (see [21, Chap. 10]) and then B is continuous from $Y = L^2(\Gamma_1)$ to $\{0\} \times H^{-3/2}(\Omega) \subset \mathcal{D}(A^{1/2})^*$.

Note that in this case, the control operator B is unbounded. The fact that B is admissible follows from a hidden regularity result, proved for instance in [15]. Also note that Theorem 1.6 and Corollary 1.7 do not apply.

Again, the exact controllability property for (5.10) in time T^* is equivalent to the so-called *Geometric Control Condition* (GCC in short), which asserts that all the rays of Geometric Optics in Ω touch the sub-boundary Γ_0 at a non-diffractive point in a time smaller than T^*, see [1], [3].

In the following, we assume the GCC in time T^* for Γ_0.

5.2.1. *Weighted controls.* Let $T > T^*$, choose $\delta > 0$ such that $T - 2\delta \geq T^*$ and fix a function η satisfying (1.3).

Then the functional J introduced in (1.4) is now defined on $H_0^1(\Omega) \times L^2(\Omega)$ and reads as

$$J(y_0, y_1) = \frac{1}{2} \int_0^T \int_{\Gamma_1} \eta(t) \chi(x)^2 |\partial_n y(x,t)|^2 \, d\Gamma dt + \int_0^1 z_0(x) \, y'(x,0) \, dx \\ - \langle z_1, y(\cdot, 0) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}, \tag{5.12}$$

where y is the solution of (5.4).

Then our results imply the following:

Theorem 5.4. *Assume that χ is compactly supported in Γ_1 and that η is a smooth weight function satisfying (1.3).*

Given any $(z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique minimizer (Y_0, Y_1) of J over $H_0^1(\Omega) \times L^2(\Omega)$. The function

$$V(x,t) = \eta(t) \chi(x) \partial_n Y(x,t)_{|\Gamma_1} \tag{5.13}$$

is a control function for (5.10), which is characterized as the control function which minimizes the $L^2(0,T; dt/\eta; L^2(\Gamma_1))$-norm, defined by

$$\|v\|^2_{L^2(0,T;dt/\eta;L^2(\Gamma_1))} = \int_0^T \int_{\Gamma_1} |v(x,t)|^2 d\Gamma \frac{dt}{\eta(t)}.$$

Furthermore, if the function χ is smooth, then if (z_0, z_1) belongs to $\mathcal{D}(A^s)$ for some real number $s \in \mathbb{R}_+$, the control function V given by (5.13) belongs to

$$V \in H_0^s(0,T; L^2(\Gamma_1)) \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T]; H_0^{s-k-1/2}(\Gamma_1)) \tag{5.14}$$

and $(Y_0, Y_1) \in \mathcal{D}((A^*)^s) = \mathcal{D}(A^{s+1})$. *In particular, the controlled solution* z *of* (5.10) *then belongs to*

$$(z, z') \in C^s([0,T]; L^2(\Omega) \times H^{-1}(\Omega)) \bigcap_{k=0}^{\lfloor s \rfloor} C^k([0,T]; H^{s-k}(\Omega) \times H^{s-1-k}(\Omega)). \tag{5.15}$$

Note that, again, we have identified $L^2(\Omega)$ with its topological dual. As in the case of distributed controls, this artificially creates a shift between the spaces X, X^*, $\mathcal{D}(A^j)$ and $\mathcal{D}((A^*)^j)$.

Proof of Theorem 5.4. The only statements which are not direct consequences of Theorem 1.4 are the regularity properties (5.14), (5.15).

Indeed, these are consequences of (3.19) and Corollary 1.5, and of the explicit computation of the spaces $\mathcal{Z}_j$ for $j \in \mathbb{N}$ defined in (1.16).

These can indeed be computed. For, we compute A^*, defined on $X^* = H_0^1(\Omega) \times L^2(\Omega)$ (note that the fact that X and X^* are not the same is due to the classical identification of $L^2(\Omega)$ with its dual) with domain $\mathcal{D}(A^*) = H^2 \cap H_0^1(\Omega) \cap H_0^1(\Omega)$:

$$A^* = -\begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad B^* \begin{pmatrix} y^0 \\ y^1 \end{pmatrix} = \left(\partial_n y^0\right)_{|\Gamma_1}.$$

It is then easy to check that B^* maps $\mathcal{D}(A^*)$ to $H_0^{1/2}(\Gamma_1)$, and more generally, B^* maps $\mathcal{D}((A^*)^j)$ to $H_0^{j-1/2}(\Gamma_1)$ (Note that, in this step, assuming that the function χ is compactly supported in Γ_1 is needed to guarantee this property). This proves (5.14).

Then the map $v \mapsto \tilde{z}$ defined in (5.11) maps $H_0^{j-1/2}(\Gamma_1)$ to $H^j(\Omega)$.

Therefore

$$A^{-1} B B^* \mathcal{D}((A^*)^j) \subset H^j(\Omega) \times H^{j-1}(\Omega),$$

from which we conclude easily that $\mathcal{Z}_j \subset H^j(\Omega) \times H^{j-1}(\Omega)$ for all $j \geq 0$ and (5.15) follows by interpolation and Corollary 1.5. □

5.2.2. *Unbounded control operator with a constant time-weight.* In this paper we have proved the regularity of controls in two different cases: smooth compactly supported time-weights or the case of bounded control operators with a constant weight. The case of a constant time-weight function with an unbounded control operator cannot be treated by the methods discussed here. The reason is the impossibility of getting bounds on the term (4.4) above, that arises when η does not vanish on $t = 0, T$, in the case where B is unbounded. To analyze the possibility of addressing this case, we discuss the particular example of the 1d wave equation with boundary control.

Let us consider the 1d wave equation controlled by the boundary in time $T = 4$, in which explicit computations can be done:

$$\begin{cases} z'' - z_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ z(0,t) = 0, \ z(1,t) = v(t), & 0 < t < T, \\ z(x,0) = z^0(x), \ z'(x,0) = z^1(x), & 0 < x < 1, \end{cases} \tag{5.16}$$

and we fix the time $T = 4$, which is larger than the time of controllability, corresponding to $T^* = 2$, which is the time needed by the waves to go from $x = 1$ to $x = 0$ and bounce back at $x = 0$.

Note that here, the function χ introduced in (5.10) does not appear anymore. This is due to the standard choice $\chi(0) = 0$ and $\chi(1) = 1$, which fits the assumptions of the previous paragraph on χ.

We refer for instance to [15] for the proof of the admissibility and observability properties of (5.16).

The application of the classical *Hilbert Uniqueness Method* in this case reads as follows, see [15].

Define, for $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the functional

$$J(y_0, y_1) = \frac{1}{2} \int_0^T |y_x(1,t)|^2 \, dt + \int_0^1 z_0(x) \, y'(x,0) \, dx \\ - \langle z_1, y(\cdot, 0) \rangle_{H^{-1}(0,1) \times H_0^1(0,1)}, \quad (5.17)$$

where y is the solution of

$$\begin{cases} y'' - y_{xx} = 0, & 0 < x < 1, \ t \in \mathbb{R}, \\ y(0,t) = y(1,t) = 0, & t \in \mathbb{R}, \\ y(x,T) = y_0(x), \ y'(x,T) = y_1(x), & 0 < x < 1. \end{cases} \quad (5.18)$$

We will use the fact that, when the time horizon in which we want to control is $T = 4$ (actually it is true for any even integer), the functional J acts diagonally on the Fourier coefficients of the solutions y of (5.18) and then the minimizer of J can be computed explicitly.

Indeed, the spectrum of the operator $-\partial_{xx}^2$ with Dirichlet boundary conditions is simply given by a sequence of eigenvectors w^k corresponding to the eigenvalues λ^k, which can be computed explicitly:

$$-w_{xx}^k = \lambda^k w^k, \quad \text{on } (0,1), \qquad w^k(x) = \sqrt{2} \sin(k\pi x), \quad \lambda^k = (k\pi)^2, \quad k \in \mathbb{N}.$$

Then, writing

$$(y_0, y_1) = \sqrt{2} \sum_{k=1}^{\infty} (\hat{y}_0^k, \hat{y}_1^k) \sin(k\pi x), \quad (5.19)$$

one easily checks that

$$y(x,t) = \sqrt{2} \sum_{k=1}^{\infty} \left(\hat{y}^k e^{ik\pi t} + \hat{y}^{-k} e^{-ik\pi t} \right) \sin(k\pi x), \quad (5.20)$$

with

$$\hat{y}^k = \frac{1}{2} \left(\hat{y}_0^k - i \frac{\hat{y}_k^1}{k\pi} \right), \qquad \hat{y}^{-k} = \frac{1}{2} \left(\hat{y}_0^k + i \frac{\hat{y}_k^1}{k\pi} \right).$$

Therefore, using Parseval's identity,

$$\begin{aligned} \frac{1}{2} \int_0^T |y_x(1,t)|^2 \, dt &= \int_0^4 \left| \sum_{|k|=1}^{\infty} \hat{y}^k k\pi (-1)^k e^{ik\pi t} \right|^2 dt \\ &= 4 \sum_{|k|=1}^{\infty} k^2 \pi^2 |\hat{y}^k|^2 = 2 \sum_{k=1}^{\infty} \left(k^2 \pi^2 |\hat{y}_0^k|^2 + |\hat{y}_1^k|^2 \right). \end{aligned}$$

But we can also expand the initial datum $(z_0, z_1) \in L^2(0,1) \times H^{-1}(0,1)$ in Fourier series as

$$(z_0, z_1) = \sqrt{2} \sum_{k=1}^{\infty} (\hat{z}_0^k, \hat{z}_1^k) \sin(k\pi x), \quad (5.21)$$

with

$$\sum_{k=1}^{\infty} \left(|\hat{z}_0^k|^2 + \frac{|\hat{z}_1^k|^2}{k^2 \pi^2} \right) < \infty.$$

Thus, for (y_0, y_1) as in (5.19), since the solutions of (5.18) are 2-periodic,

$$J(y_0, y_1) = 2\sum_{k=1}^{\infty} \left(k^2\pi^2|\hat{y}_0^k|^2 + |\hat{y}_1^k|^2\right) + \sum_{k=1}^{\infty} \left(\hat{z}_0^k\hat{y}_1^k - \hat{z}_1^k\hat{y}_0^k\right). \tag{5.22}$$

Therefore the minimizer (Y_0, Y_1) of J can be given as

$$(Y_0, Y_1) = \sqrt{2}\sum_{k=1}^{\infty} \left(\hat{Y}_0^k, \hat{Y}_1^k\right)\sin(k\pi x), \quad \text{with} \quad \begin{cases} \hat{Y}_0^k = \dfrac{\hat{z}_1^k}{4k^2\pi^2}, \\ \hat{Y}_1^k = -\dfrac{\hat{z}_0^k}{4}, \end{cases} \tag{5.23}$$

and the control function V is simply

$$V(t) = \frac{\sqrt{2}}{4}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\hat{z}_1^k}{k\pi}\cos(k\pi t) - \hat{z}_k^0\sin(k\pi t)\right).$$

In particular, it is obvious that, for $(z_0, z_1) \in H_0^1(0,1) \times L^2(0,1)$, this method yields $(Y_0, Y_1) \in H^2 \cap H_0^1(0,1) \times H_0^1(0,1)$ and $V \in H^1(0,T)$.

However,

$$V(0) = \frac{\sqrt{2}}{4}\sum_{k=1}^{\infty}(-1)^k\frac{\hat{z}_1^k}{k\pi}.$$

Therefore, the controlled solution z of (5.16) with that control function cannot be a strong solution if

$$\sum_{k=1}^{\infty}(-1)^k\frac{\hat{z}_1^k}{k\pi} \neq 0,$$

whatever the regularity of the initial datum to be controlled is, because the compatibility condition $z(1,0) = V(0)$ does not hold.

In this sense, the method we propose in this article completes the original Hilbert Uniqueness Method by providing control functions such that the regularity properties of the initial data to be controlled are automatically transferred to the controls and controlled trajectories.

Though the controlled solution is not a strong solution in general, we have seen that the control function V computed by the classical Hilbert Uniqueness Method in time $T = 4$ for the 1d wave equation (5.16) belongs to $H^1(0,T)$ when the initial datum to be controlled belongs to $H_0^1(0,1) \times L^2(0,1)$.

This might seem surprising according to the computations in Section 4, but it is not, due to the fact that solutions of (5.18) are 2-periodic (see for instance (5.20)). This periodicity guarantees that the time boundary terms in (4.2) are the same at time $t = 0$ and $t = T$ and cancel each other. Thus the term (4.4) vanishes.

Therefore, Theorem 4.1 and, similarly, Theorem 1.6 still hold in that case, when the time T is an even integer.

The case of a general T is more delicate. Indeed, in that case, the periodicity of solutions does not guarantee the term (4.4) to vanish. In the particular case under consideration, this issue could be completely analysed using the Fourier series representation of solutions, and under suitable number theoretical assumptions on T. But these techniques would hardly be extendable to the multi-dimensional case. Thus, in general, the case in which η is constant and B is unbounded is open.

5.2.3. *The case of smooth data with non-vanishing boundary trace.* To complete our analysis, we also comment the case in which we want to control (5.16) for smooth initial data $(z_0, z_1) \in H^s(0,1) \cap H^{s-1}(0,1)$ (with $s \geq 1$) which only satisfy the conditions

$$\left\{ \begin{array}{ll} z_0(0) = (\partial_{xx})^j z_0(0) = 0, & j \leq s/2 - 1/4, \\ z_1(0) = (\partial_{xx})^k z_1(0) = 0, & k \leq s/2 - 3/4, \end{array} \right. \tag{5.24}$$

since they obviously are necessary to obtain smooth controlled trajectories of (5.16).

Such initial data do not belong *a priori* to the space $\mathcal{D}(A^s)$, which furthermore requires the following boundary conditions at $x = 1$:

$$\left\{ \begin{array}{ll} z_0(1) = (\partial_{xx})^j z_0(1) = 0, & j \leq s/2 - 1/4, \\ z_1(1) = (\partial_{xx})^k z_1(1) = 0, & k \leq s/2 - 3/4. \end{array} \right. \tag{5.25}$$

In this case, the two methods above (with or without the weight function) apply, since such pair belongs to the space $L^2(0,1) \times H^{-1}(0,1)$.

However, though these data are regular, none of the above methods will provide controls satisfying the compatibility condition $z_0(1) = v(0)$ for all (z_0, z_1) smooth: For the weighted method we propose in this article, this compatibility condition simply reads $z_0(1) = 0$, whereas as we have explained above, without the weight function, there is no way to impose *a priori* the value of the control at time $t = 0$. Thus, in general, the corresponding controlled trajectories of (5.16) will not be strong solutions of (5.16).

There are however other ways of building smooth controls with the right compatibility conditions, see [1], [21, Cor. 11.3.9]:

Proposition 5.5. *Assume $T > 2$. Let $s \in \mathbb{N}$ and $(z_0, z_1) \in H^s(0,1) \cap H^{s-1}(0,1)$ be satisfying (5.24).*

Then there exists a control function $v \in H^s(0,T)$ satisfying $(\partial_t)^j v(T) = 0$ for $j \leq s-1$ such that the corresponding controlled trajectory z of (5.16) is a strong solution and satisfies:

$$(z, z') \in \bigcap_{k=0}^{s} C^k([0,T]; H^{s-k}(0,1) \times H^{s-1-k}(0,1)). \tag{5.26}$$

Note that Proposition 5.5 yields the same result as Theorem 5.4 as far as the regularity property (5.15) is concerned.

However, the control provided in Proposition 5.5 is not constructed by an explicit minimization process independent of the functional setting we are considering, which was precisely the scope of the present work.

Note that the results of this article however yield another construction for the control in Proposition 5.5. For $(z_0, z_1) \in H^s(0,1) \times H^{s-1}(0,1)$ satisfying (5.24), we can proceed as follows: during a short period $[0, \delta]$, we smoothly steer the solution of (5.16) to $(z(\delta), z'(\delta)) \in \mathcal{D}(A^s)$. This can be done using the results in [13] by choosing a smooth v satisfying the compatibility conditions

$$\left\{ \begin{array}{ll} v^{(2j)}(0) = (\partial_{xx})^j z_0(1), & j \leq s/2 - 1/4, \\ v^{(2k+1)}(0) = (\partial_{xx})^k z_1(1), & k \leq s/2 - 3/4, \end{array} \right. , \qquad v^{(j)}(\delta) = 0, \quad \forall j \in \mathbb{N}.$$

Then we use the approach given in Theorem 5.4 to steer the solution of (5.16) from $(z(\delta), z'(\delta)) \in \mathcal{D}(A^s)$ to $(0,0)$.

Let us remark that, here, again the control function is designed by a specific method depending on the smoothness result we want to obtain.

We do not know if one can construct a method which yields intrinsically a control satisfying the "good" compatibility and regularity conditions (i.e. the ones required for the smoothness of the solutions [13]), whatever the regularity of the initial data to be controlled is.

6. **Further comments.** This newly developed method of ensuring regularity of controls for smooth data can be applied in various other contexts, and be further developed in several different manners. We discuss them below.

1. Our method does not apply when considering heat type equations, which generate a semi-group which cannot be extended as a group.

Indeed, our proof of Theorem 1.4, even the formal one, does not apply in that case, because of the lack of convenient estimates for the term

$$\int_0^T \eta'(t)\langle B^*Y(t), B^*Y'(t)\rangle_U \, dt$$

in (3.2), or (equivalently when $\eta \in C^2(\mathbb{R})$, see Remark 3.1)

$$-\int_0^T \eta''(t) \left\|B^*Y(t)\right\|_U^2 \, dt \tag{6.1}$$

in (3.18).

Thus, to obtain suitable estimates on

$$\int_0^T \eta(t) \left\|B^*Y'(t)\right\|_X^2 \, dt,$$

one should be able to bound (6.1) from above.

A first idea could be to choose η'' nonnegative near T (near 0, this term is easy to handle because of the regularization property of the heat equation), but this is not compatible with the conditions (1.3) on η. One could then try to compare η'' and η and use (1.11). However, again, it is impossible to get a function $\eta \in C_0^\infty(0,T)$ with $\eta \neq 0$, η''/η being bounded.

Such question might seem irrelevant for the heat equation at first glance since the heat equation is regularizing instantaneously. However, to our knowledge, the only available result for the regularity of the control for the heat equation states that the HUM control belongs to $L^2(0,T;U)$ and to $H^1(0,T-\varepsilon;U)$ (and even $C^\infty([0,T-\varepsilon];U)$) for each $\varepsilon > 0$, but its behavior near the time $t = T$ still has not been clarified.

This has also important impact when developing efficient numerical algorithms for computing controls for the heat equation because of the intrinsic ill-posedness of the problem at $t = T$, see [17].

2. The result stated in Theorem 1.4 implies in particular that the controlled trajectory is a strong solution. This is an important point. In particular, one can design from it and the results on the observability of space semi-discrete wave equations (see [10], [24], [6], ...) a numerical method which yields discrete controls for the space semi-discrete wave equation, which converge to an exact control for the continuous wave equation, and for which we can compute explicitly convergence rates when the initial datum to be controlled is more regular. This will be explained in a forthcoming article.

3. In the present paper, we have only treated the wave equation as the sole example. However, our techniques apply to many other conservative models such

as the Schrödinger and plate equations, the system of elasticity, etc., on which there is also extensive literature about their controllability properties, see [25].

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