

Uniformly exponentially stable approximations for a class of damped systems

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Abstract

We consider time semi-discrete approximations of a class of exponentially stable infinite dimensional systems modeling, for instance, damped vibrations. It has recently been proved that for time semi-discrete systems, due to high frequency spurious components, the exponential decay property may be lost as the time step tends to zero. We prove that adding a suitable numerical viscosity term in the numerical scheme, one obtains approximations that are uniformly exponentially stable. This result is then combined with previous ones on space semi-discretizations to derive similar results on fully-discrete approximation schemes. Our method is mainly based on a decoupling argument of low and high frequencies, the low frequency observability property for time semi-discrete approximations of conservative linear systems and the dissipativity of the numerical viscosity on the high frequency components. Our methods also allow to deal directly with stabilization properties of fully discrete approximation schemes without numerical viscosity, under a suitable CFL type condition on the time and space discretization parameters.

Résumé

Nous étudions diverses discrétisations en temps de systèmes de dimension infinie intervenant dans la modélisation de systèmes amortis. Récemment, il a été prouvé que les systèmes semi-discrétisés en temps peuvent ne pas satisfaire des propriétés de décroissance exponentielle de l'énergie, uniformément en le pas de temps, à cause de hautes fréquences parasites.

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Nous prouvons que l'addition d'une viscosité numérique permet de rétablir ces propriétés de décroissance de l'énergie. Notre méthode, combinée avec les résultats existants sur les discrétisations spatiales, nous permet d'étendre notre étude aux cas de schémas numériques totalement discrétisés. Nous nous basons principalement sur un découplage des basses et hautes fréquences, l'observabilité des basses fréquences pour des systèmes discrétisés en temps, et la dissipativité introduite par la viscosité numérique pour les hautes fréquences. Notre méthode s'adapte aussi à l'étude directe de schémas numériques totalement discrétisés, sans viscosité numérique, mais avec une condition CFL sur les paramètres de discrétisation en temps et en espace.

1 Introduction

Let X and Y be Hilbert spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Let $A : \mathcal{D}(A) \subset X \rightarrow X$ be a skew-adjoint operator with compact resolvent and $B \in \mathcal{L}(X, Y)$.

We consider the system described by

$$\dot{z} = Az - B^*Bz, \quad t \geq 0, \quad z(0) = z_0 \in X. \quad (1.1)$$

Here and henceforth, a dot ($\dot{\cdot}$) denotes differentiation with respect to time t . The element $z_0 \in X$ is the initial state, and $z(t)$ is the state of the system.

Most of the linear equations modeling the damped vibrations of elastic structures can be written in the form (1.1). Some other relevant models, as the damped Schrödinger equations, fit in this setting as well.

We define the energy of the solutions of system (1.1) by

$$E(t) = \frac{1}{2} \|z(t)\|_X^2, \quad t \geq 0, \quad (1.2)$$

which satisfies

$$\frac{dE}{dt}(t) = -\|Bz(t)\|_Y^2, \quad t \geq 0. \quad (1.3)$$

In this paper, we assume that system (1.1) is exponentially stable, that is there exist positive constants μ and ν such that any solution of (1.1) satisfies

$$E(t) \leq \mu E(0) \exp(-\nu t), \quad t \geq 0. \quad (1.4)$$

Our goal is to develop a theory allowing to get, as a consequence of (1.4), exponential stability results for time-discrete systems.

We start considering the following natural time-discretization scheme for the continuous system (1.1). For any $\Delta t > 0$, we denote by z^k the approximation of the solution z of system (1.1) at time $t_k = k\Delta t$, for $k \in \mathbb{N}$, and introduce the following *implicit midpoint* time discretization of system (1.1):

$$\begin{cases} \frac{z^{k+1} - z^k}{\Delta t} = A\left(\frac{z^k + z^{k+1}}{2}\right) - B^*B\left(\frac{z^k + z^{k+1}}{2}\right), & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (1.5)$$

As in (1.2), we can define the discrete energy by

$$E^k = \frac{1}{2} \|z^k\|_X^2, \quad k \in \mathbb{N}, \quad (1.6)$$

that satisfies the dissipation law

$$\frac{E^{k+1} - E^k}{\Delta t} = - \left\| B \left(\frac{z^k + z^{k+1}}{2} \right) \right\|_Y^2, \quad k \in \mathbb{N}. \quad (1.7)$$

The results in [28], in the context of the conservative wave equation, which is a particular instance of (1.1) with $B = 0$, show that we cannot expect in general to find positive constants μ_0 and ν_0 such that

$$E^k \leq \mu_0 E^0 \exp(-\nu_0 k \Delta t), \quad k \in \mathbb{N}, \quad (1.8)$$

holds for any solution of (1.9) uniformly with respect to $\Delta t > 0$. Indeed, it was proved in [28] that spurious high-frequency modes may arise when discretizing in time the wave equation, which propagate with an arbitrarily small velocity and that, when the operator B is localized somewhere in the domain where waves propagate, cannot be observed uniformly with respect to Δt . This constitutes an obstruction to the stabilization property (1.8) as well.

Therefore, in order to get a uniform decay, it seems natural to add in system (1.5) a suitable extra numerical viscosity term to damp these high-frequency spurious components. When doing it at the right scale, the new system we obtain is as follows

$$\begin{cases} \frac{\tilde{z}^{k+1} - z^k}{\Delta t} = A \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) - B^* B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{z^{k+1} - \tilde{z}^{k+1}}{\Delta t} = (\Delta t)^2 A^2 z^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (1.9)$$

This system introduces, indeed, numerical viscosity at the right scale since the spurious high-frequency modes arising in [28] precisely correspond to solutions for which $(\Delta t)A$ is of unit order or more.

Let us also remark that system (1.9) can be rewritten as

$$\begin{aligned} \frac{z^{k+1} - z^k}{\Delta t} &= A \left(\frac{z^k + z^{k+1}}{2} \right) - B^* B \left(\frac{z^k + z^{k+1}}{2} \right) + (\Delta t)^2 A^2 z^{k+1} \\ &\quad - \frac{(\Delta t)^3}{2} A^3 z^{k+1} + \frac{(\Delta t)^3}{2} B^* B A^2 z^{k+1}, \end{aligned} \quad (1.10)$$

which is consistent with system (1.1).

To motivate system (1.9), one can compare it with the time continuous system

$$\dot{z} = Az - B^* B z + (\Delta t)^2 A^2 z, \quad (1.11)$$

which generates the semigroup $S(t) = \exp(t(A - B^*B + (\Delta t)^2 A^2))$. In (1.9), \tilde{z}^{k+1} corresponds to an approximation of $\exp(\Delta t(A - B^*B))z^k$ and z^{k+1} to an approximation of $\exp((\Delta t)^3 A^2)\tilde{z}^{k+1}$. Doing this, z^{k+1} is an approximation of $S(\Delta t)z^k \simeq \exp((\Delta t)^3 A^2)\exp(\Delta t(A - B^*B))z^k$. Thus, system (1.9) can be viewed as an alternating direction time-discrete approximation of (1.11), for which dissipation properties have been derived in the recent article [14].

Note that this numerical scheme is based on the decomposition of the operator $A - B^*B + (\Delta t)^2 A^2$ into its conservative and dissipative parts, that we treat differently. Indeed, the midpoint scheme is appropriate for conservative systems since it preserves the norm conservation property. This is not the case for dissipative systems, since midpoint schemes do not preserve the dissipative properties of high frequency solutions. Therefore, we rather use an implicit Euler scheme, which efficiently preserves these dissipative properties.

In Subsection 2.3, we will consider other possible discretization schemes, variants of (1.9), which still preserve the conservative properties of $\exp(tA)$ and the dissipative effects of $\exp(t(\Delta t)^2 A^2)$. We will also present other possible choices for the numerical viscosity term.

The energy of (1.9), still defined by (1.6), now satisfies

$$\begin{cases} \tilde{E}^{k+1} = E^k - \Delta t \left\| B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) \right\|_Y^2, & k \in \mathbb{N}, \\ E^{k+1} + (\Delta t)^3 \|Az^{k+1}\|_X^2 + \frac{(\Delta t)^6}{2} \|A^2 z^{k+1}\|_X^2 = \tilde{E}^{k+1}, & k \in \mathbb{N}. \end{cases} \quad (1.12)$$

Putting these identities together, we get

$$E^{k+1} + (\Delta t)^3 \|Az^{k+1}\|_X^2 + \frac{(\Delta t)^6}{2} \|A^2 z^{k+1}\|_X^2 + \Delta t \left\| B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) \right\|_Y^2 = E^k. \quad (1.13)$$

The convergence of the solutions of (1.9) towards those of the original system (1.1) when $\Delta t \rightarrow 0$ holds in a suitable topology. Indeed, the scheme is stable in view of (1.12), and its consistency is obvious. Therefore its convergence (in the classical sense of numerical analysis) is guaranteed: When $\Delta t \rightarrow 0$, the solutions $z_{\Delta t}$ of (1.9), extended in a standard way as piecewise affine functions on \mathbb{R}_+ , converge to the solution z of (1.1) in $L^2((0, T); X)$.

The main result of this paper is that system (1.9) enjoys a uniform stabilization property. It reads as follows:

Theorem 1.1. *Assume that system (1.1) is exponentially stable, i.e. satisfies (1.4) with constants μ and ν , and that $B \in \mathfrak{L}(X, Y)$.*

Then there exist two positive constants μ_0 and ν_0 depending only on μ , ν and $\|B\|_{\mathfrak{L}(X, Y)}$ such that any solution of (1.9) satisfies (1.8) with constants μ_0 and ν_0 uniformly with respect to the discretization parameter $\Delta t > 0$.

Our strategy is based on the fact that the uniform exponential decay properties of the energy for systems (1.1) and (1.9) respectively are equivalent to

uniform observability properties for the conservative system

$$\dot{y} = Ay, \quad t \in \mathbb{R}, \quad y(0) = y_0 \in X, \quad (1.14)$$

and its time semi-discrete viscous version

$$\begin{cases} \frac{\tilde{u}^{k+1} - u^k}{\Delta t} = A\left(\frac{u^k + \tilde{u}^{k+1}}{2}\right), & k \in \mathbb{N}, \\ \frac{u^{k+1} - \tilde{u}^{k+1}}{\Delta t} = (\Delta t)^2 A^2 u^{k+1}, & k \in \mathbb{N}, \\ u^0 = u_0, \end{cases} \quad (1.15)$$

At the continuous level the observability property consists in the existence of a time $T > 0$ and a positive constant $k_T > 0$ such that

$$k_T \|y_0\|_X^2 \leq \int_0^T \|By(t)\|_Y^2 dt, \quad (1.16)$$

for every solution of (1.14) (see [16] and Lemma 2.3 below).

A similar argument can be applied to the semi-discrete system (1.9). Namely, the uniform exponential decay (1.8) of the energy of solutions of (1.9) is equivalent to the following observability inequality: there exist positive constants T and c such that, for any $\Delta t > 0$, every solution u of (1.15) satisfies

$$\begin{aligned} c \|u_0\|_X^2 \leq \Delta t \sum_{k\Delta t \in [0, T]} \|Bu^k\|_Y^2 + \Delta t \sum_{k\Delta t \in [0, T]} (\Delta t)^2 \|Au^{k+1}\|_X^2 \\ + \Delta t \sum_{k\Delta t \in [0, T]} (\Delta t)^5 \|A^2 u^{k+1}\|_X^2. \end{aligned} \quad (1.17)$$

Note that, since the operator $(\Delta t)^2 A^2$ is unbounded, we cannot use the standard arguments in [16], which state the equivalence between the uniform exponential decay of the energy for (1.9) and uniform observability properties such as (1.17) for solutions of the conservative system

$$\frac{y^{k+1} - y^k}{\Delta t} = A\left(\frac{y^k + y^{k+1}}{2}\right), \quad k \in \mathbb{N}, \quad y^0 = y_0, \quad (1.18)$$

or, equivalently,

$$\frac{\tilde{y}^{k+1} - y^k}{\Delta t} = A\left(\frac{y^k + \tilde{y}^{k+1}}{2}\right), \quad y^{k+1} = \tilde{y}^{k+1} \quad k \in \mathbb{N}, \quad y^0 = y_0. \quad (1.19)$$

Let us now give some insights of the proof of (1.17) for solutions of (1.15). The main idea is to decompose the solution u of (1.15) into its low and high frequency parts, that we handle separately. We first use a uniform observability inequality proven in [12] for solutions of (1.18) in a filtered space, which yields a partial observability inequality for the low frequency components of solutions of

(1.15). Second, using the explicit dissipativity of (1.15) at high frequencies, we deduce a partial observability inequality for the high frequency components. Together, these two partial observability inequalities yield the needed observability property (1.17) leading to the uniform exponential decay result.

Our results yield also uniform exponential decay rates for families of equations of the form (1.1), with pairs of operators (A, B) , within a class in which the exponential decay rate of the continuous system (1.1) is known to be uniform.

One of the interesting applications of this fact is that our results can be combined with the existing ones derived for *space* semi-discrete approximation schemes of various PDE models entering in the abstract frame (1.1) as [5, 6, 13, 11, 24, 27, 23] (see [32] for more references). Indeed, knowing that some *space* semi-discrete approximation schemes of (1.1) are exponentially stable, uniformly with respect to the space mesh size, this fact, combined with Theorem 1.1, allows deducing uniform exponential decay properties for the corresponding *fully* discrete approximation schemes.

Our methods can also be applied directly to fully discrete approximation schemes under a suitable CFL type condition on the time and space discretization parameters. This can be done without adding a numerical viscosity term since the CFL condition by itself rules out the high frequency components. As we will see in the examples, this CFL condition might be very strong and yield severe restrictions, which do not appear when adding numerical viscosity as in (1.9) (see Theorem 1.1).

As said above, these approaches require observability properties such as (1.16) to hold uniformly (with respect to the space discretization parameter) for solutions of the space semi-discrete schemes for *any* initial data. However, it often occurs in applications that the space semi-discrete schemes are uniformly observable only for *filtered* initial data corresponding to low frequencies (see [18, 31, 13, 32]). We therefore adapt our methods to this case, and prove that adding a numerical viscosity term which is strong enough to efficiently damp out the high frequency components, one obtains uniformly exponentially stable fully discrete approximation schemes. When doing this, we also prove that, when considering space semi-discrete approximation schemes that are uniformly observable in filtered low-frequency subspaces, adding a suitable numerical viscosity term makes the space semi-discrete approximation schemes uniformly (with respect to the space discretization parameter) exponentially stable. This generalizes the results [27, 25, 13], where particular instances of viscosity terms have been used. This also generalizes [14], where it was proven that if (1.1) is exponentially stable, then adding a suitable viscosity term does not deteriorate the exponential stability of solutions.

In this sense, the two approaches presented in this article are complementary.

Note however that we cannot apply these methods when the damped operator B is not bounded, as in [26], where the wave equation is damped by a

feedback law on the boundary. Dealing with unbounded damping operators B needs further work.

The results in this paper on the uniform stabilization of time-discrete approximation schemes with numerical viscosity term are related to several previous ones. The following ones are worth mentioning. In [27, 26, 23, 13] numerical viscosity is added to guarantee the uniform exponential decay for finite-difference space semi-discrete approximation schemes of the wave equation. Similar results, in an abstract setting, with a stronger viscous damping term, have been proved in [25]. Similar techniques have also been employed to obtain uniform dispersive estimates for numerical approximation schemes to Schrödinger equations in [17].

Let us also mention the recent work [12], where observability issues were discussed for time and fully discrete approximation schemes of (1.18). The results of [12] will be used in the present work to derive observability properties for system (1.18) within the class of conveniently filtered low frequency data. Since they constitute a key point of our proofs, we recall them in Section 2.

Despite all the existing literature, this article seems to be the first one to provide a systematic way of transferring exponential decay properties from the continuous to the time-discrete setting.

The outline of this paper is as follows.

In Section 2, we recall the results of [12] and prove Theorem 1.1. Section 3 is devoted to explain how we can deduce uniform stabilization results for the fully discrete approximation schemes combining Theorem 1.1 and known results on uniform stabilization for space semi-discrete approximations. We also present an abstract setting specifically designed to address stabilization issues for fully discrete approximation schemes without viscosity. In Section 4, we present some concrete applications in the context of the wave equation for which several uniformly exponentially stable schemes are derived. Finally, some further comments and open problems are collected in Section 5.

2 Stabilization of time-discrete systems

This section is organized as follows. We first recall the results of [12] on the observability of the time-discrete conservative system (1.18). Second, we prove Theorem 1.1. Third, we present several variants of the numerical scheme (1.9) that lead to uniform exponential decay results similar to Theorem 1.1.

2.1 Observability of time-discrete conservative systems

We first need to introduce some notations.

Since A is a skew-adjoint operator with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : j \in \mathbb{N}\}$, where $(\mu_j)_{j \in \mathbb{N}}$ is a sequence of real

numbers such that $|\mu_j| \rightarrow \infty$ when $j \rightarrow \infty$. Set $(\Phi_j)_{j \in \mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j \in \mathbb{N}}$, that is

$$A\Phi_j = i\mu_j\Phi_j. \quad (2.1)$$

Moreover, define

$$\mathcal{C}_s(A) = \text{span} \{ \Phi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s \}. \quad (2.2)$$

The following was proved in [12]:

Theorem 2.1. *Assume that $B \in \mathfrak{L}(\mathcal{D}(A), Y)$, that is*

$$\|Bz\|_Y^2 \leq C_B^2 \left(\|Az\|_X^2 + \|z\|_X^2 \right), \quad \forall z \in \mathcal{D}(A), \quad (2.3)$$

and that A and B satisfy the following hypothesis:

$$\left\{ \begin{array}{l} \text{There exist constants } M, m > 0 \text{ such that} \\ M^2 \|(i\omega I - A)y\|^2 + m^2 \|By\|_Y^2 \geq \|y\|^2, \quad \forall \omega \in \mathbb{R}, \quad y \in \mathcal{D}(A). \end{array} \right. \quad (2.4)$$

Then, for any $\delta > 0$, there exists T_δ such that for any $T > T_\delta$, there exists a positive constant $k_{T,\delta}$, independent of Δt , that depends only on m, M, C_B, T and δ , such that for $\Delta t > 0$ small enough, the solution y^k of (1.18) satisfies

$$k_{T,\delta} \|y^0\|_X^2 \leq \Delta t \sum_{k\Delta t \in [0, T]} \left\| B \left(\frac{y^k + y^{k+1}}{2} \right) \right\|_Y^2, \quad \forall y^0 \in \mathcal{C}_{\delta/\Delta t}(A). \quad (2.5)$$

Moreover, T_δ can be taken to be such that

$$T_\delta = \pi \left[M^2 \left(1 + \frac{\delta^2}{4} \right)^2 + m^2 C_B^2 \frac{\delta^4}{16} \right]^{1/2}, \quad (2.6)$$

where C_B is as in (2.3).

In the sequel, when there is no ambiguity, we will use the simplified notation $\mathcal{C}_{\delta/\Delta t}$ instead of $\mathcal{C}_{\delta/\Delta t}(A)$.

Note that if $B \in \mathfrak{L}(X, Y)$, then the operator B is also in $\mathfrak{L}(\mathcal{D}(A), Y)$, and (2.3) holds. Thus the assumption (2.3) is satisfied in the abstract setting we are working on.

Hypothesis (2.4) is the so-called resolvent estimate, which has been proved in [4, 22] to be equivalent to the continuous observability inequality (1.16) for the conservative system (1.14) for suitable positive constants T and k_T , which turns out to be equivalent to the exponential decay property (1.4) for the continuous damped system (1.1).

To be more precise, it was proved in [22] that if the operator B is bounded, then the observability property (1.16) implies hypothesis (2.4) with

$$m = \sqrt{\frac{2T}{k_T}}, \quad M = T \|B\|_{\mathfrak{L}(X, Y)} \sqrt{\frac{T}{2k_T}}, \quad (2.7)$$

where k_T is as in (1.16).

Observe that Theorem 2.1 guarantees that, as soon as the observability inequality (1.16) holds for the continuous system (1.14), then its time-discrete counterpart holds uniformly for the solutions of the time discrete systems (1.18) within the class of filtered solutions $\mathcal{C}_{\delta/\Delta t}(A)$ involving only the low-frequency components corresponding to the eigenvalues $|\mu_i| \leq \delta/\Delta t$. This fact will play a key role in the proof of Theorem 1.1.

2.2 Proof of Theorem 1.1

In this Subsection, we assume that system (1.1) is exponentially stable and that $B \in \mathfrak{L}(X, Y)$, i.e. there exists a constant K_B such that

$$\|Bz\|_Y \leq K_B \|z\|_X, \quad \forall z \in X. \quad (2.8)$$

The proof is divided into several steps. First, we write carefully the energy identity for z solution of (1.9). Second, we observe that the resolvent estimate (2.4) holds, from which we deduce that (2.5) holds as well for solutions of system (1.18) in the filtered space $\mathcal{C}_{\delta/\Delta t}$. Third, we derive the observability inequality (1.17) for solutions of (1.15). Finally, we deduce that the time-discrete systems (1.9) are uniformly exponentially stable.

2.2.1 The energy identity

Lemma 2.2. *For any $\Delta t > 0$ and $z^0 \in X$, the solution z of (1.9) satisfies*

$$\begin{aligned} \|z^{k_2}\|_X^2 + 2\Delta t \sum_{j=k_1}^{k_2-1} \left\| B \left(\frac{z^j + \tilde{z}^{j+1}}{2} \right) \right\|_Y^2 + 2\Delta t \sum_{j=k_1}^{k_2-1} (\Delta t)^2 \|Az^{j+1}\|_X^2 \\ + \Delta t \sum_{j=k_1}^{k_2-1} (\Delta t)^5 \|A^2 z^{j+1}\|_X^2 = \|z^{k_1}\|_X^2, \quad \forall k_1 < k_2. \end{aligned} \quad (2.9)$$

The proof simply consists in summing the identities in (1.13) from $k = l_1$ to $k = l_2 - 1$. Especially, it implies that $\|z^k\|_X^2$ is decreasing, which confirms the dissipativity of the time-discrete system.

2.2.2 The resolvent estimate

Lemma 2.3. *Under the assumptions of Theorem 1.1, the resolvent estimate (2.4) holds, with constants m and M that depend only on μ and ν given by (1.4).*

Proof. The proof is based on [16].

Since system (1.1) is exponentially stable, inequality (1.4) holds. In particular, there exists a positive constant $T > 0$ such that $2E(T) \leq E(0)$. But

equality (1.3) implies that any solution z of (1.1) satisfies

$$E(T) + \int_0^T \|Bz(t)\|_Y^2 dt = E(0),$$

and therefore that

$$\int_0^T \|Bz(t)\|_Y^2 dt \geq \frac{1}{4} \|z_0\|_X^2. \quad (2.10)$$

Let us now show that, as a consequence of this, (1.16) holds for the solution of (1.14) as well.

Given $y_0 \in X$, let y and z be the solutions of (1.14) and (1.1) with initial data y_0 . Then $w = z - y$ satisfies

$$\dot{w} = Aw - B^*Bw - B^*By, \quad t \in \mathbb{R}, \quad w(0) = 0.$$

Multiplying by w and integrating in time, we obtain that

$$\begin{aligned} \frac{1}{2} \|w(T)\|_X^2 + \int_0^T \|Bw(t)\|_Y^2 dt &\leq \int_0^T | \langle Bw(t), By(t) \rangle_Y | dt \\ &\leq \frac{1}{2} \int_0^T \left(\|Bw(t)\|_Y^2 + \|By(t)\|_Y^2 \right) dt. \end{aligned}$$

In particular,

$$\int_0^T \|Bw(t)\|_Y^2 dt \leq \int_0^T \|By(t)\|_Y^2 dt.$$

This inequality, combined with (2.10), leads to

$$\begin{aligned} \frac{1}{4} \|y_0\|_X^2 &\leq \int_0^T \|Bz(t)\|_Y^2 dt \\ &\leq 2 \int_0^T \left(\|Bw(t)\|_Y^2 + \|By(t)\|_Y^2 \right) dt \\ &\leq 3 \int_0^T \|By(t)\|_Y^2 dt. \end{aligned} \quad (2.11)$$

It follows that (1.16) holds, and the resolvent estimate (2.4) holds with m and M as in (2.7), according to the results in [22]. \square

Applying Theorem 2.1, for any $\delta > 0$, choosing a time $T^* > T_\delta$ (where T_δ is defined in (2.6)) there exists a positive constant $k_{T^*,\delta}$ such that inequality (2.5) holds for any solution y of (1.18) with $y^0 \in \mathcal{C}_{\delta/\Delta t}$. In the sequel, we fix a positive number $\delta > 0$ (for instance $\delta = 1$), and $T^* = 2T_\delta$.

2.2.3 Uniform observability inequalities

Lemma 2.4. *There exists a constant $c > 0$ such that (1.17) holds with $T = T^*$ for all solutions u of (1.15) uniformly with respect to Δt .*

Proof. In the sequel we deal with the solutions u of (1.15), for which we prove (1.17) for $T = T^* = 2T_\delta$. The proof presented below is inspired in previous work [14] from the authors, where similar arguments have been used in the continuous setting.

As said in the introduction, we decompose the solution u of (1.15) into its low and high frequency parts. To be more precise, we consider

$$u_l = \pi_{\delta/\Delta t} u, \quad u_h = (I - \pi_{\delta/\Delta t})u, \quad (2.12)$$

where $\delta > 0$ is the positive number that have been chosen above, and $\pi_{\delta/\Delta t}$ is the orthogonal projection on $\mathcal{C}_{\delta/\Delta t}$ defined in (2.2). Here the notation u_l and u_h stands for the low and high frequency components, respectively.

Note that both u_l and u_h are solutions of (1.15).

Besides, u_h lies in the space $\mathcal{C}_{\delta/\Delta t}^\perp$, in which the following property holds:

$$\Delta t \|Ay\|_X \geq \delta \|y\|_X, \quad \forall y \in \mathcal{C}_\delta^\perp. \quad (2.13)$$

The low frequencies. In a first step, we compare u_l with y_l solution of (1.18) with initial data $y_l(0) = u_l(0)$. Now, set $w_l = u_l - y_l$. From (2.5), which is valid for solutions of (1.18) with initial data in $\mathcal{C}_{\delta/\Delta t}$, we get

$$\begin{aligned} k_{T^*,\delta} \|u_l^0\|_X^2 = k_{T^*,\delta} \|y_l^0\|_X^2 &\leq 2\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_l^k + \tilde{w}_l^{k+1}}{2}\right) \right\|_Y^2 \\ &\quad + 2\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{w_l^k + \tilde{w}_l^{k+1}}{2}\right) \right\|_Y^2. \end{aligned} \quad (2.14)$$

In the sequel, to simplify the notation, $c > 0$ will denote a positive constant that may change from line to line, but which does not depend on Δt .

Let us then estimate the last term in the right hand side of (2.14). To this end, we write the equation satisfied by w_l , which can be deduced from (1.18) and (1.15):

$$\begin{cases} \frac{\tilde{w}_l^{k+1} - w_l^k}{\Delta t} = A\left(\frac{w_l^k + \tilde{w}_l^{k+1}}{2}\right), & k \in \mathbb{N}, \\ \frac{w_l^{k+1} - \tilde{w}_l^{k+1}}{\Delta t} = (\Delta t)^2 A^2 u_l^{k+1}, & k \in \mathbb{N}, \\ w_l^0 = 0. \end{cases} \quad (2.15)$$

The energy estimates for w_l give

$$\begin{cases} \|\tilde{w}_l^{k+1}\|_X^2 = \|w_l^k\|_X^2, \\ \|w_l^{k+1}\|_X^2 = \|\tilde{w}_l^{k+1}\|_X^2 - 2(\Delta t)^3 \langle A u_l^{k+1}, A\left(\frac{\tilde{w}_l^{k+1} + w_l^{k+1}}{2}\right) \rangle_X. \end{cases} \quad (2.16)$$

Note that w_l^k and \tilde{w}_l^{k+1} belong to $\mathcal{C}_{\delta/\Delta t}$ for all $k \in \mathbb{N}$, since u_l and y_l both belong to $\mathcal{C}_{\delta/\Delta t}$. Therefore, the energy estimates for w_l lead, for $k \in \mathbb{N}$, to

$$\begin{aligned} \|w_l^k\|_X^2 &= -2\Delta t \sum_{j=1}^k (\Delta t)^2 \langle Au_l^j, A\left(\frac{w_l^j + \tilde{w}_l^{j+1}}{2}\right) \rangle_X \\ &\leq \Delta t \sum_{j=1}^k (\Delta t)^2 \|Au_l^j\|_X^2 + \delta^2 \Delta t \sum_{j=1}^k \left\| \frac{w_l^j + \tilde{w}_l^{j+1}}{2} \right\|_X^2 \\ &\leq \Delta t \sum_{j=1}^k (\Delta t)^2 \|Au_l^j\|_X^2 + \delta^2 \Delta t \sum_{j=0}^k \|w_l^j\|_X^2, \end{aligned}$$

where we used the first line of (2.16).

Grönwall's Lemma applies and allows to deduce from (2.14) and the fact that the operator B is bounded, the existence of a positive c independent of Δt , such that

$$c \|u_l^0\|_X^2 \leq \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_l^k + \tilde{u}_l^{k+1}}{2}\right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Au_l^k\|_X^2.$$

Besides,

$$\begin{aligned} \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_l^k + \tilde{u}_l^{k+1}}{2}\right) \right\|_Y^2 &\leq 2\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_l^k + \tilde{u}_l^{k+1}}{2}\right) \right\|_Y^2 \\ &\quad + 2\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_h^k + \tilde{u}_h^{k+1}}{2}\right) \right\|_Y^2 \end{aligned}$$

and, since u_h^k and \tilde{u}_h^{k+1} belong to $\mathcal{C}_{\delta/\Delta t}^\perp$ for all k , we get from (2.13) that

$$\begin{aligned} \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_h^k + \tilde{u}_h^{k+1}}{2}\right) \right\|_Y^2 &\leq K_B^2 \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| \frac{u_h^k + \tilde{u}_h^{k+1}}{2} \right\|_X^2 \\ &\leq K_B^2 \Delta t \sum_{k\Delta t \in [0, T^*]} \|u_h^k\|_X^2 \leq \frac{K_B^2}{\delta^2} \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Au_h^k\|_X^2 + K_B^2 \Delta t \|u_h^0\|_X^2, \end{aligned}$$

since, from the first line of (1.15),

$$\|\tilde{u}_h^{k+1}\|_X^2 = \|u_h^k\|_X^2, \quad \forall k \in \mathbb{N}.$$

It follows that there exists $c > 0$ independent of Δt such that

$$\begin{aligned} c \|u_l^0\|_X^2 &\leq \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{u_l^k + \tilde{u}_l^{k+1}}{2}\right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Au_l^k\|_X^2 \\ &\quad + \Delta t \|u_h^0\|_X^2. \end{aligned} \quad (2.17)$$

The high frequencies. We now discuss briefly the decay properties of solutions u_h of (1.15) with initial data $u_h^0 \in \mathcal{C}_{\delta/\Delta t}^\perp$. In this case, we easily check that for all $k \in \mathbb{N}$, $u_h^k \in \mathcal{C}_{\delta/\Delta t}^\perp$. But, as in (1.13), we have

$$\begin{aligned} \|(I - (\Delta t)^3 A^2)u_h^{k+1}\|_X^2 &= \|u_h^{k+1}\|_X^2 + 2(\Delta t)^3 \|Au_h^{k+1}\|_X^2 \\ &\quad + (\Delta t)^6 \|A^2 u_h^{k+1}\|_X^2 = \|\tilde{u}_h^{k+1}\|_X^2 = \|u_h^k\|_X^2, \quad k \in \mathbb{N}. \end{aligned} \quad (2.18)$$

Due to the property (2.13), we get

$$(1 + 2(\Delta t)\delta^2) \|u_h^{k+1}\|_X^2 \leq \|u_h^k\|_X^2.$$

We deduce that

$$\|u_h^{k+1}\|_X^2 \leq \frac{1}{1 + 2(\Delta t)\delta^2} \|u_h^k\|_X^2, \quad k \in \mathbb{N},$$

which implies

$$\|u_h^k\|_X^2 \leq \left(\frac{1}{1 + 2(\Delta t)\delta^2} \right)^k \|u_h^0\|_X^2, \quad k \in \mathbb{N}. \quad (2.19)$$

Especially, taking $k^* = \lceil T^*/\Delta t \rceil$, we get a constant $\gamma < 1$ independent of $\Delta t > 0$ such that

$$\|u_h^{k^*}\|_X^2 \leq \gamma \|u_h^0\|_X^2.$$

Since we also have from (2.18) that, for $k \in \mathbb{N}$,

$$\|u_h^k\|_X^2 + 2\Delta t \sum_{j=0}^{k-1} (\Delta t)^2 \|Au_h^{j+1}\|_X^2 + \Delta t \sum_{j=0}^{k-1} (\Delta t)^5 \|A^2 u_h^{j+1}\|_X^2 = \|u_h^0\|_X^2,$$

taking $k = k^* = \lceil T^*/\Delta t \rceil$, we deduce the existence of a positive constant C , which depends only on T^* and δ (namely $C = (1 - \gamma)/2$), such that

$$C \|u_h^0\|_X^2 \leq \Delta t \sum_{j=0}^{k^*-1} (\Delta t)^2 \|Au_h^{j+1}\|_X^2 + \Delta t \sum_{j=0}^{k^*-1} (\Delta t)^5 \|A^2 u_h^{j+1}\|_X^2, \quad (2.20)$$

holds uniformly with respect to $\Delta t > 0$ for any solution of (1.15) with initial data $u^0 \in \mathcal{C}_{\delta/\Delta t}^\perp$.

Combining (2.17) and (2.20) yields Lemma 2.4, since u_h and u_l lie in orthogonal spaces with respect to the scalar products $\langle \cdot, \cdot \rangle_X$ and $\langle A \cdot, A \cdot \rangle_X$. \square

2.2.4 Proof of Theorem 1.1

Proof of Theorem 1.1. Here we follow the argument in [16, 14].

We decompose z solution of (1.9) as $z = u + w$ where u is the solution of the system (1.15) with initial data $u^0 = z^0$. Applying Lemma 2.4 to $u = z - w$, we get

$$\begin{aligned}
c \|z^0\|_X^2 &\leq 2 \left(\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Az^{k+1}\|_X^2 \right. \\
&\quad \left. + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^5 \|A^2 z^{k+1}\|_X^2 \right) + 2 \left(\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B \left(\frac{w^k + \tilde{w}^{k+1}}{2} \right) \right\|_Y^2 \right. \\
&\quad \left. + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Aw^{k+1}\|_X^2 + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^5 \|A^2 w^{k+1}\|_X^2 \right). \quad (2.21)
\end{aligned}$$

Below, we bound the terms in the right hand-side of (2.21) involving w by the ones involving z .

The function w satisfies

$$\begin{cases} \frac{\tilde{w}^{k+1} - w^k}{\Delta t} = A \left(\frac{w^k + \tilde{w}^{k+1}}{2} \right) - B^* B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{w^{k+1} - \tilde{w}^{k+1}}{\Delta t} = (\Delta t)^2 A^2 w^{k+1}, & k \in \mathbb{N}, \\ w^0 = 0. \end{cases} \quad (2.22)$$

Multiplying the first line of (2.22) by $w^k + \tilde{w}^{k+1}$ and taking the norm of each member in the second one, we get the following energy identities for $k \in \mathbb{N}$:

$$\begin{aligned}
\|\tilde{w}^{k+1}\|_X^2 &= \|w^k\|_X^2 - 2\Delta t \langle B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), B \left(\frac{w^k + \tilde{w}^{k+1}}{2} \right) \rangle_Y, \\
\|w^{k+1}\|_X^2 + 2(\Delta t)^3 \|Aw^{k+1}\|_X^2 + (\Delta t)^6 \|A^2 w^{k+1}\|_X^2 &= \|\tilde{w}^{k+1}\|_X^2. \quad (2.23)
\end{aligned}$$

In particular, this gives

$$\begin{aligned}
\|w^{k+1}\|_X^2 + 2(\Delta t)^3 \|Aw^{k+1}\|_X^2 + (\Delta t)^6 \|A^2 w^{k+1}\|_X^2 \\
+ 2\Delta t \langle B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), B \left(\frac{w^k + \tilde{w}^{k+1}}{2} \right) \rangle_Y = \|w^k\|_X^2.
\end{aligned}$$

Using that B is bounded, we get

$$\begin{aligned}
&\|w^k\|_X^2 + 2(\Delta t) \sum_{j=0}^{k-1} (\Delta t)^2 \|Aw^{j+1}\|_X^2 + (\Delta t) \sum_{j=0}^{k-1} (\Delta t)^5 \|A^2 w^{j+1}\|_X^2 \\
&\leq \Delta t \sum_{j=0}^{k-1} \left\| B \left(\frac{z^j + \tilde{z}^{j+1}}{2} \right) \right\|_Y^2 + \frac{K_B^2}{2} (\Delta t) \sum_{j=0}^{k-1} \left(\|w^j\|_X^2 + \|\tilde{w}^{j+1}\|_X^2 \right). \quad (2.24)
\end{aligned}$$

But the second line in (2.23) gives that

$$\begin{aligned} \Delta t \sum_{j=0}^{k-1} \|\tilde{w}^{j+1}\|_X^2 &= \Delta t \sum_{j=0}^{k-1} \|w^{j+1}\|_X^2 + 2(\Delta t)^2 \sum_{j=0}^{k-1} (\Delta t)^2 \|Aw^{j+1}\|_X^2 \\ &\quad + (\Delta t)^2 \sum_{j=0}^{k-1} (\Delta t)^5 \|A^2 w^{j+1}\|_X^2. \end{aligned} \quad (2.25)$$

Therefore, for Δt small enough, (2.24) gives

$$\begin{aligned} \|w^k\|_X^2 + \Delta t \sum_{j=0}^{k-1} (\Delta t)^2 \|Aw^{j+1}\|_X^2 + \frac{\Delta t}{2} \sum_{j=0}^{k-1} (\Delta t)^5 \|A^2 w^{j+1}\|_X^2 \\ \leq \Delta t \sum_{j=0}^{k-1} \left\| B\left(\frac{z^j + \tilde{z}^{j+1}}{2}\right) \right\|_Y^2 + K_B^2 \Delta t \sum_{j=0}^{k-1} \|w^j\|_X^2. \end{aligned} \quad (2.26)$$

Grönwall's inequality then gives a constant G , that depends only on K_B and T^* , such that

$$\begin{aligned} \sup_{k\Delta t \in [0, T^*]} \left\{ \|w^k\|_X^2 \right\} + \Delta t \sum_{k\Delta t \in]0, T^*]} (\Delta t)^2 \|Aw^{k+1}\|_X^2 \\ + \Delta t \sum_{k\Delta t \in]0, T^*]} (\Delta t)^5 \|A^2 w^{k+1}\|_X^2 \leq G \Delta t \sum_{j=0}^{k-1} \left\| B\left(\frac{z^j + \tilde{z}^{j+1}}{2}\right) \right\|_Y^2. \end{aligned}$$

Combined with (2.25), we get that

$$\begin{aligned} \Delta t \sum_{k\Delta t \in]0, T^*]} \left(\|w^k\|_X^2 + \|\tilde{w}^{k+1}\|_X^2 \right) + \Delta t \sum_{k\Delta t \in]0, T^*]} (\Delta t)^2 \|Aw^{k+1}\|_X^2 \\ + \Delta t \sum_{k\Delta t \in]0, T^*]} (\Delta t)^5 \|A^2 w^{k+1}\|_X^2 \leq G \Delta t \sum_{j=0}^{k-1} \left\| B\left(\frac{z^j + \tilde{z}^{j+1}}{2}\right) \right\|_Y^2. \end{aligned} \quad (2.27)$$

Combining (2.21), (2.27) and the fact that B is bounded, we get the existence of a constant c such that

$$\begin{aligned} c \|z^0\|_X^2 \leq \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B\left(\frac{z^k + \tilde{z}^{k+1}}{2}\right) \right\|_Y^2 + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^2 \|Az^{k+1}\|_X^2 \\ + \Delta t \sum_{k\Delta t \in [0, T^*]} (\Delta t)^5 \|A^2 z^{k+1}\|_X^2. \end{aligned} \quad (2.28)$$

Finally, using the energy identity (2.9), we get that

$$\left\| z^{T^*/\Delta t} \right\|_X^2 \leq (1 - c) \|z^0\|_X^2. \quad (2.29)$$

The semi-group property then implies Theorem 1.1. \square

Remark 2.5. Our proof of Theorem 1.1 needs to introduce a parameter $\delta > 0$, that we can choose arbitrarily. It would be natural to look for the choice of $\delta > 0$ yielding the best estimate in the decay rate of the energy. However, our method, based on the arguments of [16], does not give a good approximation of the decay rate of the energy. This is a drawback of this method, which also appears in the continuous setting.

2.3 Some variants

Other discretization schemes. Other discretization schemes for system (1.1) are possible. For instance, we can consider the following one:

$$\begin{cases} \frac{z_1^{k+1} - z^k}{\Delta t} = A\left(\frac{z^k + z_1^{k+1}}{2}\right), & k \in \mathbb{N}, \\ \frac{z^{k+1} - z_1^{k+1}}{\Delta t} = -B^*Bz^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (2.30)$$

As for system (1.5), the results of [28], in the context of the conservative wave equation, allow proving the existence of spurious high-frequency waves, which do not propagate. This suffices to show the lack of uniform exponential decay for (2.30).

Therefore, we need to add a numerical viscosity term. We have at least two choices to introduce this numerical viscosity: Either we consider

$$\begin{cases} \frac{z_1^{k+1} - z^k}{\Delta t} = A\left(\frac{z^k + z_1^{k+1}}{2}\right), & k \in \mathbb{N}, \\ \frac{z^{k+1} - z_1^{k+1}}{\Delta t} = -B^*Bz^{k+1} + (\Delta t)^2 A^2 z^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0, \end{cases} \quad (2.31)$$

or

$$\begin{cases} \frac{z_1^{k+1} - z^k}{\Delta t} = A\left(\frac{z^k + z_1^{k+1}}{2}\right), & k \in \mathbb{N}, \\ \frac{z_2^{k+1} - z_1^{k+1}}{\Delta t} = -B^*Bz_2^{k+1}, & k \in \mathbb{N}, \\ \frac{z^{k+1} - z_2^{k+1}}{\Delta t} = (\Delta t)^2 A^2 z^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (2.32)$$

The proof above of the uniform exponential decay rate can be adapted to both systems. The low frequency components can be observed similarly. The same decoupling argument between low and high frequencies can be applied as well. Indeed, putting $B = 0$ into systems (2.31) and (2.32) yields again system (1.15). Therefore we can get the same results as for system (1.9).

Theorem 2.6. *Assume that system (1.1) is exponentially stable, i.e. satisfies (1.4) with constants μ and ν and that $B \in \mathcal{L}(X, Y)$.*

Then there exist two positive constants μ_0 and ν_0 depending only on μ , ν and $\|B\|_{\mathfrak{L}(X,Y)}$, such that any solution of (2.31) or of (2.32) satisfies (1.8) with constants μ_0 and ν_0 uniformly with respect to the discretization parameter $\Delta t > 0$.

We skip the proof since it is similar to the previous one.

Other viscosity operators. Other viscosity operators could have been chosen. In our approach, we used the viscosity term $(\Delta t)^2 A^2$, which is unbounded, but we could have considered the viscosity operator

$$(\Delta t)\mathcal{V}_{\Delta t} = \frac{(\Delta t)^2 A^2}{I - (\Delta t)^2 A^2}, \quad (2.33)$$

which is well defined, since A^2 is a definite negative operator, and commutes with A . This choice presents the advantage that the viscosity operator now is bounded, keeping the properties of being small at frequencies of order less than $1/\Delta t$ and of order 1 on frequencies of order $1/\Delta t$ and more. Again, the same proof as the one presented above works.

The following result constitutes a generalization of Theorem 1.1, and applies to a wide range of viscosity operators, and, in particular, to (2.33).

Theorem 2.7. *Assume that system (1.1) is exponentially stable, and that $B \in \mathfrak{L}(X, Y)$.*

Consider a viscosity operator $\mathcal{V}_{\Delta t}$ such that there exists $\delta > 0$ such that:

1. $\mathcal{V}_{\Delta t}$ defines a self-adjoint negative definite operator.
2. The operators $\pi_{\delta/\Delta t}$ and $\mathcal{V}_{\Delta t}$ commute.
3. There exist two positive constants $c > 0$ and $C > 0$ such that

$$\begin{cases} \sqrt{\Delta t} \left\| \left(\sqrt{-\mathcal{V}_{\Delta t}} \right) z \right\|_X \leq C \|z\|_X, \quad \forall z \in \mathcal{C}_{\delta/\Delta t}, \\ \sqrt{\Delta t} \left\| \left(\sqrt{-\mathcal{V}_{\Delta t}} \right) z \right\|_X \geq c \|z\|_X, \quad \forall z \in \mathcal{C}_{\delta/\Delta t}^\perp, \end{cases}$$

uniformly with respect to $\Delta t > 0$.

Then the solutions of

$$\begin{cases} \frac{\tilde{z}^{k+1} - z^k}{\Delta t} = A \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) - B^* B \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{z^{k+1} - \tilde{z}^{k+1}}{\Delta t} = (\Delta t) \mathcal{V}_{\Delta t} z^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (2.34)$$

are exponentially uniformly decaying in the sense of (1.8).

A similar result holds for the corresponding variants of systems (2.31) and (2.32).

3 Stabilization of time-discrete systems depending on a parameter

This section is devoted to study time-discrete approximation schemes of abstract systems of the form (1.1) depending on a parameter, that can be for instance the space-mesh size when dealing with fully discrete approximation schemes, in which case A is a space discretization of a partial differential operator. As we shall see, the results of the previous section apply.

Furthermore, in the context of fully discrete systems, we shall also show that introducing a suitable CFL type condition, it is unnecessary to add a numerical viscosity term to obtain the uniform exponential decay of the energy. This is so, roughly, because the CFL condition itself rules out the high frequency components without the need of numerical viscosity.

As said in the introduction, this approach requires observability properties to hold uniformly with respect to the space discretization parameter for solutions of the space semi-discrete schemes for *any* initial data. However, in numerous applications, the space semi-discrete approximation schemes are only observable at low frequencies. We therefore develop our arguments to deal with this case adding a stronger numerical viscosity operator to efficiently damp out the high-frequencies which are not ruled out in the time continuous setting. Simultaneously, we prove a result for space semi-discrete approximation schemes which, to our knowledge, had not been stated so far in such a general setting, even if some instances can be found in [27, 25, 13].

Again, the strategy we propose is strongly based on the methods and results in [12], especially Theorem 2.1 given above. Applications to the stabilization of numerical approximation schemes for the damped wave equation are given in Section 4.

3.1 The general case

To state our result, it is convenient to introduce the following class of pairs of operators (A, B) :

Definition 3.1. For any $(K_B, \mu, \nu) \in (\mathbb{R}_+^*)^3$, we define $\mathfrak{D}(K_B, \mu, \nu)$ as the class of operators (A, B) satisfying:

- (A1) The operator A is skew-adjoint on some Hilbert space X , and has a compact resolvent.
- (A2) The operator B is in $\mathfrak{L}(X, Y)$, where Y is a Hilbert space, and satisfies (2.8) with constant K_B .
- (A3) System (1.1) is exponentially stable, and solutions of (1.1) satisfy (1.4) with constants μ and ν .

Note that this definition does not depend on the Hilbert spaces X and Y .

In this class, Theorems 1.1-2.6-2.7 apply and provide uniform exponential decay properties for the time semi-discrete approximation scheme (1.9). This can be deduced from the explicit dependence of the constants entering in Theorems 1.1-2.6-2.7, which only depend on K_B , μ and ν . At this point, the fact that the class $\mathcal{D}(K_B, \mu, \nu)$ is independent of the spaces X and Y plays a key role.

Also note that Definition 3.1 only refers to the behavior of the *continuous* system (1.1), although, as we have seen, and in particular in view of Theorem 2.1, it also has applications in what concerns time-discrete systems.

This method allows dealing with fully discrete approximation schemes. In that setting, we consider a family of operators $(A_{\Delta x}, B_{\Delta x})$, where $\Delta x > 0$ is the standard parameter associated with the space mesh-size. In this way one can use automatically the existing results for space semi-discretizations as, for instance, [1, 5, 6, 13, 11, 23, 24, 27].

Note that the work [24] is not dealing with stabilization properties, but rather with controllability properties of space semi-discrete schemes. However, it is standard that these two properties (controllability and stabilization) are very close, since both are equivalent to observability properties. Therefore, these works can be adapted to study the stabilization properties as well. We refer to the survey article [32] for more details and more references.

Remark 3.2. We emphasize that this approach is based on the systematic use of existing results for space semi-discretizations. One could proceed all the way around, first applying the results in this paper to derive uniform stabilization results for time discrete approximation schemes and then discretizing the space variables. For doing this, however, due to the more complex dependence of the PDE and its space semi-discretizations on the space variables, there is no systematic way of transferring results from the continuous to the discrete setting. In this sense, the method we propose here of using the existing results for space semi-discretizations to later apply the results in this paper about time discretizations is much more easier to be implemented and yields better results.

3.2 Stabilization of fully discrete approximation schemes without viscosity

This subsection is devoted to prove a particular result for fully discrete approximation schemes under a CFL type assumption on the space and time discretization parameters, which does not require adding numerical viscosity terms. We observe, however, that this approach requires, often, restrictions on Δt that can be avoided by adding numerical viscosity terms.

Theorem 3.3. *Let $(A_{\Delta x}, B_{\Delta x})_{\Delta x > 0}$ be a family of operators defined on Hilbert spaces $X_{\Delta x}$ endowed with a norm $\|\cdot\|_{\Delta x}$. Assume that there exist positive constants K_B , μ and ν such that, for all $\Delta x > 0$, $(A_{\Delta x}, B_{\Delta x}) \in \mathcal{D}(K_B, \mu, \nu)$.*

Then, for any $\eta > 0$, there exist positive constants μ_η and ν_η such that the

solutions of

$$\begin{cases} \frac{z_{\Delta x}^{k+1} - z_{\Delta x}^k}{\Delta t} = A_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) - B_{\Delta x}^* B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ z_{\Delta x}^0 = z_{0, \Delta x} \in X_{\Delta x}, \end{cases} \quad (3.1)$$

satisfy

$$\|z_{\Delta x}^k\|_{\Delta x}^2 \leq \mu_\eta \|z_{\Delta x}^0\|_{\Delta x}^2 \exp(-\nu_\eta k \Delta t), \quad k \geq 0, \quad (3.2)$$

uniformly with respect to $\Delta t > 0$ and $\Delta x > 0$ provided that

$$\|A_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})} \leq \frac{\eta}{\Delta t}. \quad (3.3)$$

Remark 3.4. In practical applications, the operator $A_{\Delta x}$ is often a space discretization of an unbounded operator A , for which we typically have a bound of the form $\|A_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})} \simeq C(\Delta x)^{-\sigma}$ for some positive exponent σ . In this case, condition (3.3) is guaranteed as soon as

$$\frac{C}{(\Delta x)^\sigma} \leq \frac{\eta}{\Delta t}.$$

The CFL condition (3.3) therefore imposes the ratio $\Delta t / (\Delta x)^\sigma$ to be uniformly bounded when Δx and Δt go to 0.

Remark 3.5. This theorem implies that we do not need to add a numerical viscosity term on the time-discrete approximation schemes to get a uniform exponential decay of the energies if we impose a CFL type condition on the discretization parameters Δx and Δt .

Proof. The proof of Theorem 3.3 is actually easier than the one of Theorem 1.1, since we do not need the decomposition (2.12) into low and high frequency components. In some sense, the CFL rules out the high frequency components.

First, we derive the energy identity for solutions of (3.1):

$$\|z_{\Delta x}^l\|_{\Delta x}^2 = \|z_{\Delta x}^0\|_{\Delta x}^2 - 2\Delta t \sum_{k=0}^{l-1} \left\| B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2, \quad l \in \mathbb{N}. \quad (3.4)$$

Second, since $(A_{\Delta x}, B_{\Delta x}) \in \mathfrak{D}(K_B, \mu, \nu)$, the resolvent estimates (2.4) involving $A_{\Delta x}$ and $B_{\Delta x}$ hold uniformly with respect to $\Delta x > 0$, due to Lemma 2.3.

Then, applying Theorem 2.1 with $\delta = \eta$, because of assumption (3.3) that implies that $\mathcal{C}_{\eta/\Delta t}(A_{\Delta x}) = X_{\Delta x}$, we get a time $T^* > 0$ and a positive constant k_{T^*} independent of $\Delta x > 0$ such that any solution $y_{\Delta x}$ of

$$\begin{cases} \frac{y_{\Delta x}^{k+1} - y_{\Delta x}^k}{\Delta t} = A_{\Delta x} \left(\frac{y_{\Delta x}^k + y_{\Delta x}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ y_{\Delta x}^0 = y_{0, \Delta x} \in X_{\Delta x}, \end{cases} \quad (3.5)$$

satisfies

$$k_{T^*} \|y_{\Delta x}^0\|_{\Delta x}^2 \leq \Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B_{\Delta x} \left(\frac{y_{\Delta x}^k + y_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2. \quad (3.6)$$

Now, let $z_{0, \Delta x} \in X_{\Delta x}$, and consider the solutions $z_{\Delta x}$ of (3.1) and $y_{\Delta x}$ of (3.5) with initial data $y_{0, \Delta x} = z_{0, \Delta x}$. Set $w_{\Delta x} = z_{\Delta x} - y_{\Delta x}$. Then

$$k_{T^*} \|z_{\Delta x}^0\|_{\Delta x}^2 \leq 2\Delta t \sum_{k\Delta t \in [0, T^*]} \left(\left\| B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2 + \left\| B_{\Delta x} \left(\frac{w_{\Delta x}^k + w_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2 \right). \quad (3.7)$$

Therefore, we only need to bound the last term. This is easier than in (2.21). Indeed, $w_{\Delta x}$ satisfies

$$\frac{w_{\Delta x}^{k+1} - w_{\Delta x}^k}{\Delta t} = A_{\Delta x} \left(\frac{w_{\Delta x}^k + w_{\Delta x}^{k+1}}{2} \right) - B_{\Delta x}^* B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right), \quad k \in \mathbb{N}, \quad (3.8)$$

with $w_{\Delta x}^0 = 0$.

The energy estimates on $w_{\Delta x}$ now give, for $l \in \mathbb{N}$

$$\|w_{\Delta x}^l\|_{\Delta x}^2 = -2\Delta t \sum_{k=0}^{l-1} \langle B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right), B_{\Delta x} \left(\frac{w_{\Delta x}^k + w_{\Delta x}^{k+1}}{2} \right) \rangle_{Y_{\Delta x}},$$

and then

$$\|w_{\Delta x}^l\|_{\Delta x}^2 \leq \Delta t \|B_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, Y_{\Delta x})}^2 \sum_{k=0}^{l-1} \left\| \frac{w_{\Delta x}^k + w_{\Delta x}^{k+1}}{2} \right\|_{\Delta x}^2 + \Delta t \sum_{k=0}^{l-1} \left\| B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2.$$

Since $\|B_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, Y_{\Delta x})} \leq K_B$, applying Grönwall's Lemma, we obtain a constant G independent of $\Delta x > 0$ such that

$$\Delta t \sum_{k\Delta t \in [0, T^*]} \|w_{\Delta x}^k\|_{\Delta x}^2 \leq G\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2.$$

This last inequality implies with (3.7) that

$$k_{T^*} \|z_{\Delta x}^0\|_{\Delta x}^2 \leq 2(1 + K_B^2 G)\Delta t \sum_{k\Delta t \in [0, T^*]} \left\| B_{\Delta x} \left(\frac{z_{\Delta x}^k + z_{\Delta x}^{k+1}}{2} \right) \right\|_{Y_{\Delta x}}^2.$$

Plugging this inequality in (3.4) for $l^* = \lceil T^*/\Delta t \rceil$ gives

$$\|z_{\Delta x}^{l^*}\|_{\Delta x}^2 \leq \|z_{\Delta x}^0\|_{\Delta x}^2 \left(1 - \frac{k_{T^*}}{1 + K_B^2 G}\right).$$

As previously, setting

$$\alpha = \left(1 - \frac{k_{T^*}}{1 + K_B^2 G}\right),$$

which is independent of Δt , we obtain that

$$\|z_{\Delta x}^l\|_{\Delta x}^2 \leq \|z_{\Delta x}^0\|_{\Delta x}^2 \exp\left(\left(\frac{l\Delta t}{T^*} - 1\right) \ln(\alpha)\right), \quad \forall l \in \mathbb{N},$$

which proves the result. \square

Remark 3.6. As before, the proof of Theorem 3.3 can also be carried out for the time-discrete scheme

$$\begin{cases} \frac{\tilde{z}_{\Delta x}^{k+1} - z_{\Delta x}^k}{\Delta t} = A_{\Delta x} \left(\frac{z_{\Delta x}^k + \tilde{z}_{\Delta x}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{z_{\Delta x}^{k+1} - \tilde{z}_{\Delta x}^{k+1}}{\Delta t} = -B_{\Delta x}^* B_{\Delta x} z_{\Delta x}^{k+1}, & k \in \mathbb{N}, \\ z_{\Delta x}^0 = z_{0, \Delta x} \in X_{\Delta x}, \end{cases} \quad (3.9)$$

under the CFL condition (3.3).

3.3 Stabilization of fully discrete approximation schemes with viscosity

In this Subsection, we consider the case in which the space semi-discrete systems are uniformly observable for initial data lying in filtered subspaces, as it occurs often, see [18, 31, 13, 32].

Theorem 3.7. *Let $(A_{\Delta x}, B_{\Delta x})_{\Delta x > 0}$ be a family of operators defined on Hilbert spaces $X_{\Delta x}$ endowed with the norms $\|\cdot\|_{\Delta x}$.*

Assume that there exists a constant K_B such that for all $\Delta x > 0$, the operator norm $\|B_{\Delta x}\|_{\mathcal{L}(X_{\Delta x}, Y_{\Delta x})}$ is bounded by K_B .

Assume that there exist positive constants η , σ , T and k_T such that for all initial data $y_0 \in \mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})$, the solution y of

$$\dot{y} = A_{\Delta x} y, \quad t \in \mathbb{R}, \quad y(0) = y_0 \in \mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x}), \quad (3.10)$$

satisfies

$$k_T \|y_0\|_{\Delta x}^2 \leq \int_0^T \|B_{\Delta x} y(t)\|_{Y_{\Delta x}}^2 dt. \quad (3.11)$$

Set $\varepsilon = \max\{\Delta t, (\Delta x)^\sigma\}$.

Consider a viscosity operator \mathcal{V}_ε such that:

1. \mathcal{V}_ε defines a self-adjoint negative definite operator.
2. The operators $\pi_{1/\varepsilon}$ and \mathcal{V}_ε commute.
3. There exist two positive constants $c > 0$ and $C > 0$ such that

$$\begin{cases} \sqrt{\varepsilon} \left\| \left(\sqrt{-\mathcal{V}_\varepsilon} \right) z \right\|_{\Delta x} \leq C \|z\|_{\Delta x}, \quad \forall z \in \mathcal{C}_{1/\varepsilon}(A_{\Delta x}), \\ \sqrt{\varepsilon} \left\| \left(\sqrt{-\mathcal{V}_\varepsilon} \right) z \right\|_{\Delta x} \geq c \|z\|_{\Delta x}, \quad \forall z \in \mathcal{C}_{1/\varepsilon}(A_{\Delta x})^\perp, \end{cases}$$

uniformly with respect to $\varepsilon > 0$.

Then the solutions of

$$\begin{cases} \frac{\tilde{z}^{k+1} - z^k}{\Delta t} = A_{\Delta x} \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right) - B_{\Delta x}^* B_{\Delta x} \left(\frac{z^k + \tilde{z}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{z^{k+1} - \tilde{z}^{k+1}}{\Delta t} = \varepsilon \mathcal{V}_\varepsilon z^{k+1}, & k \in \mathbb{N}, \\ z^0 = z_0. \end{cases} \quad (3.12)$$

are exponentially uniformly decaying in the sense of (3.2).

Sketch of the proof. The proof can be done similarly as the one of Theorems 1.1-2.7. The main difference in the proof is that the low and high-frequency components are separated by the frequency $1/\varepsilon$ instead of $1/\Delta t$.

As explained in [12], the observability inequalities (3.11) in the filtered spaces $\mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})$ imply observability inequalities (2.5) for solutions of (1.18) with initial data lying in $\mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x}) \cap \mathcal{C}_{1/\Delta t}(A_{\Delta x}) = \mathcal{C}_{1/\varepsilon}(A_{\Delta x})$. The proof of this fact simply consists in the following remark: the uniform observability inequalities (3.11) in the filtered spaces $\mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})$ imply uniform resolvent estimates (2.4) for data in $\mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})$, and Theorem 2.1, due to the explicit dependence of the constants in (2.5) on the constants m and M appearing in (2.4), yields the result.

Second, we replace system (1.15) by

$$\begin{cases} \frac{\tilde{u}^{k+1} - u^k}{\Delta t} = A_{\Delta x} \left(\frac{u^k + \tilde{u}^{k+1}}{2} \right), & k \in \mathbb{N}, \\ \frac{u^{k+1} - \tilde{u}^{k+1}}{\Delta t} = \varepsilon \mathcal{V}_\varepsilon u^{k+1}, & k \in \mathbb{N}, \\ u^0 = u_0, \end{cases} \quad (3.13)$$

and consider u_l and u_h defined by

$$u_l = \pi_{1/\varepsilon} u, \quad u_h = (I - \pi_{1/\varepsilon}) u,$$

instead of (2.12).

The rest of the proof follows line to line that of Lemma 2.4 and is left to the reader. \square

Theorem 3.7 also yields an interesting corollary for time-continuous systems:

Corollary 3.8. *Let $(A_{\Delta x}, B_{\Delta x})_{\Delta x > 0}$ be a family of operators defined on Hilbert spaces $X_{\Delta x}$ endowed with the norms $\|\cdot\|_{\Delta x}$.*

Assume that there exists a constant K_B such that for all $\Delta x > 0$, the operator norm $\|B_{\Delta x}\|_{\mathcal{L}(X_{\Delta x}, Y_{\Delta x})}$ is bounded by K_B .

Assume that there exist positive constants η , σ , T and k_T such that for all initial data $y_0 \in \mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})$, the solution y of (3.10) satisfies (3.11).

Consider a viscosity operator $\mathcal{V}_{\Delta x}$ such that:

1. $\mathcal{V}_{\Delta x}$ defines a self-adjoint negative definite operator.
2. The operators $\pi_{\eta/(\Delta x)^\sigma}$ and $\mathcal{V}_{\Delta x}$ commute.
3. There exist two positive constants $c > 0$ and $C > 0$ such that

$$\begin{aligned} (\Delta x)^{\sigma/2} \left\| \left(\sqrt{-\mathcal{V}_{\Delta x}} \right) z \right\|_{\Delta x} &\leq C \|z\|_{\Delta x}, \quad \forall z \in \mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x}), \\ (\Delta x)^{\sigma/2} \left\| \left(\sqrt{-\mathcal{V}_{\Delta x}} \right) z \right\|_{\Delta x} &\geq c \|z\|_{\Delta x}, \quad \forall z \in \mathcal{C}_{\eta/(\Delta x)^\sigma}(A_{\Delta x})^\perp, \end{aligned}$$

uniformly with respect to $\Delta x > 0$.

Then the solutions of

$$\begin{cases} \dot{z} = A_{\Delta x} z - B_{\Delta x}^* B_{\Delta x} z + (\Delta x)^\sigma \mathcal{V}_{\Delta x} z, & t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases} \quad (3.14)$$

are exponentially uniformly decaying in the sense of (1.4).

Indeed, this can be deduced from Theorem 3.7 by letting $\Delta t \rightarrow 0$.

Corollary 3.8 can be seen as a generalization of [14], where similar results have been derived for viscous approximations of (1.1). In [14], the same result is obtained but the assumptions differ in one essential point: The observability inequality (1.16) for solutions of (1.14) is assumed to hold for *any* initial data, and not only in a filtered space as in Corollary 3.8. Thus, in [14], no assumption is required on the viscosity parameter.

Though, the proof in [14] can be easily adapted to prove Corollary 3.8 directly for time continuous systems.

Also remark that some instances of applications of variants of Corollary 3.8 can be found in several different articles dealing with space semi-discrete damped systems [27, 25, 23, 13].

In Subsection 4.3.1, we will indicate without proof how one can deduce the results in [27, 23] from the results in [18] and the methods developed in [14] and here.

Remark 3.9. Corollary 3.8 yields optimal results in the following sense: If system (3.14) is exponentially decaying for $\mathcal{V}_{\Delta x} = -|A_{\Delta x}|$, which always satisfies the assumptions of Corollary 3.8, uniformly with respect to the space discretization parameter, then there exists $\varepsilon > 0$ such that any solution y of (3.10) with initial

data in $\mathcal{C}_{\varepsilon/(\Delta x)^\sigma}(A_{\Delta x})$ satisfies (3.11). Indeed, in this case, following the proof of Lemma 2.3, one can prove that there exist a time $T > 0$ and a constant $k_T > 0$ such that, for any $\Delta x > 0$, any solution y of (3.10) satisfies

$$k_T \|y_0\|_{\Delta x}^2 \leq \int_0^T \|B_{\Delta x} y(t)\|_{Y_{\Delta x}}^2 dt + \int_0^T (\Delta x)^\sigma \left\| \left(\sqrt{|A_{\Delta x}|} \right) y(t) \right\|_{\Delta x}^2 dt.$$

In particular, if the initial data lies in $\mathcal{C}_{\varepsilon/(\Delta x)^\sigma}(A_{\Delta x})$, we have that

$$k_T \|y_0\|_{\Delta x}^2 \leq \int_0^T \|B_{\Delta x} y(t)\|_{Y_{\Delta x}}^2 dt + \varepsilon T \|y_0\|_{\Delta x}^2,$$

and then, taking $\varepsilon = k_T/2T$, we recover (3.11).

4 Applications

The goal of this section is to present several applications of Theorems 1.1-3.3 to the damped wave equation. Of course, the Schrödinger and plate equations, and the system of elasticity, among others, enter in this frame too, but the applications to these other models will be presented elsewhere.

4.1 The time-discrete damped wave equation

Consider a smooth non-empty open bounded domain $\Omega \subset \mathbb{R}^d$.

We consider the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta_x u + \sigma(x)^2 u_t = 0, & x \in \Omega, \quad t \geq 0, \\ u(x, t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x, 0) = u_0 \in H_0^1(\Omega), \quad u_t(x, 0) = v_0 \in L^2(\Omega), & x \in \Omega, \end{cases} \quad (4.1)$$

where $\sigma : \Omega \rightarrow \mathbb{R}_+$ is a non-negative bounded function which is strictly positive in some open non-empty subset $\omega \subset \Omega$: There exists $\alpha > 0$ such that

$$\sigma^2(x) \geq \alpha, \quad \forall x \in \omega. \quad (4.2)$$

The energy of (4.1)

$$E(t) = \frac{1}{2} \int_{\Omega} \left[|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 \right] dx, \quad (4.3)$$

satisfies the dissipation law

$$\frac{dE}{dt}(t) = - \int_{\omega} \sigma(x)^2 |\partial_t u(t, x)|^2 dx, \quad \forall t \in [0, T]. \quad (4.4)$$

It is well-known that the energy of (4.3) decays exponentially if the set ω satisfies a geometric condition, namely the so-called *Geometric Control Condition*, introduced in [2, 3]: there exists a time $T > 0$ such that all the rays of Geometric Optics in Ω enter the set ω in a time smaller than T .

To show that system (4.1) enters in the abstract setting of this paper, let us recall that it is equivalent to

$$\dot{Z} = AZ - B^*BZ, \text{ with } Z = \begin{pmatrix} u \\ v \end{pmatrix}, A = \begin{pmatrix} 0 & Id \\ \Delta_x & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & \sigma \end{pmatrix}. \quad (4.5)$$

In this setting, A is a skew-adjoint unbounded operator on the Hilbert space $X = H_0^1(\Omega) \times L^2(\Omega)$, with domain $\mathcal{D}(A) = H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$. From the assumptions (4.2) on σ , the operator B is obviously continuous on X .

Besides, the energy (4.3) of (4.1) reads as $\|Z(t)\|_X^2/2$.

Then, we introduce the following time semi-discrete approximation scheme:

$$\begin{cases} \frac{\tilde{Z}^{k+1} - Z^k}{\Delta t} = \begin{pmatrix} 0 & Id \\ \Delta_x & 0 \end{pmatrix} \left(\frac{Z^k + \tilde{Z}^{k+1}}{2} \right) \\ \quad \quad \quad - \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \left(\frac{Z^k + \tilde{Z}^{k+1}}{2} \right), & k \in \mathbb{N}^*, \\ \frac{Z^{k+1} - \tilde{Z}^{k+1}}{\Delta t} = (\Delta t)^2 \begin{pmatrix} \Delta_x & 0 \\ 0 & \Delta_x \end{pmatrix} Z^{k+1}, & k \in \mathbb{N}^*, \\ Z^0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \end{cases} \quad (4.6)$$

We then define the energy as in (1.6).

According to Theorem 1.1, we get:

Theorem 4.1. *Assume that the damping function σ satisfies (4.2) for a non-empty open set $\omega \subset \Omega$, that satisfies the Geometric Control Condition.*

Then there exist positive constants ν_0 and μ_0 such that any solution of (4.6) satisfies (1.8) uniformly with respect to the discretization parameter $\Delta t > 0$.

4.2 A fully discrete damped wave equation: The mixed finite element method

Here we present an application to a fully discrete approximation scheme. To present our results properly, we first need to recall some properties of the *space* semi-discrete wave equation.

We now consider the damped wave equation (4.1) in 1d, that is with $\Omega = (0, 1)$. We still assume that the damping function σ is non-negative, bounded, and satisfies (4.2). Note that in this case the *Geometric Control Condition* is automatically satisfied, and therefore the decay of the energy of (4.1) is exponential.

When semi-discretizing equation (4.1) in *space*, it may happen that the *space* semi-discrete approximations are *not* exponentially stable uniformly with respect to the space discretization parameter. This has been observed in many cases, for instance in [15, 18, 21, 13]. We refer to the review article [32] for more references.

A possible cure has been proposed in [1] and analyzed in [5, 6, 11] based on a mixed finite element method, on which we will focus now.

Let N be a nonnegative integer. Set $\Delta x = 1/(N + 1)$ and consider the subdivision of $(0, 1)$ given by

$$0 = x_0 < x_1 < \cdots < x_j = j\Delta x < \cdots < x_{N+1} = 1.$$

Let us present the space semi-discrete approximation scheme of (4.1) in 1d, on $(0, 1)$, derived from the mixed finite element method (see [1, 5, 6, 11])

$$\left\{ \begin{array}{l} \frac{\ddot{u}_{j-1} + 2\ddot{u}_j + \ddot{u}_{j+1}}{4} - \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} + \frac{1}{4} \left(\sigma_{j-1/2}^2 (\dot{u}_{j-1} + \dot{u}_j) \right. \\ \left. + \sigma_{j+1/2}^2 (\dot{u}_j + \dot{u}_{j+1}) \right) = 0, \quad (t, j) \in \mathbb{R}_+ \times \{1, \dots, N\}, \\ u_0(t) = u_{N+1}(t) = 0, \quad t \in \mathbb{R}_+, \\ u_j(0) = u_{j,0}, \quad \dot{u}_j(0) = v_{j,0}, \quad j \in \{1, \dots, N\}, \end{array} \right. \quad (4.7)$$

where $\sigma_{j+1/2}^2$ is an approximation of σ^2 on $[j\Delta x, (j+1)\Delta x]$.

The energy of solutions of (4.7) is defined by

$$E_{\Delta x}(t) = \frac{\Delta x}{2} \sum_{j=0}^N \left(\left| \frac{\dot{u}_j + \dot{u}_{j+1}}{2} \right|^2 + \left| \frac{u_{j+1} - u_j}{\Delta x} \right|^2 \right). \quad (4.8)$$

Following [1, 5, 6, 11], one can prove that the energy $E_{\Delta x}$ is exponentially stable, uniformly with respect to $\Delta x > 0$, when σ satisfies (4.2).

Let us check that system (4.7) is a particular instance of the abstract setting we provided.

Define the $N \times N$ matrix $M_{\Delta x}$ by

$$M_{\Delta x}(i, j) = \begin{cases} 1/2 & \text{if } i = j, \\ 1/4 & \text{if } |i - j| = 1, \\ 0 & \text{else,} \end{cases}$$

which is invertible, self-adjoint and positive definite.

The space semi-discrete approximation scheme (4.7) can be written as

$$M_{\Delta x} \ddot{U}_{\Delta x} + A_{0, \Delta x} U_{\Delta x} + C_{1, \Delta x} \dot{U}_{\Delta x} = 0, \quad t \in \mathbb{R}_+,$$

where $A_{0, \Delta x}$ is a positive definite matrix $N \times N$, which represents the Laplace discrete operator, and $C_{1, \Delta x}$ is the $N \times N$ matrix

$$C_{1, \Delta x}(i, j) = \begin{cases} (\sigma_{j+1/2}^2 + \sigma_{j-1/2}^2)/4 & \text{if } i = j, \\ \sigma_{i+1/2}^2/4 & \text{if } i + 1 = j, \\ \sigma_{i-1/2}^2/4 & \text{if } i - 1 = j, \\ 0 & \text{else.} \end{cases}$$

System (4.7) can be rewritten as

$$\dot{Z}_{\Delta x} = A_{\Delta x} Z_{\Delta x} - C_{\Delta x} Z_{\Delta x}, \quad t \in \mathbb{R}_+, \quad (4.9)$$

where $Z_{\Delta x}$, $A_{\Delta x}$ and $C_{\Delta x}$ denote

$$\begin{aligned} Z_{\Delta x} &= \begin{pmatrix} U_{\Delta x} \\ V_{\Delta x} \end{pmatrix}, \quad A_{\Delta x} = \begin{pmatrix} 0 & Id \\ -M_{\Delta x}^{-1} A_{0, \Delta x} & 0 \end{pmatrix}, \\ C_{\Delta x} &= \begin{pmatrix} 0 & 0 \\ 0 & M_{\Delta x}^{-1} C_{1, \Delta x} \end{pmatrix}. \end{aligned} \quad (4.10)$$

Remark that the matrix $A_{\Delta x}$ is skew-adjoint on the energy space $X_{\Delta x} = \mathbb{R}^{2N}$ endowed with the norm

$$\begin{aligned} \left\| \begin{pmatrix} U_{\Delta x} \\ V_{\Delta x} \end{pmatrix} \right\|_{\Delta x}^2 &= \Delta x \sum_{j=0}^N \left(\left| \frac{V_{\Delta x, j} + V_{\Delta x, j+1}}{2} \right|^2 + \left| \frac{U_{\Delta x, j+1} - U_{\Delta x, j}}{\Delta x} \right|^2 \right) \\ &= \langle M_{\Delta x} V_{\Delta x}, V_{\Delta x} \rangle_{* \Delta x} + \langle A_{0, \Delta x} U_{\Delta x}, U_{\Delta x} \rangle_{* \Delta x}, \end{aligned}$$

where the scalar product $\langle \cdot, \cdot \rangle_{* \Delta x}$ is the classical discrete L^2 scalar product, corresponding to the discrete L^2 norm

$$\|V_{\Delta x}\|_{* \Delta x}^2 = \Delta x \sum_{j=1}^N |V_{\Delta x, j}|^2. \quad (4.11)$$

Note that, in this setting, the energy (4.8) of solutions of (4.7) coincides with the energy $\|Z_{\Delta x}(t)\|_{\Delta x}^2/2$ of solutions of (4.9).

Let us check that $C_{\Delta x}$ has the form $B_{\Delta x}^* B_{\Delta x}$ for some $N \times N$ matrix $B_{\Delta x}$. According to Choleski's decomposition, we only have to check that $C_{\Delta x}$ is a selfadjoint positive matrix on $X_{\Delta x}$. For generic vectors $Z_{1\Delta x}$ and $Z_{2\Delta x}$ as in (4.10), we have:

$$\begin{aligned} \langle C_{\Delta x} Z_{1\Delta x}, Z_{2\Delta x} \rangle_{\Delta x} &= \langle M_{\Delta x} M_{\Delta x}^{-1} C_{1\Delta x} V_{1\Delta x}, V_{2\Delta x} \rangle_{* \Delta x} \\ &= \langle C_{1\Delta x} V_{1\Delta x}, V_{2\Delta x} \rangle_{* \Delta x} \\ &= \Delta x \sum_{j=0}^N \sigma_{j+1/2}^2 \left(\frac{V_{1\Delta x, j} + V_{1\Delta x, j+1}}{2} \right) \left(\frac{V_{2\Delta x, j} + V_{2\Delta x, j+1}}{2} \right). \end{aligned} \quad (4.12)$$

This last expression shows that $C_{\Delta x}$ is a selfadjoint positive operator on $X_{\Delta x}$. Therefore there exists $B_{\Delta x}$ such that $B_{\Delta x}^* B_{\Delta x} = C_{\Delta x}$. Besides, classical linear algebra implies that

$$\|C_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})} = \|B_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})}^2.$$

From the computations above, and especially (4.12), we have

$$\|C_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})} = \sup_{\substack{\|Z_{1\Delta x}\|_{\Delta x} \leq 1, \\ \|Z_{2\Delta x}\|_{\Delta x} \leq 1}} \{ \langle C_{\Delta x} Z_{1\Delta x}, Z_{2\Delta x} \rangle_{\Delta x} \} \leq \|\sigma^2\|_{L^\infty}. \quad (4.13)$$

We are then in the abstract setting given in Section 3: Hypothesis (A1) and (A2) of Definition 3.1 have been checked above, and (A3) has been proved in [5] (see [1, 6, 11] for related results).

4.2.1 Method I: Adding a numerical viscosity term in time

We add a numerical viscosity term to the scheme above, corresponding to (1.9). In this case, the fully discrete approximation scheme reads:

$$\left\{ \begin{array}{l} \frac{\tilde{u}_j^{k+1} - u_j^k}{\Delta t} = \frac{v_j^k + \tilde{v}_j^{k+1}}{2}, \\ \frac{1}{4\Delta t} \left((\tilde{v}_{j-1}^{k+1} + 2\tilde{v}_j^{k+1} + \tilde{v}_{j+1}^{k+1}) - (v_{j-1}^k + 2v_j^k + v_{j+1}^k) \right) = \\ \frac{1}{2(\Delta x)^2} (\tilde{u}_{j+1}^{k+1} + u_{j+1}^k - 2\tilde{u}_j^{k+1} - 2u_j^k + \tilde{u}_{j-1}^{k+1} + u_{j-1}^k) \\ - \frac{1}{8}\sigma_{j+1/2}^2 \left((v_j^k + v_{j+1}^k) + (\tilde{v}_j^{k+1} + \tilde{v}_{j+1}^{k+1}) \right) \\ - \frac{1}{8}\sigma_{j-1/2}^2 \left((v_{j-1}^k + v_j^k) + (\tilde{v}_{j-1}^{k+1} + \tilde{v}_j^{k+1}) \right), \\ \frac{1}{4\Delta t} \left((u_{j-1}^{k+1} + 2u_j^{k+1} + u_{j+1}^{k+1}) - (\tilde{u}_{j-1}^{k+1} + 2\tilde{u}_j^{k+1} + \tilde{u}_{j+1}^{k+1}) \right) \\ = \left(\frac{\Delta t}{\Delta x} \right)^2 (u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}), \\ \frac{1}{4\Delta t} \left((v_{j-1}^{k+1} + 2v_j^{k+1} + v_{j+1}^{k+1}) - (\tilde{v}_{j-1}^{k+1} + 2\tilde{v}_j^{k+1} + \tilde{v}_{j+1}^{k+1}) \right) \\ = \left(\frac{\Delta t}{\Delta x} \right)^2 (v_{j+1}^{k+1} - 2v_j^{k+1} + v_{j-1}^{k+1}), \end{array} \right. \quad (4.14)$$

which holds for $(k, j) \in \mathbb{N} \times \{1, \dots, N\}$, with the boundary conditions

$$u_0^k = u_{N+1}^k = v_0^k = v_{N+1}^k = 0, \quad \forall k \in \mathbb{N}, \quad (4.15)$$

and the initial data

$$u_j^0 = u_{j,0}, \quad v_j^0 = v_{j,0}, \quad \forall j \in \{1, \dots, N\}. \quad (4.16)$$

Here u_j^k and v_j^k respectively denote approximations of the functions u and \dot{u} in $x_j = j\Delta x$ at time $k\Delta t$.

As an application of Theorem 1.1, we get:

Theorem 4.2. *The energy*

$$E_{\Delta x}^k = \frac{\Delta x}{2} \sum_{j=0}^N \left(\left| \frac{v_j^k + v_{j+1}^k}{2} \right|^2 + \left| \frac{u_{j+1}^k - u_j^k}{\Delta x} \right|^2 \right), \quad k \in \mathbb{N},$$

of solutions of (4.14) is exponentially decaying, uniformly with respect to $\Delta t > 0$ and $\Delta x > 0$, in the sense of (3.2).

4.2.2 Method II: Imposing a CFL condition

Here we want to use Theorem 3.3 to derive uniform properties on the following fully discrete system, obtained by discretizing in time system (4.9) using (3.1):

$$\left\{ \begin{array}{l} \frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{v_j^k + v_j^{k+1}}{2}, \\ \frac{1}{4\Delta t} \left((v_{j-1}^{k+1} + 2v_j^{k+1} + v_{j+1}^{k+1}) - (v_{j-1}^k + 2v_j^k + v_{j+1}^k) \right) = \\ \frac{1}{2(\Delta x)^2} (u_{j+1}^{k+1} + u_{j+1}^k - 2u_j^{k+1} - 2u_j^k + u_{j-1}^{k+1} + u_{j-1}^k) \\ - \frac{1}{8}\sigma_{j+1/2}^2 \left((v_j^k + v_{j+1}^k) + (v_j^{k+1} + v_{j+1}^{k+1}) \right) \\ - \frac{1}{8}\sigma_{j-1/2}^2 \left((v_{j-1}^k + v_j^k) + (v_{j-1}^{k+1} + v_j^{k+1}) \right), \end{array} \right. \quad (4.17)$$

which holds for $(k, j) \in \mathbb{N} \times \{1, \dots, N\}$, with the boundary conditions (4.15) and initial data (4.16).

To apply Theorem 3.3, we need to estimate the norm of the matrix $A_{\Delta x}$ defined in (4.10). Actually, its spectrum is given in [5]: The eigenvalues of $A_{\Delta x}$ are

$$\lambda_{\pm l, \Delta x} = \pm \frac{2i}{\Delta x} \tan \left(l \Delta x \frac{\pi}{2} \right), \quad l \in \{1, \dots, N\}.$$

Since $A_{\Delta x}$ is skew-adjoint on $X_{\Delta x}$, its operator norm is given by its highest eigenvalue:

$$\|A_{\Delta x}\|_{\mathcal{L}(X_{\Delta x}, X_{\Delta x})} = \frac{2}{\Delta x} \tan \left((1 - \Delta x) \frac{\pi}{2} \right) \underset{\Delta x \rightarrow 0}{\simeq} \frac{4}{\pi(\Delta x)^2}.$$

As a consequence of Theorem 3.3, we get:

Theorem 4.3. *The energy*

$$E_{\Delta x}^k = \frac{\Delta x}{2} \sum_{j=0}^N \left(\left| \frac{v_j^k + v_{j+1}^k}{2} \right|^2 + \left| \frac{u_{j+1}^k - u_j^k}{\Delta x} \right|^2 \right), \quad k \in \mathbb{N},$$

of solutions of (4.17) is exponentially decaying, uniformly with respect to $\Delta t > 0$ and $\Delta x > 0$, in the sense of (3.2) provided there exists a constant η such that

$$\Delta t \leq \eta(\Delta x)^2. \quad (4.18)$$

Remark 4.4. In this case, the CFL condition (4.18) is very restrictive for practical computations. Therefore, in practice, the fully discrete scheme (4.14) that involves a numerical viscosity term, for which no CFL condition is needed, seems preferable.

4.3 A fully discrete damped wave equation: A viscous finite difference approximation

We now describe how our results may be combined with those of [27, 23], which add numerical viscosity in the discretization with respect to the space-variable, to derive a uniformly exponentially stable fully discrete scheme.

The finite difference space semi-discrete approximation scheme of system (4.1) is as follows

$$\begin{cases} \ddot{u}_j - \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} + \sigma_j^2 \dot{u}_j = 0, & t \in \mathbb{R}_+, j \in \{1, \dots, N\}, \\ u_0(t) = u_{N+1}(t) = 0, & t \in \mathbb{R}_+, \\ u_j(0) = u_{j,0}, \quad \dot{u}_j(0) = v_{j,0}, & j \in \{1, \dots, N\}, \end{cases} \quad (4.19)$$

where σ_j , $u_{j,0}$, $v_{j,0}$ and u_j are, respectively, approximations of the functions σ , u_0 , v_0 at the point x_j .

The energy of system (4.19), given by

$$E_{\Delta x}(t) = \frac{\Delta x}{2} \sum_{j=0}^N \left(|\dot{u}_j(t)|^2 + \left| \frac{u_{j+1}(t) - u_j(t)}{\Delta x} \right|^2 \right), \quad (4.20)$$

is dissipated according to the law

$$\frac{dE_{\Delta x}}{dt}(t) = -\Delta x \sum_{j=1}^N \sigma_j^2 |\dot{u}_j(t)|^2.$$

However, due to spurious high frequency solutions that are created by the numerical scheme, the energies $E_{\Delta x}$ do not decay exponentially uniformly with respect to Δx (see [18, 27]), except in the particular case $\omega = (0, 1)$: If $\omega \neq (0, 1)$, there are no positive constants μ and ν such that the inequality

$$E_{\Delta x}(t) \leq \mu E_{\Delta x}(0) \exp(-\nu t), \quad t \geq 0, \quad (4.21)$$

holds for any $\Delta x > 0$ and for any solution of (4.19).

Therefore, to get a uniform decay rate of the energies $E_{\Delta x}$ (with respect to $\Delta x > 0$), an extra numerical viscosity term was added in [27]:

$$\begin{cases} \ddot{u}_j - \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} + \sigma_j^2 \partial_t u_j \\ \quad - (\Delta x)^2 \left(\frac{\dot{u}_{j+1} - 2\dot{u}_j + \dot{u}_{j-1}}{(\Delta x)^2} \right) = 0, & t \in \mathbb{R}_+, j \in \{1, \dots, N\}, \\ u_0(t) = u_{N+1}(t) = 0, & t \in \mathbb{R}_+, \\ u_j(0) = u_{j,0}, \quad u'_j(0) = v_{j,0}, & j \in \{1, \dots, N\}. \end{cases} \quad (4.22)$$

For this system, the energy, still defined by (4.20), is now dissipated according to the law:

$$\frac{dE_{\Delta x}}{dt}(t) = -\Delta x \sum_{j=1}^N \sigma_j^2 |\dot{u}_j(t)|^2 - (\Delta x)^3 \sum_{j=0}^N \left(\frac{u_{j+1}(t) - u_j(t)}{\Delta x} \right)^2.$$

It was proved in [27] that, if σ satisfies (4.2), the energy of the solutions of (4.22) is exponentially stable uniformly with respect to the mesh size $\Delta x > 0$, in the sense that there exist positive constants μ and ν such that (4.21) holds for any $\Delta x > 0$ and for any solution of (4.22).

Besides, one can check that system (4.22) can be written as

$$\ddot{U}_{\Delta x} + A_{0,\Delta x} U_{\Delta x} + B_{0,\Delta x}^* B_{0,\Delta x} \dot{U}_{\Delta x} + (\Delta x)^2 A_{0,\Delta x} \dot{U}_{\Delta x} = 0, \quad t \in \mathbb{R}_+, \quad (4.23)$$

where $U_{\Delta x} = (u_1, \dots, u_j, \dots, u_N)^*$, $A_{0,\Delta x}$ is a positive definite matrix, which represents the discrete Laplace operator, and $B_{0,\Delta x}$ is the $N \times N$ matrix defined by:

$$B_{0,\Delta x} = \left(\text{diag}(\sigma_j) \right).$$

4.3.1 Exponential decay for the time continuous system (4.23)

In this Subsection, we indicate how one can prove the uniform exponential decay result for solutions of (4.23) using the combination of the results in [18] and the methods introduced in [14] and further developed in Corollary 3.8.

Let us first recall the results in [14]. Let H be a Hilbert space endowed with the norm $\|\cdot\|_H$. Let $A_0 : \mathcal{D}(A_0) \rightarrow H$ be a self-adjoint positive operator with compact resolvent and $C \in \mathfrak{L}(H, Y)$.

We then consider the initial value problem

$$\begin{cases} \ddot{u} + A_0 u + \varepsilon A_0 \dot{u} + B^* B \dot{u} = 0, & t \geq 0, \\ u(0) = u_0 \in \mathcal{D}(A_0^{1/2}), & \dot{u}(0) = u_1 \in H. \end{cases} \quad (4.24)$$

The energy of solutions of (4.24) is given by

$$E(t) = \frac{1}{2} \|\dot{u}(t)\|_H^2 + \frac{1}{2} \left\| A_0^{1/2} u(t) \right\|_H^2, \quad (4.25)$$

and satisfies

$$\frac{dE}{dt}(t) = - \|B\dot{u}(t)\|_Y^2 - \varepsilon \left\| A_0^{1/2} \dot{u}(t) \right\|_H^2. \quad (4.26)$$

Theorem 4.5. *Assume that system (4.24) with $\varepsilon = 0$ is exponentially stable and satisfies (1.4) for some positive constants μ and ν , and that $B \in \mathfrak{L}(H, Y)$.*

Then there exist two positive constants μ_0 and ν_0 depending only on $\|B\|_{\mathfrak{L}(H, Y)}$, ν and μ such that any solution of (4.24) satisfies (1.4) with constants μ_0 and ν_0 uniformly with respect to the viscosity parameter $\varepsilon \in [0, 1]$.

We now introduce the spectrum of A_0 . Since A_0 is self-adjoint positive definite with compact resolvent, its spectrum is discrete and $\sigma(A_0) = \{\lambda_j^2 : j \in \mathbb{N}\}$, where λ_j is an increasing sequence of real positive numbers such that $\lambda_j \rightarrow \infty$ when $j \rightarrow \infty$. Set $(\Psi_j)_{j \in \mathbb{N}}$ an orthonormal basis of eigenvectors of A_0 associated to the eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}}$.

For convenience, similarly as in (2.2), we define

$$\mathfrak{C}_s = \text{span} \{ \Psi_j : \text{the corresponding } \lambda_j \text{ satisfies } |\lambda_j| \leq s \}. \quad (4.27)$$

We claim that the proof of Theorem 4.5 in [14] also proves the following Theorem:

Theorem 4.6. *Let $\varepsilon \in (0, 1]$. Assume that system*

$$\ddot{u} + A_0 u = 0, \quad t \geq 0, \quad u(0) = u_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{u}(0) = u_1 \in H. \quad (4.28)$$

is exactly observable within the class $\mathfrak{C}_{1/\sqrt{\varepsilon}}$ in the following sense: there exist a time $T^ > 0$ and a positive constant $k_* > 0$ such that any solution u of (4.28) with initial data $(u_0, u_1) \in \mathfrak{C}_{1/\sqrt{\varepsilon}}^2$ satisfies*

$$k_* \left(\left\| A_0^{1/2} u_0 \right\|_H^2 + \|u_1\|_H^2 \right) \leq \int_0^{T^*} \|B\dot{u}(t)\|_Y^2 dt.$$

Then there exist two positive constants μ and ν depending only on $\|B\|_{\mathfrak{L}(H, Y)}$, T^ and k_* such that any solution of (4.24) satisfies (1.4).*

In [18], it has been proved that there exist positive constants T^* and k_* such that for all $\Delta x > 0$, the solution of

$$\ddot{U}_{\Delta x} + A_{0, \Delta x} U_{\Delta x} = 0, \quad t \geq 0, \quad (4.29)$$

with initial data $(U_{0, \Delta x}, U_{1, \Delta x}) \in \mathfrak{C}_{1/\Delta x}(A_{\Delta x})^2$ satisfies

$$k_* \left(\left\| A_{0, \Delta x}^{1/2} U_{0, \Delta x} \right\|_{* \Delta x}^2 + \|U_{1, \Delta x}\|_{* \Delta x}^2 \right) \leq \int_0^{T^*} \left\| B_{\Delta x} \dot{U}_{\Delta x}(t) \right\|_{* \Delta x}^2 dt.$$

Setting $X_{* \Delta x} = \mathbb{R}^N$ endowed with the norm $\|\cdot\|_{* \Delta x}$, one easily checks that $\|B_{\Delta x}\|_{\mathfrak{L}(X_{* \Delta x}, X_{* \Delta x})}$ is bounded uniformly in $\Delta x > 0$.

Theorem 4.6 then applies, and proves that systems (4.23) are exponentially stable uniformly with respect to $\Delta x > 0$.

Remark 4.7. Note that this method also applies in higher dimension, using for instance the results in [31] which state uniform observability properties for finite difference approximation schemes of a 2d wave equation. Doing this, we recover the results in [27] in 2d.

We now go on analyzing (4.22). We rewrite system (4.22) as

$$\dot{Z}_{\Delta x} = A_{\Delta x} Z_{\Delta x} - B_{\Delta x}^* B_{\Delta x} Z_{\Delta x}, \quad t \in \mathbb{R}_+, \quad (4.30)$$

where

$$\begin{aligned} Z_{\Delta x} &= \begin{pmatrix} U_{\Delta x} \\ V_{\Delta x} \end{pmatrix}, \quad A_{\Delta x} = \begin{pmatrix} 0 & Id \\ -A_{0, \Delta x} & 0 \end{pmatrix}, \\ B_{\Delta x} &= \left(0 \quad \sqrt{B_{0, \Delta x} + (\Delta x)^2 A_{0, \Delta x}} \right). \end{aligned} \quad (4.31)$$

One can check that the operator $A_{\Delta x}$ is skew-adjoint on the vector space $X_{\Delta x} = \mathbb{R}^{2N}$ endowed with the norm $\|\cdot\|_{\Delta x}$:

$$\left\| \begin{pmatrix} U_{\Delta x} \\ V_{\Delta x} \end{pmatrix} \right\|_{\Delta x}^2 = \Delta x \sum_{j=0}^N \left(|v_j|^2 + \left| \frac{u_{j+1} - u_j}{\Delta x} \right|^2 \right), \quad (4.32)$$

where $U_{\Delta x} = (u_1, \dots, u_j, \dots, u_N)^*$ and $V_{\Delta x} = (v_1, \dots, v_j, \dots, v_N)^*$, with the convention $u_0 = u_{N+1} = 0$.

Note that the original energy (4.20) of system (4.22) coincides with the quantity $\|Z_{\Delta x}\|_{\Delta x}^2/2$ of solutions of (4.30), with the notation above.

We then need to check that the operator $B_{\Delta x}$ is a bounded map from $X_{\Delta x}$ to $X_{*\Delta x}^2 = \mathbb{R}^{2N}$, where $X_{*\Delta x} = \mathbb{R}^N$ is endowed with the classical discrete L^2 norm $\|\cdot\|_{*\Delta x}$ given in (4.11). Since σ is assumed to be in $L^\infty(0, 1)$, we obviously have

$$\|\text{diag}(\sigma_j)V_{\Delta x}\|_{*\Delta x} \leq \|\sigma\|_{L^\infty} \|V_{\Delta x}\|_{*\Delta x}.$$

Besides, using a standard interpolation property,

$$\left\| (\Delta x) A_{0,\Delta x}^{1/2} V_{\Delta x} \right\|_{*\Delta x}^2 \leq \|V_{\Delta x}\|_{*\Delta x} \left\| (\Delta x)^2 A_{0,\Delta x} V_{\Delta x} \right\|_{*\Delta x} \leq 4 \|V_{\Delta x}\|_{*\Delta x}^2,$$

since

$$(\Delta x)^2 A_{0,\Delta x} V_{\Delta x} = W_{\Delta x}, \quad \text{with } w_j = v_{j+1} - 2v_j + v_{j-1}, \quad \forall j \in \{1, \dots, N\}.$$

Combining these last inequalities, we get the uniform bound

$$\|B_{\Delta x}\|_{\mathcal{L}(X_{\Delta x}, X_{*\Delta x}^2)} \leq 2 + \|\sigma\|_{L^\infty}.$$

We are therefore in the setting of Section 3: We checked hypothesis (A1) and (A2) of Definition 3.1 for the operators $A_{\Delta x}$ and $B_{\Delta x}$, and (A3) comes from the results of [27].

We now present the applications of the abstract methods in Section 3 to this particular setting.

4.3.2 Method I: Adding a numerical viscosity term in time

We introduce the fully discrete approximation scheme, corresponding to (1.9), given by

$$\left\{ \begin{array}{l} \frac{\tilde{u}_j^{k+1} - u_j^k}{\Delta t} = \frac{v_j^k + \tilde{v}_j^{k+1}}{2}, \\ \frac{\tilde{v}_j^{k+1} - v_j^k}{\Delta t} = \frac{1}{2(\Delta x)^2} (\tilde{u}_{j+1}^{k+1} + u_{j+1}^k - 2\tilde{u}_j^{k+1} - 2u_j^k + \tilde{u}_{j-1}^{k+1} + u_{j-1}^k) \\ - \frac{1}{2}\sigma_j^2(v_j^k + \tilde{v}_j^{k+1}) + \frac{1}{2}(\tilde{v}_{j+1}^{k+1} + v_{j+1}^k - 2\tilde{v}_j^{k+1} - 2v_j^k + \tilde{v}_{j-1}^{k+1} + v_{j-1}^k), \\ \frac{u_j^{k+1} - \tilde{u}_j^{k+1}}{\Delta t} = \left(\frac{\Delta t}{\Delta x}\right)^2 (u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}), \\ \frac{v_j^{k+1} - \tilde{v}_j^{k+1}}{\Delta t} = \left(\frac{\Delta t}{\Delta x}\right)^2 (v_{j+1}^{k+1} - 2v_j^{k+1} + v_{j-1}^{k+1}), \end{array} \right. \quad (4.33)$$

which holds for $(k, j) \in \mathbb{N} \times \{1, \dots, N\}$, with the boundary conditions (4.15) and the initial data (4.16). Here again, u_j^k and v_j^k respectively denote approximations of the functions u and \dot{u} in $x_j = j\Delta x$ at time $k\Delta t$.

This fully discrete approximation scheme coincides with the system (1.9) with $A = A_{\Delta x}$ and $B = B_{\Delta x}$.

Applying Theorem 1.1, we get:

Theorem 4.8. *The energy*

$$E_{\Delta x}^k = \frac{\Delta x}{2} \sum_{j=0}^N \left(|v_j^k|^2 + \left| \frac{u_{j+1}^k - u_j^k}{\Delta x} \right|^2 \right) \quad (4.34)$$

of solutions of system (4.33) is exponentially decaying, uniformly with respect to both parameters $\Delta x > 0$ and $\Delta t > 0$. To be more precise, there exist positive constants ν_0 and μ_0 such that the energies of solutions (4.33) satisfy (3.2).

Note that in Theorem 4.8, no CFL condition is required.

4.3.3 Method II: Imposing a CFL condition

Again, we consider the space semi-discrete approximation (4.22) (or equivalently (4.30)) of (4.1), that we now discretize in time using the midpoint scheme

(1.5): For all $(k, j) \in \mathbb{N} \times \{1, \dots, N\}$,

$$\left\{ \begin{array}{l} \frac{u_j^{k+1} - u_j^k}{\Delta t} = \frac{v_j^k + v_j^{k+1}}{2}, \\ \frac{v_j^{k+1} - v_j^k}{\Delta t} = \frac{1}{2(\Delta x)^2} (u_{j+1}^{k+1} + u_{j+1}^k - 2u_j^{k+1} - 2u_j^k + u_{j-1}^{k+1} + u_{j-1}^k) \\ - \frac{1}{2} \sigma_j^2 (v_j^k + v_j^{k+1}) + \frac{1}{2} (v_{j+1}^{k+1} + v_{j+1}^k - 2v_j^{k+1} - 2v_j^k + v_{j-1}^{k+1} + v_{j-1}^k), \end{array} \right. \quad (4.35)$$

with the boundary conditions (4.15), and initial data (4.16).

The discrete energies are defined by (4.34) as before. Note that this scheme is simpler than (4.33), since it does not contain numerical viscosity terms in time.

To use Theorem 3.3, we need to estimate the norm $\|A_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})}$.

Actually, if

$$Z_{1\Delta x} = \begin{pmatrix} U_{1\Delta x} \\ V_{1\Delta x} \end{pmatrix}, \quad Z_{2\Delta x} = \begin{pmatrix} U_{2\Delta x} \\ V_{2\Delta x} \end{pmatrix},$$

then

$$\begin{aligned} \langle Z_{1\Delta x}, A_{\Delta x} Z_{2\Delta x} \rangle_{\Delta x} &= \Delta x \sum_{j=0}^N \left(\frac{u_{1\Delta x, j+1} - u_{1\Delta x, j}}{\Delta x} \right) \left(\frac{v_{2\Delta x, j+1} - v_{2\Delta x, j}}{\Delta x} \right) \\ &\quad - \Delta x \sum_{j=1}^N v_{1\Delta x, j} \left(\frac{u_{2\Delta x, j+1} - 2u_{2\Delta x, j} + u_{2\Delta x, j-1}}{(\Delta x)^2} \right). \end{aligned}$$

In particular,

$$\begin{aligned} (\Delta x)^2 \left| \langle Z_{1\Delta x}, A_{\Delta x} Z_{2\Delta x} \rangle_{\Delta x} \right|^2 &\leq \left(\Delta x \sum_{j=0}^N \left(\frac{u_{1\Delta x, j+1} - u_{1\Delta x, j}}{\Delta x} \right)^2 \right) \left(\Delta x \sum_{j=0}^N \left(v_{2\Delta x, j+1} - v_{2\Delta x, j} \right)^2 \right) \\ &\quad + \left(\Delta x \sum_{j=1}^N |v_{1\Delta x, j}|^2 \right) \left(\Delta x \sum_{j=0}^N \left(\frac{u_{2\Delta x, j+1} - u_{2\Delta x, j}}{\Delta x} - \frac{u_{2\Delta x, j} - u_{2\Delta x, j-1}}{\Delta x} \right)^2 \right), \end{aligned}$$

that gives

$$\left| \langle Z_{1\Delta x}, A_{\Delta x} Z_{2\Delta x} \rangle_{\Delta x} \right| \leq \frac{2}{\Delta x} \|Z_{1\Delta x}\|_{\Delta x} \|Z_{2\Delta x}\|_{\Delta x}.$$

This proves that $\|A_{\Delta x}\|_{\mathfrak{L}(X_{\Delta x}, X_{\Delta x})} \leq 2/\Delta x$. Actually, in this case, we know the eigenvalues and eigenvectors explicitly (see for instance [18]), and therefore this norm can be computed explicitly to be $2 \sin((1 - \Delta x)\pi/2)/\Delta x$.

As a corollary of Theorem 3.3, we get:

Theorem 4.9. *Given $\eta > 0$, if we impose the CFL type condition*

$$\Delta t \leq \eta \Delta x, \quad (4.36)$$

then there exist positive constants ν_η and μ_η such that the energy of solutions of (4.35) satisfies (3.2), uniformly with respect to the discretization parameters $\Delta x > 0$ and $\Delta t > 0$.

Remark 4.10. Here it seems more natural to use the discretization (4.35) than (4.33) since the CFL condition (4.36) is not very restrictive.

Note that the results we presented here for the 1d wave equation can be adapted to deal with 2d wave equations in a square as in [27] or more general domains as in [23].

4.3.4 Method III: Discretizing with only one viscosity term

We are in the setting of Theorem 3.7, and therefore we can use only one viscosity term: Set $\varepsilon = \max\{\Delta t, \Delta x\}$ and consider

$$\left\{ \begin{array}{l} \frac{\tilde{u}_j^{k+1} - u_j^k}{\Delta t} = \frac{v_j^k + \tilde{v}_j^{k+1}}{2}, \\ \frac{\tilde{v}_j^{k+1} - v_j^k}{\Delta t} = \frac{1}{2(\Delta x)^2} (\tilde{u}_{j+1}^{k+1} + u_{j+1}^k - 2\tilde{u}_j^{k+1} - 2u_j^k + \tilde{u}_{j-1}^{k+1} + u_{j-1}^k) \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad - \frac{1}{2} \sigma_j^2 (v_j^k + \tilde{v}_j^{k+1}), \\ \frac{u_j^{k+1} - \tilde{u}_j^{k+1}}{\Delta t} = \left(\frac{\varepsilon}{\Delta x} \right)^2 (u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}), \\ \frac{v_j^{k+1} - \tilde{v}_j^{k+1}}{\Delta t} = \left(\frac{\varepsilon}{\Delta x} \right)^2 (v_{j+1}^{k+1} - 2v_j^{k+1} + v_{j-1}^{k+1}), \end{array} \right. \quad (4.37)$$

which holds for $(k, j) \in \mathbb{N} \times \{1, \dots, N\}$, with the boundary conditions (4.15) and initial data (4.16).

Theorem 4.11. *Setting $\varepsilon = \max\{\Delta t, \Delta x\}$, the energy $E_{\Delta x}^k$ defined in (4.34) of solutions of system (4.37) is exponentially decaying, uniformly with respect to both parameters $\Delta x > 0$ and $\Delta t > 0$. To be more precise, there exist positive constants ν_0 and μ_0 such that the energy of solutions (4.33) satisfies (3.2).*

Remark 4.12. The main advantage of (4.37) over (4.33) is the presence of only one viscosity operator. In other words, (4.33) dissipates too much.

The advantage of (4.37) over (4.35) consists in the absence of CFL condition, which makes (4.37) more robust in practice.

5 Further comments

1. As we mentioned in the introduction, our methods and results require the assumption that the damping operator B is bounded. This is due to the method we employ, which is based on the equivalence between the exponential decay of the energy and the observability properties of the conservative system, that requires the damping operator to be bounded. That is the case, even in the continuous setting. However, in several relevant applications, as for instance when dealing with the problem of boundary stabilization of the wave equation (see [20]), the feedback law is unbounded, and our method does not apply. This issue requires further work.

2. Another drawback of our method is that it provides an explicit estimate of the exponential decay rate of the energy of the time semi-discrete approximation systems, which is far from sharp in general. Again, this also happens in the continuous case, since we deduce stabilization properties from the study of the observability properties of the corresponding conservative systems. In the continuous case, the computation of the decay rate of the energy is technically involved and requires to work directly on the damped system. We refer to the works [7, 8, 19] that deal with these questions for damped wave equations.

In our context, it would be also relevant to ask if one can choose the numerical viscosity term such that the time-discrete damped systems are exponentially stable, uniformly with respect to the time discretization parameter, and such that the decay rate of the energy of these time discrete systems coincides with the one of the continuous system. To our knowledge, this issue is still open. Let us mention the work [13], which gives a partial answer to this question for space semi-discrete approximation schemes of the 1d Perfectly Matched Layers equations, which correspond to a particular instance of damped wave equations.

3. In this article, we assumed exponential decay properties for the continuous damped systems under consideration. However, there are several important models of vibrations where the energy decay rate is polynomial or even logarithmic within the class of solutions with initial data in $\mathcal{D}(A)$ instead of X . That is the case for instance for networks of vibrating strings [9] or damped wave equations, when the damping operator is effective on a subdomain where the *Geometric Control Condition* is not fulfilled [2, 19]. One could ask if there is a systematic discretization method for these systems that preserves these decay properties. To our knowledge, this issue is widely open. The time semi-discrete schemes provided here are good candidates to preserve these decay properties.

4. The same questions arise when discretizing in time semilinear wave equations. For instance, in [10] (see also [29, 30]), the exponential decay property of solutions of semilinear wave equations in \mathbb{R}^3 with a damping term which is effective on the exterior of a ball are analyzed. Under suitable properties of the nonlinearity, it is proved that the exponential decay of the energy holds locally uniformly for finite energy solutions. It would be interesting to analyze whether the same exponential decay property holds, uniformly with respect to the time-step, for the numerical schemes analyzed in this article in this semilinear setting.

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